Generalized-homogeneous deviation means

By Z. DARÓCZY and ZS. PÁLES (Debrecen)

1. Introduction. Let $\varphi\colon \mathbf{R}_+ \to \mathbf{R}$ be a strictly monotone and continuous function. For the quasiarithmetic mean $M_{n,\varphi}\colon \mathbf{R}_+^n \to \mathbf{R}_+$ $(n\in\mathbf{N})$ defined by

$$M_{n,\varphi}(\underline{x}) := \varphi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right] \tag{1.1}$$

whenever $\underline{x}=(x_1, \dots, x_n)\in\mathbf{R}_+^n$, $n\in\mathbf{N}$ the following basic result is well known (see HARDY—LITTLEWOOD—PÓLYA [8, Theorem 84.]): *If* (1.1) *is homogeneous* i.e. $M_{n,\varphi}(t\underline{x})=tM_{n,\varphi}(\underline{x})$ *for* $t\in\mathbf{R}_+$, $\underline{x}\in\mathbf{R}_+^n$, $n\in\mathbf{N}$ *then there exists* $a\in\mathbf{R}$, *such that*

$$M_{n,\varphi}(\underline{x}) = M_{n,a}(\underline{x}) \tag{1.2}$$

for each $\underline{x}\in\mathbf{R}_+^n$ *and* $n\in\mathbf{N}$ *where*

$$M_{n,a}(\underline{x}) := \begin{cases} \left[\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i^a\right]^{1/a}, & \text{if } a\neq 0,\\ \left[\displaystyle\prod_{i=1}^{n} x_i\right]^{1/n}, & \text{if } a=0.\end{cases} \tag{1.3}$$

One of the known generalizations of quasiarithmetic means is the concept of *deviation means* (see DARÓCZY [3], [4]).

The function $E\colon \mathbf{R}_+^2 \to \mathbf{R}$ is called a *deviation* on $\mathbf{R}_+$ if the following properties are satisfied:

(E1) The function $y\to E(x, y)$ is strictly monotone decreasing and continuous for any fixed $x\in\mathbf{R}_+$.

(E2) $E(x, x)=0$ for any $x\in\mathbf{R}$.

Denote by $\varepsilon(\mathbf{R}_+)$ the set of all deviations defined on $\mathbf{R}_+$. It is known (see DARÓCZY [3]) that, for arbitrary $n\in\mathbf{N}$ and $\underline{x}=(x_1, \dots, x_n)\in\mathbf{R}_+^n$, the equation

$$\sum_{i=1}^{n} E(x_i, y) = 0 \tag{1.4}$$

has a unique solution $y_0\in\mathbf{R}_+$ such that $\min\{x_i \mid 1\leqq i\leqq n\}\leqq y_0\leqq\max\{x_i \mid 1\leqq i\leqq n\}$ is also satisfied. Therefore the symmetrical value $y_0=: \mathfrak{M}_{n,E}(\underline{x})$ is called the *E-deviation mean* of $\underline{x}$.

If $\varphi\colon \mathbf{R}_+\to\mathbf{R}$ is a strictly monotone and continuous function then

$$E_\varphi(x, y) := \varepsilon[\varphi(x)-\varphi(y)], \quad (x, y\in\mathbf{R}_+) \tag{1.5}$$

is a deviation on $\mathbf{R}_+$, where $\varepsilon=1$ if φ increases and $\varepsilon=-1$ if φ decreases. It is easy to see that

$$\mathfrak{M}_{n, E_\varphi}(\underline{x}) = M_{n,\varphi}(\underline{x}) \tag{1.6}$$

i.e. quasiarithmetic means are deviation means, too. Therefore, the following question seems to be natural: What can we state about homogeneous deviation means? The following result answers this question (DARÓCZY [3]): *If* $E\in\varepsilon(\mathbf{R}_+)$ *and* $\mathfrak{M}_{n,E}(t\,\underline{x})=t\mathfrak{M}_{n,E}(\underline{x})$ *for any* $t\in\mathbf{R}_+$, $\underline{x}\in\mathbf{R}_+^n$, $n\in\mathbf{N}$, *then there exists* $h\colon \mathbf{R}_+\to\mathbf{R}_+$ *with* $h(1)=1$ *and* $f\colon \mathbf{R}_+\to\mathbf{R}$ *with* $\operatorname{sgn} f(x)=\operatorname{sgn}(1-x)$, $(x\in\mathbf{R}_+)$ *such that*

$$E(x, y) = h(y) f\left(\frac{x}{y}\right). \tag{1.7}$$

This result is unsatisfactory from many points of view. On the one hand it is not reversible since (1.7) is not a deviation for any given h and f, on the other hand there still exist too many homogeneous deviation means. These problems lead us to introduce a new homogeneity property.

Definition. Let $E\in\varepsilon(\mathbf{R}_+)$. The deviation mean $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ is said to be *k-homogeneous* if

$$\mathfrak{M}_{nk,E}(\underline{t}\circ\underline{x}) = \mathfrak{M}_{k,E}(\underline{t})\mathfrak{M}_{n,E}(\underline{x}) \tag{1.8}$$

for any $\underline{t}=(t_1, \dots, t_k)\in\mathbf{R}_+^k$ and $\underline{x}\in\mathbf{R}_+^n$, $n\in\mathbf{N}$ and where

$$\underline{t}\circ\underline{x} = (t_1x_1, \dots, t_1x_n, \dots, t_kx_1, \dots, t_kx_n)\in\mathbf{R}_+^{kn}. \tag{1.9}$$

We remark that k-homogeneous $(k\geqq 2)$ deviation means are also 1-homogeneous means i.e. homogeneous means since substituting $\underline{t}=(t, \dots, t)$ $(t\in\mathbf{R}_+)$ in (1.8) we obtain

$$\mathfrak{M}_{k,E}(\underline{t}) = t \quad\text{and}\quad \mathfrak{M}_{kn,E}(\underline{t}\circ\underline{x}) = \mathfrak{M}_{n,E}(t\underline{x}).$$

On the other hand it can easily be seen that homogeneous quasiarithmetic means are k-homogeneous means, too, since

$$M_{kn,a}(\underline{t}\circ\underline{x}) = \left(\frac{1}{kn}\sum_{j=1}^{k}\sum_{i=1}^{n}(t_jx_i)^a\right)^{1/a} = \left(\frac{1}{k}\sum_{j=1}^{k}t_j^a\frac{1}{n}\sum_{i=1}^{n}x_i^a\right)^{1/a} = M_{k,a}(\underline{t})M_{n,a}(\underline{x}) \tag{1.10}$$

if $a\neq 0$. Either by letting $a\to 0$ in (1.10) or by a direct computation we obtain that (1.10) is valid for $a=0$ as well.

Deviation means which are k-homogeneous for any k are called *multiplicative* means. The class of these means is known (DARÓCZY—PÁLES [7, Theorem 9]): *If* $E\in\varepsilon(\mathbf{R}_+)$ *and* $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ *is multiplicative then there exist a multiplicative function* $m\colon \mathbf{R}_+\to\mathbf{R}_+$ (i.e. $m(xy)=m(x)m(y)$ *if* $x, y\in\mathbf{R}_+$) *and a constant* $a\in\mathbf{R}\setminus\{0\}$ *such that either*

$$\mathfrak{M}_{n,E}(\underline{x}) = \exp\left[\frac{\sum_{i=1}^{n}m(x_i)\ln x_i}{\sum_{i=1}^{n}m(x_i)}\right], \quad (\underline{x}\in\mathbf{R}_+^n) \tag{1.11}$$

or

$$(1.12)\qquad \mathfrak{M}_{n,E}(\underline{x})=\left[\frac{\sum_{i=1}^{n} m(x_i)\,x_i^a}{\sum_{i=1}^{n} m(x_i)}\right]^{1/a},\quad (\underline{x}\in\mathbf{R}_+^n).$$

Conversely, the means standing on the right hand side of (1.11) *and* (1.12) *are multiplicative deviation means.*

In the present article we shall investigate the k-homogeneous deviation means for some fixed $k\geqq 2$. We shall prove a surprising result:

If E is differentiable with respect to its second variable and its derivative is nonvanishing then $\mathfrak{M}_{n,E}$ $(n\in\mathbf{N})$ is a k-homogeneous deviation mean if and only if there exist a multiplicative function m and a constant $a\in\mathbf{R}\setminus\{0\}$ such that either (1.11) *or* (1.12) *is satisfied.*

With the help of this result we can easily see that the k-homogeneity of a regular deviation mean (for some fixed $k\geqq 2$) implies the multiplicativity of this mean.

2. Basic functional equation. In this section we deduce a functional equation which plays an essential role in our discussion.

Theorem 1. *Let $E\in\varepsilon(\mathbf{R}_+)$ and assume that $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ is a k-homogeneous mean for some fixed $k\geqq 2$. Then the functions*

$$(2.1)\qquad f(x):=E(x,1),\quad \mu(x):=\mathfrak{M}_{k,E}(x,1,\dots,1),\quad x\in\mathbf{R}_+$$

satisfy the functional equation

$$(2.2)\qquad \frac{f(xy)}{f(x)f(y)}+\frac{k-1}{f(x)}=\frac{f(\mu(x)y)}{f(\mu(x))f(y)}$$

for any $x,y\in\mathbf{R}_+\setminus\{1\}$.

Proof. By the definition of deviation means, $\mathfrak{M}_{n,E}$ is k-homogeneous if and only if

$$(2.3)\qquad \sum_{i=1}^{n}\sum_{j=1}^{k} E\big(t_jx_i,\mathfrak{M}_{k,E}(\underline{t})\,\mathfrak{M}_{n,E}(\underline{x})\big)=0$$

for any $\underline{t}=(t_1,\dots,t_k)\in\mathbf{R}_+^k$, $\underline{x}=(x_1,\dots,x_n)\in\mathbf{R}_+^n$ $n\in\mathbf{N}$. Let $\underline{t}\in\mathbf{R}_+^k$ be fixed and define $F_{\underline{t}}\colon \mathbf{R}_+^2\to\mathbf{R}$ by

$$(2.4)\qquad F_{\underline{t}}(x,y):=\sum_{j=1}^{k} E\big(t_jx,\mathfrak{M}_{k,E}(\underline{t})\,y\big).$$

Then $F_{\underline{t}}\in\varepsilon(\mathbf{R}_+)$ since it is strictly monotone decreasing and continuous in the second variable, further, by the homogeneity of $\mathfrak{M}_{n,E}$,

$$F_{\underline{t}}(x,x)=\sum_{j=1}^{k} E\big(t_jx,\mathfrak{M}_{k,E}(\underline{t})\,x\big)=\sum_{j=1}^{k} E\big(t_jx,\mathfrak{M}_{k,E}(x\underline{t})\big)=0.$$

Applying (2.4) the equation (2.3) turns into

$$\sum_{i=1}^{n} F_{\underline{t}}(x_i, \mathfrak{M}_{n,E}(\underline{x})) = 0,$$

from which we obtain

$$\mathfrak{M}_{n,E}(\underline{x}) = \mathfrak{M}_{n,F_{\underline{t}}}(\underline{x}) \tag{2.5}$$

for any $\underline{x}\in\mathbf{R}_+^n$, $n\in\mathbf{N}$. It is known (see DARÓCZY—PÁLES [6, Theorem 1]) that (2.5) is satisfied if and only if

$$F_{\underline{t}}(u, v)E(w, v) = F_{\underline{t}}(w, v)E(u, v) \tag{2.6}$$

for any $0<u\leqq v\leqq w$.

We see immediately that (2.6) is valid not only in the region $0<u\leqq v\leqq w$ but for any $0<u, v, w$. E.g. let $0<u\leqq v$ and $0<w\leqq v$. Choose $w^*>v$ and apply (2.6) for the values $0<u\leqq v<w^*$ and $0<w\leqq v<w^*$. Then we have

$$F_{\underline{t}}(u, v)E(w^*, v) = F_{\underline{t}}(w^*, v)E(u, v), \tag{2.7}$$

$$F_{\underline{t}}(w^*, v)E(w, v) = F_{\underline{t}}(w, v)E(w^*, v). \tag{2.8}$$

Multiplying (2.7) by (2.8) and dividing the resulting equation by

$$E(w^*, v)F_{\underline{t}}(w^*, v) > 0$$

we obtain (2.6) just for $0<u, w\leqq v$. In the case $0<v\leqq u, w$ the proof of (2.6) is completely similar.

Now let $x, y\in\mathbf{R}_+\setminus\{1\}$ be arbitrary and substitute into (2.6)

$$u = y\mu(x), \quad v = 1, \quad w = \mu(x),$$

$$\underline{t} = \left(\frac{x}{\mu(x)}, \frac{1}{\mu(x)}, \ldots, \frac{1}{\mu(x)}\right)\in\mathbf{R}_+^k.$$

Then, since $\mathfrak{M}_{k,E}$ is a homogeneous mean,

$$\mathfrak{M}_{k,E}(\underline{t}) = \mathfrak{M}_{k,E}\left(\frac{x}{\mu(x)}, \frac{1}{\mu(x)}, \ldots, \frac{1}{\mu(x)}\right) = \frac{1}{\mu(x)}\mathfrak{M}_{k,E}(x, 1, \ldots, 1) = 1$$

i.e., using the notations (2.1), it follows from (2.6) that

$$[f(xy)+(k-1)f(y)]f(\mu(x)) = f(x)f(\mu(x)y).$$

Dividing both sides by $f(x)f(\mu(x))f(y)\neq 0$ we obtain the desired equation (2.2). □

3. The elimination of μ. In the present section we deduce a functional equation that contains only the unknown function $f(x)=E(x, 1)$. We shall need the following definition (DARÓCZY [3]).

Definition. The function $E\colon \mathbf{R}_+^2\to\mathbf{R}$ is called a differentiable deviation on $\mathbf{R}_+$ if it satisfies the following properties:

(E*1) For each fixed $x, y\in\mathbf{R}_+$

$$E_2(x, y) := \frac{\partial E(x, y)}{\partial y}$$

exists and is negative.

(E*2) $E(x, x)=0$ for any $x\in\mathbf{R}_+$.

Denote by $\varepsilon^*(\mathbf{R}_+)$ the set of all differentiable deviations on $\mathbf{R}_+$. Then it is obvious that $\varepsilon^*(\mathbf{R}_+)\subset\varepsilon(\mathbf{R}_+)$.

Lemma. *Let $E\in\varepsilon^*(\mathbf{R}_+)$ and suppose $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ to be a homogeneous mean. Then, using the notation $f(x):=E(x,1)$, $(x\in\mathbf{R}_+)$, the function*

$$x \to \frac{f(xy)}{f(x)f(y)}-\frac{f(xz)}{f(x)f(z)} =: G(x, y, z) \tag{3.1}$$

is continuous on the set $\mathbf{R}_+\setminus\{1\}$ and the limit exists at $x=1$ for each fixed $y, z\in\mathbf{R}_+\setminus\{1\}$.

Proof. Since $\mathfrak{M}_{n,E}$ is a homogeneous mean, applying the second theorem mentioned in the introduction, we obtain that there exist $h\colon \mathbf{R}_+\to\mathbf{R}_+$ with $h(1)=1$ such that

$$E(x, y) = h(y) f\left(\frac{x}{y}\right) \tag{3.2}$$

for $x, y\in\mathbf{R}_+$. With the help of this relation we get

$$G(x, y, z) = \frac{E\left(y, \frac{1}{x}\right)}{E\left(1, \frac{1}{x}\right)E(y, 1)} - \frac{E\left(z, \frac{1}{x}\right)}{E\left(1, \frac{1}{x}\right)E(z, 1)} =$$

$$= \frac{1}{E(y, 1)}\frac{E\left(y, \frac{1}{x}\right)-E(y, 1)}{E\left(1, \frac{1}{x}\right)-E(1, 1)} - \frac{1}{E(z, 1)}\frac{E\left(z, \frac{1}{x}\right)-E(z, 1)}{E\left(1, \frac{1}{x}\right)-E(1, 1)}$$

for $x, y, z\in\mathbf{R}_+\setminus\{1\}$. It follows from the last expression that $x\to G(x, y, z)$ is a continuous function, and $\lim\limits_{x\to 1} G(x, y, z)$ exists since $E\in\varepsilon^*(\mathbf{R}_+)$. □

Theorem 2. *Let $E\in\varepsilon^*(\mathbf{R}_+)$ and assume that $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ is a k-homogeneous mean for some fixed $k\geqq 2$. Then, using the notation $f(x)=E(x,1)$,*

$$2\frac{f(xy)}{f(x)f(y)} = \frac{f(x^2)}{f^2(x)}+\frac{f(y^2)}{f^2(y)} \tag{3.3}$$

for any $x, y\in\mathbf{R}_+\setminus\{1\}$.

Proof. Let $y, z\in\mathbf{R}_+\setminus\{1\}$ be fixed values and define the function $\tilde{G}_{y,z}\colon \mathbf{R}_+\to\mathbf{R}$ with the help of G as follows

$$\begin{aligned}\tilde{G}_{y,z}(x) &:= G(x, y, z),\ x\in\mathbf{R}_+\setminus\{1\},\\ &:= \lim_{t\to 1} G(t, y, z), \quad x = 1.\end{aligned}$$

Then, by the Lemma, $\tilde{G}_{y,z}$ is a continuous function. Now we prove that $\tilde{G}_{y,z}$ is identically constant i.e. $\tilde{G}_{y,z}(x)=\tilde{G}_{y,z}(1)$ for any $x\in\mathbf{R}_+$.

Let $x_0>1$ be an arbitrary but fixed value, further let

$$H_{x_0} := \{x\in[1, x_0] \,|\, \tilde{G}_{y,z}(x) = \tilde{G}_{y,z}(x_0)\}.$$

Then the continuity of $\tilde{G}_{y,z}$ implies that H_{x_0} is closed, and since $x_0\in H_{x_0}$, H_{x_0} is nonvoid.

Now apply the functional equation (2.2) for the values x, y and x, z. Subtracting the equations obtained we get

$$\tilde{G}_{y,z}(x) = \tilde{G}_{y,z}(\mu(x))$$

for $x\in\mathbf{R}_+$. It follows from this relation that

$$\mu(H_{x_0})\subset H_{x_0}. \tag{3.4}$$

Denote by $\bar{x}_0$ the greatest lower bound of the set H_{x_0}. Since H_{x_0} is closed we have $\bar{x}_0\in H_{x_0}$. Therefore, by (3.4), we obtain

$$\mu(\bar{x}_0)\in H_{x_0}. \tag{3.5}$$

Now we prove that $1<\bar{x}_0$ cannot be valid. By the definition of $\mu(\bar{x}_0)$ we have

$$E(\bar{x}_0, \mu(\bar{x}_0))+(k-1)E(1, \mu(\bar{x}_0)) = 0.$$

Since $k\geqq 2$, it follows from this relation that

$$1 < \mu(\bar{x}_0) < \bar{x}_0 \tag{3.6}$$

provided that $1<\bar{x}_0$. However (3.6) and (3.5) contradict the definition of $\bar{x}_0$. Thus we have proved $1=\bar{x}_0$ i.e. $1\in H_{x_0}$. Therefore $\tilde{G}_{y,z}(x_0)=\tilde{G}_{y,z}(1)$.

If $x_0<1$ then it can analogously be seen that $\tilde{G}_{y,z}(x_0)=\tilde{G}_{y,z}(1)$ is also satisfied.

Now notice the relation $\tilde{G}_{y,z}(1)=-\tilde{G}_{z,y}(1)$ if $y, z\in\mathbf{R}_+\setminus\{1\}$. By its help we obtain the equation

$$G(x, y, x) = \tilde{G}_{y,x}(x)=\tilde{G}_{y,x}(1) = -\tilde{G}_{x,y}(1) = -\tilde{G}_{x,y}(y) = -G(y, x, y), \tag{3.7}$$

for $x, y\in\mathbf{R}_+\setminus\{1\}$. Taking into consideration the notion of G we get at once (3.3) from (3.7). □

4. The solution of (3.3). In this section we determine all the solutions $f\colon\mathbf{R}_+\to\mathbf{R}$ of (3.3) having the property $\operatorname{sgn} f(x)=\operatorname{sgn}(x-1)$, $x\in\mathbf{R}_+$.

Theorem 3. *Assume that the function* $f\colon \mathbf{R}_+\to\mathbf{R}$ *satisfies* $\operatorname{sgn} f(x)=\operatorname{sgn}(x-1)$ *for* $x\in\mathbf{R}_+$ *and* (3.3) *for* $x, y\in\mathbf{R}_+\setminus\{1\}$. *Then there exists a constant* $b\geqq 0$ *such* *that, for the function*

$$g(x)=\frac{f(x^2)}{f^2(x)},\quad x\in\mathbf{R}_+\setminus\{1\}, \tag{4.1}$$

we have

$$g(xy)[g(x)+g(y)]-g(x)g(y)=b^2 \tag{4.2}$$

for any $x, y\in\mathbf{R}_+\setminus\{1\}$.

PROOF. Let, for $x, y\in\mathbf{R}_+\setminus\{1\}$,

$$F(x, y):=2\frac{f(xy)}{f(x)f(y)}.$$

Then it is obvious that F satisfies the equation

$$F(xy, z)F(x, y)=F(x, yz)F(y, z) \tag{4.3}$$

On the other hand, by (3.3), we have

$$F(x, y)=g(x)+g(y).$$

Therefore it follows from (4.3) that

$$[g(xy)+g(z)][g(x)+g(y)]=[g(x)+g(yz)][g(y)+g(z)]$$

for $x, y, z\in\mathbf{R}_+\setminus\{1\}$. Using the notation

$$G(x, y):=g(xy)[g(x)+g(y)]-g(x)g(y)$$

we obtain

$$G(x, y)=G(y, z) \tag{4.4}$$

for $x, y, z\in\mathbf{R}_+\setminus\{1\}$. The repeated application of (4.4) yields

$$G(x, y)=G(y, z)=G(z, u) \tag{4.5}$$

for $x, y, z, u\in\mathbf{R}_+\setminus\{1\}$. It is obvious from (4.5) that G is identically constant, i.e. there exists $c\in\mathbf{R}$ so that

$$g(xy)[g(x)+g(y)]-g(x)g(y)=c \tag{4.6}$$

if $x, y\in\mathbf{R}_+\setminus\{1\}$.

To complete the proof of the theorem it is enough to show that $c\geqq 0$. Then $c=b^2$ for a suitable $b\geqq 0$.

Assume, on the contrary, that $c<0$. Then, by (4.6),

$$g(xy)g(x)<g(y)[g(x)-g(xy)].$$

If $x, y>1$ then $g(x), g(y), g(xy)>0$ therefore we get

$$g(x)>g(xy). \tag{4.7}$$

Substituting into (4.7) $x:=x^n$, $y:=x$ we obtain

$$g(x^n)>g(x^nx)=g(x^{n+1}).$$

On the other hand $g(x^n)>0$ hence the following limit exists

$$\lim_{n\to\infty} g(x^n) =: \bar{g}(x) \geqq 0.$$

Substituting $y:=x^n$ into (4.6) and calculating the limit as $n\to\infty$, we get

$$\bar{g}^2(x) = c.$$

However this is a contradiction, since $c<0$. □

First we discuss the functional equation (4.2) in the case $b=0$.

Theorem 4. *Assume that the function* $g\colon]1, \infty[\to \mathbf{R}_+$ *satisfies the equation*

$$g(xy)[g(x)+g(y)]-g(x)g(y) = 0 \tag{4.8}$$

for $x, y>1$. *Then there exists a positive constant d such that*

$$g(x) = \frac{1}{d \ln x} \tag{4.9}$$

for $x>1$.

Proof. Using the notation

$$l(x) := \frac{1}{g(x)}, \quad x > 1,$$

we obtain from (4.8) that

$$l(xy) = \frac{1}{g(xy)} = \frac{g(x)+g(y)}{g(x)\,g(y)} = \frac{1}{g(x)}+\frac{1}{g(y)} = l(x)+l(y)$$

if $x, y>1$. Let, for $t>0$,

$$A(t) = l(e^t).$$

Then we have

$$A(t+s) = A(t)+A(s)$$

for $t, s>0$. It is well known that there exists an additive function $\bar{A}\colon \mathbf{R}\to\mathbf{R}$ such that $\bar{A}(t)=A(t)$ if $t>0$ (see Aczél—Erdős [2], Daróczy—Losonczi [5]). On the other hand $\bar{A}(t)=A(t)>0$ if $t\in\mathbf{R}_+$, therefore there exists a constant $d>0$ such that $\bar{A}(t)=dt$ for $t\in\mathbf{R}$ (see Aczél [1]). However, for $t>0$, $l(e^t)=dt$, i.e. substituting $l=\ln x$, we obtain

$$l(x) = d \ln x$$

for $x>1$. Therefore we just get the solution (4.9). □

Theorem 5. *Let* $b>0$, *and assume that* $g\colon]1, \infty[\to\mathbf{R}_+$ *satisfies the equation*

$$g(xy)[g(x)+g(y)]-g(x)g(y) = b^2 \tag{4.10}$$

for $x, y>1$. *Then either*

(i) $$g(x) = b, \quad x > 1,$$

or

(ii) *there exists a positive constant a such that*

$$g(x) = b\frac{x^a+1}{x^a-1}$$

for $x>1$.

PROOF. For $x>1$, let

$$m(x) = \frac{g(x)-b}{g(x)+b}. \tag{4.11}$$

Then, by (4.10), we obtain

$$m(xy) = \frac{g(xy)-b}{g(xy)+b} = \frac{\dfrac{b^2+g(x)\,g(y)}{g(x)+g(y)}-b}{\dfrac{b^2+g(x)\,g(y)}{g(x)+g(y)}+b} = \left[\frac{g(x)-b}{g(x)+b}\right]\left[\frac{g(y)-b}{g(y)+b}\right] = m(x)m(y)$$

for $x, y>1$.

The function $g(x)\equiv b$ is obviously a solution of (4.10). Now try to find all the different solutions. Assume that $g(y_0)\neq b$ for some $y_0>1$. Then we prove that $g(x)\neq b$ for $x>1$. Choose $n\in\mathbf{N}$ such that $y_0^n>x$. Then, since $g(y_0)\neq b$, $m(y_0)\neq 0$, we have $0\neq m^n(y_0)=m(y_0^n)=m\left(\frac{y_0^n}{x}x\right)=m\left(\frac{y_0^n}{x}\right)m(x)$. Hence m does not vanish anywhere. Then m is positive, since

$$m(x) = m\left(\sqrt{x}\sqrt{x}\right) = m^2\left(\sqrt{x}\right) > 0.$$

Rearranging (4.11), we obtain

$$g(x) = b\frac{1+m(x)}{1-m(x)} \tag{4.12}$$

for $x>1$. Since g is positive, we have $m(x)<1$ for $x>1$. Let

$$A(t) = \ln m(e^t)$$

if $t>0$. Then it follows from the properties of m that $A(t+s)=A(t)+A(s)$ for $t, s>0$ and $A(t)<0$ if $t<0$. Therefore, as we have shown in the proof of Theorem 4, there exists a positive constant a such that $A(t)=-at$ for $t>0$. Substituting $t=\ln x$ we have

$$\ln m(x) = -a\ln x$$

i.e.

$$m(x) = x^{-a}$$

for $x>1$. Applying (4.12) we obtain immediately the solution (ii). □

Theorem 6. *Let* $f\colon \mathbf{R}_+\to\mathbf{R}$ *and assume that* $\operatorname{sgn} f(x)=\operatorname{sgn}(x-1)$ *if* $x\in\mathbf{R}_+$, *further f satisfies the equation* (3.3) *for* $x, y\in\mathbf{R}_+\setminus\{1\}$. *Then there exist a multiplicative function* $m\colon \mathbf{R}_+\to\mathbf{R}_+$ *(i.e.* $m(xy)=m(x)m(y)$ *if* $x, y\in\mathbf{R}_+$ *and positive constants a, c such that either*

$$f(x) = cm(x)\ln x \tag{j}$$

or

(jj) $$f(x) = cm(x)(x^a - 1)$$

for $x \in \mathbf{R}_+$.

PROOF. Using the notation (4.1) we easily obtain from (3.3) that

$$2\frac{f(xy)}{f(x)f(y)} = g(x) + g(y) \tag{4.13}$$

if $x, y \in \mathbf{R}_+ \setminus \{1\}$. Further, by Theorem 3, g satisfies the equation (4.2) for $x, y \in \mathbf{R}_+ \setminus \{1\}$ with a suitable constant $b \geqq 0$.

If $b=0$, then, by Theorem 4, there exists a constant $d>0$ such that $g(x) = \frac{1}{d \ln x}$, $(x>1)$. We prove this equation also for $0<x<1$. Let $0<x<1$ and $y=1/x$ in the equation (4.13). Since $f(xy)=f(x(1/x))=f(1)=0$, it follows from (4.13) that

$$g(x) = -g\left(\frac{1}{x}\right) = -\frac{1}{d \ln\left(\frac{1}{x}\right)} = \frac{1}{d \ln x}.$$

Therefore

$$2\frac{f(xy)}{f(x)f(y)} = \frac{1}{d \ln x} + \frac{1}{d \ln y} = \frac{d \ln xy}{(d \ln x)(d \ln y)} \tag{4.14}$$

for any $x, y \in \mathbf{R}_+ \setminus \{1\}$.

With the help of (4.14) it is easy to see that the function $m \colon \mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$m(x) := \frac{2f(x)}{d \ln x}, \quad x \in \mathbf{R}_+ \setminus \{1\},$$

$$:= 1, \quad x = 1$$

is multiplicative. Hence, for $x \in \mathbf{R}_+$,

$$f(x) = \frac{d}{2} m(x) \ln x = cm(x) \ln x.$$

If $b>0$, then by Theorem 5 we have two possibilities. If $g(x) \equiv b$ for any $x>1$, then applying (4.13) for $0<x<1$, $y=1/x$ we get

$$g(x) = -g\left(\frac{1}{x}\right) = -b.$$

But then, by (4.13),

$$2\frac{f\left(4 \cdot \frac{1}{2}\right)}{f(4)f\left(\frac{1}{2}\right)} = g(4) + g\left(\frac{1}{2}\right) = b - b = 0,$$

hence $f(2)=0$. Because of this contradiction there remains the case

$$g(x)=b\frac{x^a+1}{x^a-1}, \quad x>1.$$

Then, it follows from (4.13) that

$$g(x)=-g\left(\frac{1}{x}\right)=-b\frac{x^{-a}+1}{x^{-a}-1}=b\frac{x^a+1}{x^a-1}$$

for $0<x<1$. Applying (4.13) again we get

$$\frac{2f(xy)}{f(x)f(y)}=b\frac{x^a+1}{x^a-1}+b\frac{y^a+1}{y^a-1}=\frac{2b[(xy)^a-1]}{(x^a-1)(y^a-1)} \tag{4.15}$$

if $x, y\in\mathbf{R}_+\setminus\{1\}$.

It follows from (4.15) that the function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ defined by

$$m(x):=\frac{bf(x)}{x^a-1}, \quad x\in\mathbf{R}_+\setminus\{1\},$$

$$:=1, \quad x=1$$

is multiplicative. Hence

$$f(x)=\frac{1}{b}m(x)(x^a-1)=cm(x)(x^a-1)$$

for $x\in\mathbf{R}_+$.

Conversely, it can easily be checked that the obtained functions (j) and (jj) really satisfy the functional equation (3.3) for $x, y\in\mathbf{R}_+\setminus\{1\}$. □

5. k-homogeneous deviation means

Theorem 7. *Let $E\in\varepsilon^*(\mathbf{R}_+)$ and assume that $\mathfrak{M}_{n,E}\colon \mathbf{R}_+^n\to\mathbf{R}_+$ $(n\in\mathbf{N})$ is a k-homogeneous mean for some fixed $k\geqq 2$. Then there exist a multiplicative function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ (i.e. $m(xy)=m(x)m(y)$ for $x, y\in\mathbf{R}_+$) and a positive constant a such that either* (1.11) *or* (1.12) *is satisfied.*

Conversely, the means obtained are really k-homogeneous, moreover they are multiplicative means.

Proof. If $E\in\varepsilon^*(\mathbf{R}_+)$ and $\mathfrak{M}_{n,E}$ is k-homogeneous then, by Theorem 2, $f(x)=E(x,1)$ satisfies equation (3.3) for $x, y\in\mathbf{R}_+\setminus\{1\}$. Hence, by Theorem 6, there exist a multiplicative function m and positive constants a, c such that either (j) or (jj) is satisfied. On the other hand $\mathfrak{M}_{n,E}$ is a homogeneous mean, therefore there exists a function $h\colon \mathbf{R}_+\to\mathbf{R}_+$ such that $h(1)=1$ and (3.2) is valid. Hence we obtain that either

$$E(x, y) = h(y) f\left(\frac{x}{y}\right) = h(y)\, cm\left(\frac{x}{y}\right) \ln \frac{x}{y} =$$

$$= c\frac{h(y)}{m(y)} m(x)(\ln x - \ln y) =$$

$$= H(y)\, m(x)(\ln(x) - \ln y)$$

or

$$E(x, y) = h(y) f\left(\frac{x}{y}\right) = h(y)\, cm\left(\frac{x}{y}\right)\left[\left(\frac{x}{y}\right)^a - 1\right] =$$

$$= c\frac{h(y)}{m(y)\, y^a} m(x)(x^a - y^a) = H(y)\, m(x)(x^a - y^a).$$

Now let $x_1, \ldots, x_n \in \mathbf{R}_+$ $(n \in \mathbf{N})$ and consider the equation

$$\sum_{i=1}^{n} E(x_i, y) = 0.$$

Solving this equation for y in both cases we easily get that either (1.11) or (1.12) is valid.

The multiplicativity of $\mathfrak{M}_{n,E}$ of this form can easily be checked. □

Remark. Apparently, in the above manner, we have obtained the means in (1.12) only for $a>0$. However, for $a<0$

$$\left[\frac{\sum_{i=1}^{n} m(x_i) x_i^a}{\sum_{i=1}^{n} m(x_i)}\right]^{1/a} = \left[\frac{\sum_{i=1}^{n} \overline{m}(x_i) x_i^{-a}}{\sum_{i=1}^{n} \overline{m}(x_i)}\right]^{-1/a}$$

where $\overline{m}(x) = m(x) x^a$ is also a multiplicative function.

6. Open problems. In the present paper we have proved that the k-homogeneity of a deviation mean and certain regularity assumptions imply the multiplicativity of the mean. It would have some interest to find a value k and a deviation such that the generated mean is k-homogeneous but not multiplicative. It can be proved that the k-homogeneity of deviation means implies l-homogeneity if $l \leqq k$. Therefore if there exists a nonmultiplicative but k-homogeneous (for some k) deviation mean, then necessarily there also exists a nonmultiplicative 2-homogeneous mean.

In our discussion the functional equation (2.2) plays an important role. It would be very useful to know all the solutions $f\colon \mathbf{R}_+ \to \mathbf{R}$ with $\operatorname{sgn} f(x) = \operatorname{sgn}(x-1)$, $(x \in \mathbf{R}_+)$, and $\mu\colon \mathbf{R}_+ \to \mathbf{R}_+$ with $\operatorname{sgn}(\mu(x)-1) = \operatorname{sgn}(x-1)$ $(x \in \mathbf{R}_+)$ of equation (2.2).

The regularity properties of E were used only in the elimination of μ. It is easy to see that we would get all the k-homogeneous deviation means if we could deduce equation (3.1) without using regularity properties for the deviations.

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