A rational functional equation with an economic application

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0. Introduction

The purpose of this paper is to solve the functional equation

$$\text{(1)} \qquad \frac{f_1(x_1)}{\sum_{i=1}^{n} f_i(x_i)} = \lambda \frac{g_1(x_1)}{\sum_{i=1}^{n} g_i(x_i)} + (1-\lambda) \frac{h_1(x_1)}{\sum_{i=1}^{n} h_i(x_i)}$$

for $n \geqq 2$ and

$$f_i, g_i, h_i \colon X_i \to F \quad (i = 1, \dots, n).$$

Let F be a field, whereas the X_i's are allowed to be arbitrary sets. Let $\lambda \in F$ such that $\lambda \notin \{0, 1\}$. Of course, firstly we have to assume that all denominators are different from zero. Secondly, let all images $f_i(X_i)$ have sufficiently many elements, namely at least 4 elements in the case $n=2$ and 3 elements in all other cases.

It is very helpful to know all solutions of functional equation (1) for finding those of the system

$$\text{(2)} \qquad \frac{\sum_{k=1}^{m} b_k d_j(p_j, a_j, b_1, \dots, b_m)}{\sum_{i=1}^{n} d_i(p_i, a_i, b_1, \dots, b_m)} = \sum_{k=1}^{m} e_{jk}(p_1, \dots, p_n, a_1, \dots, a_n, b_k)$$

for $j=1, \dots, n$, $n \geqq 3$, $m \geqq 2$. This equation represents an aggregation problem in economics. In this context, $p_i \in P_i \subset \mathbf{R}_{++}$ denotes the price producer i requires for his product, whereas a_i (not necessary a real) represents other nonprice characteristics of commodity i, for example advertisement or goodwill; $b_k \in B_k \subset \mathbf{R}_+$ is the budget of consumer k and

$$e_{jk}(p_1, \dots, p_n, a_1, \dots, a_n, b_k) \in \mathbf{R}_+$$

denotes his expenditure for commodity j. Thus the left hand side of equation (2) is the total turnover of commodity j.

A characterization of the functional form representing the left hand side of equation (2) is given in FUNKE (1982), who was inspired by a similar characterization of BÜRK and GEHRIG (1979). It seems worthwhile to analyze the corresponding aggregation problem, i.e., to answer the questions: Is it possible to represent such a turnover function by a sum of individual expenditure functions or ot? In nother words, has functional equation (2) solutions? And if so, how do the expenditure functions look like?

Using the solutions of equation (1) one can show that all solutions of equation (2) are of a very special form. Moreover, one can show that some economically very suggestive assumptions imply that all expenditure functions have to be identical, i.e., there is no individuality of demand. Additionally, if there is no money illusion, the Engel-curves are straight lines. These results will be discussed in Section 3.

In Section 1 we deal with the basic functional equation, i.e., with equation (1) for $n=2$ and identities on the left hand side. Equation (1) can be transformed into a system of functional equations that contains an equation which has been considered by ACZÉL (1966, p. 160). The remaining equations make this problem more difficult but, as we shall see, more interesting, too. We use a direct way to solve equation (1) without applying Aczél's result. However, there are some similarities between the two approaches.

At the end of Section 2 we turn to the most general case of equation (1), whereas in Section 3 the economic applications will be given without proofs.

1. The Basic Functional Equation

Let $n=2$ and all f_i's be identities. Then equation (1) can be written as

$$\frac{x}{x+y} = \lambda \frac{a(x)}{a(x)+b(y)} + (1-\lambda) \frac{c(x)}{c(x)+d(y)} \tag{3}$$

where $a, c\colon X \subset F_0 \to F$, $b, d\colon Y \subset F_0 \to F$ and $\lambda \in F_0 \setminus \{1\}$[1]. F is a field containing sufficiently many elements. Of course, we have to require that all denominators are different from zero for all $x \in X$, $y \in Y$.

Theorem 1. *Let* $|X| \geqq 4$, $|Y| \geqq 4$ *and*

$$\prod_{x \in X} x \neq \prod_{y \in Y} y$$

for $|X|=|Y|=4$. *Then all solutions of equation* (3) *can either be written in the form*

$$\begin{cases} a(x) = r(\lambda x - \varepsilon), \\ b(y) = r(\lambda y + \varepsilon), \\ c(x) = s(\mu x + \varepsilon), \\ d(y) = s(\mu y - \varepsilon) \\ \text{with } \quad \mu = 1-\lambda,\ r, s \in F_0 \quad \text{and} \quad \varepsilon \in F \end{cases} \tag{4}$$

[1]) $F_0 := F \setminus \{0\}$ as usual.

or in the form

$$(5)\qquad \begin{cases} a(x) = \dfrac{rx}{\lambda-\varepsilon x}, \\ b(y) = \dfrac{ry}{\lambda+\varepsilon y}, \\ c(x) = \dfrac{sx}{\mu+\varepsilon x}, \\ d(y) = \dfrac{sy}{\mu-\varepsilon y} \\ \text{with } \mu = 1-\lambda,\ r, s\in F_0 \text{ and } \varepsilon\in F \text{ such that} \\ 0\notin(\lambda-\varepsilon X)\cup(\lambda+\varepsilon Y)\cup(\mu+\varepsilon X)\cup(\mu-\varepsilon Y) \end{cases}$$

or for $2\lambda=1$ *in the form*

$$(6)\qquad \begin{cases} a(x) = rp(\varepsilon x), \\ b(y) = rq(-\varepsilon y), \\ c(x) = sp(\varepsilon x), \\ d(y) = -sq(-\varepsilon y) \\ \text{for } r, s\in F_0, \text{ suitable } \varepsilon\in F_0 \text{ and } p \text{ and } q \text{ are} \\ \text{suitable root functions, i.e., inverses of the} \\ \text{standard quadratic function } z\mapsto z^2. \end{cases}$$

Conversely, all functions given above are solutions of equation (3).

Remark 1. Of course, solutions of the form (6) require both Char $F\neq 2$ and a suitable algebraic completeness of F. However, there always exist solutions of the forms (4) and (5), for example standard solutions, i.e., $\varepsilon=0$.

PROOF. The last assertion of Theorem 1 will easily be shown by plugging the functions given above into equation (3). The following proof is splitted into two parts. By using the condition $|X|\geqq 3$ and $|Y|\geqq 3$ Part 1 shows that the functions a, b, c and d either are of the form (6) or are ratios of polynomials of degree $\leqq 2$. This result will be used for both Part 2 of this proof and for the proof of Theorem 2. Additionally, by the assumption from Theorem 1, namely $|X|\geqq 4$ and $|Y|\geqq 4$, some so-called degeneration cases can be excluded. That is, the polynomials mentioned above fulfill certain conditions of non-divisibility.

Part 1: In this section, for the sake of simplicity, we omit the arguments in the functions and indicate the function values through the index of its argument. For example, we write c for $c(x)$ and b_i for $b(y_i)$. Multiplying equation (3) by its denominators yields:

$$(7)\qquad (\mu x-\lambda y)\,ad+(\lambda x-\mu y)\,bc-yac+xbd = 0.$$

For determining b and d through appropriate combinations of equation (7) for

$x_1, x_2, x_3 \in X$ we obtain the system

$$\begin{cases} P^{11}b + P^{12}d = q^1, \\ P^{21}b + P^{22}d = q^2 \end{cases} \tag{8}$$

with the monomials in y

$$P^{11} = \mu(x_2c_1 - x_1c_2)y + x_1x_2(c_2 - c_1),$$
$$P^{12} = \lambda(x_2a_1 - x_1a_2)y + x_1x_2(a_2 - a_1),$$
$$P^{21} = \mu(x_3c_2 - x_2c_3)y + x_2x_3(c_3 - c_2),$$
$$P^{22} = \lambda(x_3a_2 - x_2a_3)y + x_2x_3(a_3 - a_2),$$
$$q^1 = (x_1a_2c_2 - x_2a_1c_1)y,$$
$$q^2 = (x_2a_3c_3 - x_3a_2c_2)y.$$

For solving equation (8) we have to pay attention for zeros of the determinant

$$\psi(y) := |P^{11}P^{22} - P^{12}P^{21}|.$$

If ψ is the zero polynomial, then one of the following cases has to hold:

Case 1a: Let $x_2a_1 - x_1a_2 = x_2c_1 - x_1c_2 = 0$ hold. By appropriate combinations of equation (7) for x_1 and x_2 and through some calculations we obtain

$$b(y) = \frac{a_1}{x_1}y \quad \text{and} \quad d(y) = \frac{c_1}{x_1}y, \tag{9}$$

namely zeros of order two for the corresponding binomials in b and d, respectively. Clearly from solution (9) one similarly obtains

$$a(x) = \frac{a_1}{x_1}x \quad \text{and} \quad c(x) = \frac{c_1}{x_1}x. \tag{9'}$$

We refer to the results (9) and (9′) as the standard solution of equation (3).

Case 1b: For $\mu = \lambda$, $c_i = \eta a_i$ $(i = 1, 2, 3;\ \eta \neq 0)$

$$\frac{a_1^2}{x_1} = \frac{a_2^2}{x_2} = \frac{a_3^2}{x_3}$$

holds, that is,

$$a_i^2 = \varepsilon x_i \quad (i = 1, 2, 3).$$

This implies

$$a_i = p(\varepsilon x_i) \quad (i = 1, 2, 3),$$

where p is an appropriate inverse of the standard quadratic function $z \mapsto z^2$. Again, by using equation (7) assertion (6) is proven.

Case 1c: There exist some $\alpha, \eta, \sigma, \tau \in F_0$ such that

$$a_i = \frac{\alpha x_i}{\sigma + \tau x_i} \quad \text{and} \quad c_i = \frac{\eta \alpha x_i}{\sigma + \tau x_i} \quad (i = 1, 2, 3)$$

holds. We call this case degenerate because it yields a contradiction as will become clear from the following treatment of both b and d in case 2a.

We now assume that the determinant ψ is different from the zero polynomial, i.e., it has to be a polynomial of either degree 0, 1 or 2. If ψ is different from zero we obtain the following representation for b and d:

$$(10) \qquad b(y)=\frac{\beta(y)}{\psi(y)}y \quad \text{and} \quad d(y)=\frac{\delta(y)}{\psi(y)}y \quad (y\in Y\setminus\psi^-(0))^{*)},$$

where

$$\psi(y)=\psi_2 y^2+\psi_1 y+\psi_0 = P^{11}P^{22}-P^{12}P^{21**)}$$

$$\beta(y)=\beta_1 y+\beta_0=\frac{1}{y}(q^1P^{22}-q^2P^{12}),$$

$$\delta(y)=\delta_1 y+\delta_0=\frac{1}{y}(q^2P^{11}-q^1P^{21}).$$

Since system (8) has to be solvable for all $y\in Y$ zeros of ψ in Y also have to be zeros of β and δ. This states that ψ has at most one zero in Y; in this case we have

$$(11) \qquad b(y)=\frac{\bar\beta_0}{\bar\psi_1 y+\bar\psi_0}y \quad \text{and} \quad d(y)=\frac{\bar\delta_0}{\bar\psi_1 y+\bar\psi_0}y \quad (y\in Y\setminus\psi^-(0)).$$

This form either represents the standard solution[2]) for $\bar\psi_1=0$ or a degenerate case to be dealt with in

Case 2a: Suppose b and d are degenerate, that is, after some suitable normalizations b and d can be written as

$$b(y)=d(y)=\frac{y}{y+\eta} \quad (y\in Y\setminus\psi^-(0)).$$

Since b and d cannot be constant simultaneously $\eta\neq 0$ must hold. By equation (7) it follows:

$$y^2(-\lambda a-\mu c-ac)+y\big(x(\mu a+\lambda c+1)-\eta ac\big)+\eta\big(x(\mu a+\lambda c)-\eta ac\big)=0.$$

The left hand side of this equation must be the zero polynomial because, firstly, its degree is less than 2 and secondly, it has $|Y\setminus\psi^-(0)|\geqq 3$ zeros. Letting the coefficients of this polynomial equal zero, we obtain three functional equations for a, c and $\eta\in F_0$. These turn out to be contradictionary.

For both the sake of completeness and for further use we show that the following degenerate case cannot occur:

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*) $\psi^-(z):=\{y\in F|\psi(y)=z\}$.

**) Contrary to indicated roman letters indicated greek letters are used for coefficients rather than function values. Sometimes, however, we continue to omit the arguments in the functions. For example, we write α for $\alpha(x)=\alpha_1 x+\alpha_0$.

[2]) See case 1a above.

Case 2b: Suppose $\beta \sim \delta$, that is, $\beta = \nu\delta$ for an appropriately choosen $\nu \in F_0$. In the case of $\deg \beta = \deg \delta = 0$ by using system (8) one can show that $\psi_2 = 0$ holds, i.e.,

$$b(y) = \nu d(y) = \frac{\nu\delta_0}{\psi_1 y + \psi_0} y \quad (y \in Y).$$

In the case of $\deg \beta = \deg \delta = 1$, again, by using system (8) one can show that both β and δ divide ψ. In either case we obtain the former type of degeneration that turned out to be contradictionary.

Part 2 of the proof of Theorem 1: We have shown that, if b and d are not of type (6), both are of the form (10) for all $y \in Y$. Because of symmetry, it follows that

$$a(x) = \frac{\alpha(x)}{\varphi(x)} x \quad \text{and} \quad c(x) = \frac{\gamma(x)}{\varphi(x)} x \quad (x \in X) \tag{12}$$

must hold where

$$\varphi(x) = \varphi_2 x^2 + \varphi_1 x + \varphi_0,$$
$$\alpha(x) = \alpha_1 x + \alpha_0,$$
$$\gamma(x) = \gamma_1 x + \gamma_0.$$

These results we plug into equation (7) and obtain

$$\text{(13)} \quad \begin{cases} \omega(x, y) := (\mu x - \lambda y)\alpha(x)\varphi(x)\delta(y)\psi(y) + (\lambda x - \mu y)\gamma(x)\varphi(x)\beta(y)\psi(y) - \\ \qquad - x\alpha(x)\gamma(x)(\psi(y))^2 + y\beta(y)\delta(y)(\varphi(x))^2 = 0 \\ (x \in X, \quad y \in Y). \end{cases}$$

We now show, that ω is the zero polynomial. If this holds has to divide $x\alpha\gamma$ and ψ has to divide $y\beta\delta$. Dividing ω by $\varphi\psi$ the remaining proof will be easy.

For showing that ω is the zero polynomial, we exclusively deal with the most complicate case, that is, $|X| = |Y| = 4$ and

$$\prod_{x \in X} x \neq \prod_{y \in Y} y.$$

From equation (13) and by rules of divisibility it follows that the representation

$$\omega(x, y) = \varrho(y)\sigma(x) + \zeta(x)\tau(y) \quad (x, y \in F) \tag{14}$$

holds, where

$$\sigma(x) = x^4 + \sigma_3 x^3 + \sigma_2 x^2 + \sigma_1 x + \sigma_0 := \prod_{x_i \in X} (x - x_i),$$
$$\tau(y) = y^4 + \tau_3 y^3 + \tau_2 y^2 + \tau_1 y + \tau_0 := \prod_{y_i \in Y} (y - y_i)$$

and ϱ and ζ are appropriate polynomials of degree equal or less than 4:

$$\varrho(y) = \varrho_4 y^4 + \varrho_3 y^3 + \varrho_2 y^2 + \varrho_1 y + \varrho_0,$$
$$\zeta(x) = \zeta_4 x^4 + \zeta_3 x^3 + \zeta_2 x^2 + \zeta_1 y + \zeta_0.$$

Obviously, representation (14) is unique except for some multiples of σ and τ. We have to distinguish between the following six cases for ϱ and ζ:

Case 3a: ϱ has no zero in Y and $\varrho(y)=f(\psi(y))^2+\bar{f}\tau(y)$ for all $y\in F$ and appropriately choosen $f\in F_0$, $\bar{f}\in F$.

Case 3b: ϱ has a zero in Y.

Case 3c: ϱ neither meets case 3a nor case 3b.

Case 4a: ζ has no zero in X and $\zeta(x)=g(\varphi(x))^2+\bar{g}\sigma(x)$ for all $x\in F$ and appropriately choosen $g\in F_0$, $\bar{g}\in F$.

Case 4b: ζ has a zero in X.

Case 4c: ζ neither meets case 4a nor case 4b.

Except for the combination case 3a—case 4a it can be shown easily that φ divides $x\alpha\gamma$ and ψ divides $y\beta\delta$. Therefore the degree of ω is reducible in both x and y, i.e., ω has to be the be the zero polynomial. The remaining combination case 3a—case 4a will turn out to contradictionary. This combination yields

$$\omega(x,y)=f(\psi(y))^2\sigma(x)+g(\varphi(x))^2\tau(y)+h\sigma(x)\tau(y)\quad(x,y\in F)$$

where $h=\bar{f}+\bar{g}$. Since φ divides $\psi^2(f\sigma+x\alpha\gamma)+h\sigma\tau$ and ψ and τ do not have a common divisor φ has to divide $f\sigma+x\alpha\gamma$ and thus $h=0$ must hold. Similarly it follows that ψ has to divide $g\tau-y\beta\delta$. Furthermore, because ω has no term x^4y^4 we obtain $f+g=0$. Now several normalizations yield $\psi_2=\varphi_2=1, f=1$ and $g=-1$, that is

$$\omega(x,y)=(\psi(y))^2\sigma(x)-(\varphi(x))^2\tau(y)\quad(x,y\in F). \tag{15}$$

By the divisibility conditions mentioned above, namely

$$\varphi(x)\xi(x)=\sigma(x)+x\alpha(x)\gamma(x) \tag{16}$$

where

$$\xi(x)=\xi_2x^2+\xi_1x+\xi_0$$

and

$$\psi(y)\pi(y)=-\tau(y)-y\beta(y)\delta(y) \tag{17}$$

where

$$\pi(y)=\pi_2y^2+\pi_1y+\pi_0$$

it follows that we can divide equation (15) by $\varphi\psi$ and obtain

$$(\mu x-\lambda y)\alpha(x)\delta(y)+(\lambda x-\mu y)\gamma(x)\beta(y)-\xi(x)\psi(y)-\varphi(x)\pi(y)=0. \tag{18}$$

Since equations (16), (17) and (18) hold for all $x,y\in F$ one obtains a corresponding system of equations for the coefficients. This system implies $\sigma_0=\tau_0$,[3]) that is,

$$\prod_{x\in X}x=\prod_{y\in Y}y.$$

But this equality was excluded by assumption. Obviously, this assumption is satisfied for some suitable choice of the arguments if either $|X|\geqq 5$ or $|Y|\geqq 5$. This

[3]) For this deduction the impossibility of both $\alpha\sim\gamma$ and $\beta\sim\delta$ is very important. In this context, see case 2b.

statement completes the analysis above for domains containing more elements than 4.

By using the divisibility conditions $\varphi|x\alpha\gamma$ and $\psi|y\beta\delta$ further treatment of equation (13) yields one of the following two cases:

Case 5a: $\varphi_2=\varphi_0=\psi_2=\psi_0=0,\ \varphi_1\neq 0$ and $\psi_1\neq 0$

or

Case 5b: There exist $f, g\in F_0$ such that

$$\varphi(x)=f\alpha(x)\gamma(x) \quad \text{and} \quad \psi(y)=g\beta(y)\delta(y).$$

In both cases we divide equation (13) by $\varphi\psi$. The remaining proof, i.e., the final deduction of both the solutions of form (4) from case 5a and the solutions of form (5) from case 5b, is left to the reader.

Remark 2. The reason for excluding both $0\in X$ and $0\in Y$ is that $x=0$ or $y=0$ would be exceptions for the basic functional equation (3). For example $x=0$ and some other $x\in X$ yield

$$a(0)=c(0)=0.$$

This implies that from equation (3) we cannot get any information for b and d if $x=0$.

Remark 3. The following example shows that the assumption

$$\prod_{x\in X} x \neq \prod_{y\in Y} y$$

is an essential one:

$$F=\mathbf{R},\quad \lambda=\frac{2}{5},$$

$$a(x)=10(1+x)(5+18x+10x^2)^{-1}x,$$

$$b(y)=10(1+y)(5+17y+10y^2)^{-1}y,$$

$$c(x)=10(1-2x)(5+18x+10x^2)^{-1}x,$$

$$d(y)=10(3+y)(5+17y+10y^2)^{-1}y,$$

$$X=\{x\in\mathbf{R}|x^4+6{,}4x^3+5{,}68x^2-0{,}6x-0{,}25=0\},$$

$$Y=\{y\in\mathbf{R}|y^4+2{,}6y^3-0{,}77y^2-2{,}9y-0{,}25=0\}.$$

2. The General Case

Firstly, we consider the functional equation

$$\frac{x_1}{\sum_{i=1}^{n} x_i}=\lambda\frac{a_1(x_1)}{\sum_{i=1}^{n} a_i(x_i)}+(1-\lambda)\frac{b_1(x_1)}{\sum_{i=1}^{n} b_i(x_i)} \tag{19}$$

with $a_i, b_i\colon X_i\subset F\to F,\ i=1,\ldots,n,\ 0\notin X_1\cup X_2+\ldots+X_n$ and $\lambda\in F$ but $\lambda\notin\{0,1\}$.

Of course we continue to assume that all denominators are different from zero for all $x_i \in X_i$, $i=1, \dots, n$. The case $n=2$ has already been dealt with in the previous section.

Theorem 2. *Let* $|X_i| \geqq 3$ *and* $n \geqq 3$. *Then all solutions of equation* (19) *can be written in the form*

$$a_i(x_i) = r(\lambda x_i - \zeta_i) \quad \text{and} \quad b_i(x_i) = s(\mu x_i + \xi_i) \quad (i = 1_0 \dots, n) \tag{20}$$

with $\mu = 1-\lambda$, $r, s \in F_0$ *and* $\xi_i, \xi_i \in F$, $i=1, \dots, n$, *such that* $\zeta_1 = \xi_1$ *and*

$$\sum_{i=1}^{n} \zeta_i = \sum_{i=1}^{n} \xi_i = 0 \tag{21}$$

holds. Conversely, functions of this form are solutions of equation (19).

Proof. The second assertion is obvious. To show the first assertion we denote by E the set of all mappings

$$e = (e_2, \dots, e_n)\colon Y := \sum_{i=2}^{n} X_i \to \mathop{\times}_{i=2}^{n} X_i$$

satisfying

$$\sum_{i=2}^{n} e_i(y) = y \quad (y \in Y). \tag{22}$$

Apparently E is not empty and each representation of y as a sum of x_i's is represented by an $e \in E$. Using x for x_1, a for a_1, c for b_1 and

$$\left.\begin{aligned} b(e, y) &:= \sum_{i=2}^{n} a_i\big(e_i(y)\big) \qquad (23)\\ d(e, y) &:= \sum_{i=2}^{n} b_i\big(e_i(y)\big) \qquad (23') \end{aligned}\right\} \quad (e \in E,\ y \in Y)$$

we obtain from equation (19)

$$\frac{x}{x+y} = \lambda \frac{a(x)}{a(x)+b(e, y)} + (1-\lambda) \frac{c(x)}{c(x)+d(e, y)} \quad (x \in X_1,\ y \in Y,\ e \in E). \tag{24}$$

If e does not vary, this functional equation can be treated similarly to the basic equation (3). From the first part of the proof of Theorem 1 we already know that for $|X_1| \geqq 3$ and $|Y| \geqq 3$ [4] all solutions either are of the standard form or, for $2\lambda = 1$, of the form (6) or of the form

$$a(x) = \frac{\alpha(x)}{\varphi(x)} x, \quad c(x) = \frac{\gamma(x)}{\varphi(x)} x,$$

$$b(e, y) = \frac{\beta(y)}{\psi(y)} y, \quad d(e, y) = \frac{\delta(y)}{\psi(y)} y.$$

The construction of b and d with the values of a and c for the three different $x \in X$ shows that b and d do not depend on e. Furthermore, ω, which is defined by

[4]) The assumption $|X_i| \geqq 3$, $i=2, \dots, n$ implies $|Y| \geqq 5$.

equation (13), does not depend on y because its degree in y is less than 5 and Y contains 5 elements at least. Therefore ψ has to divide $y\beta\delta$; thus some monomials may be cancelled out. Hence

Case 6a: The following representation holds:

$$b(y) = \frac{\beta_1 y+\beta_0}{\eta_1 y+\eta_0} \quad \text{and} \quad d(y) = \frac{\delta_1 y+\delta_0}{\eta_1 y+\eta_0} \quad (y\in Y)\,^{5)6)}.$$

Since b does not depend on e we obtain by backward substitution (23) equation

$$(25) \qquad \sum_{i=2}^{n} a_i(x_i) = \frac{\beta_1 \sum_{i=2}^{n} x_i+\beta_0}{\eta_1 \sum_{i=2}^{n} x_i+\eta_0} \quad (x_i\in X_i,\ i = 2, \ldots, n).$$

Furthermore, again through backward substitution one obtains a similar equation for the b_i's. From equation (25) it follows:

$$(26) \qquad \frac{\beta_1(u+v)+\beta_0}{\eta_1(u+v)+\eta_0}-\frac{\beta_1(\bar{u}+v)+\beta_0}{\eta_1(\bar{u}+v)+\eta_0} = \frac{\beta_1(u+\bar{v})+\beta_0}{\eta_1(u+\bar{v})+\eta_0}-\frac{\beta_1(\bar{u}+\bar{v})+\beta_0}{\eta_1(\bar{u}+\bar{v})+\eta_0},$$

$$\Big(u, \bar{u}\in U := \sum_{i=2}^{k} X_i,\ v, \bar{v}\in V := \sum_{i=k+1}^{n} X_i\Big)$$

with an arbitrary, but fixed, integer k: $2\leqq k\leqq n$. Further treatment of equation (26) either yields $\eta_0\beta_1-\eta_1\beta_0=0$ or $\eta_1=0$. In both cases we have

$$(27) \qquad \sum_{i=2}^{n} a_i(x_i) = \bar{\beta}_1 \sum_{i=2}^{n} x_i+\bar{\beta}_0 \quad (\bar{\beta}_0, \bar{\beta}_1\in F).$$

Similarly we obtain

$$(27') \qquad \sum_{i=2}^{n} b_i(x_i) = \bar{\delta}_1 \sum_{i=2}^{n} x_i+\bar{\delta}_0 \quad (\bar{\delta}_0, \bar{\delta}_1\in F).$$

Separation of variables and some suitable substitutions yield

$$a_i(x_i) = r(\lambda x_i-\zeta_i) \quad \text{and} \quad b_i(x_i) = s(\mu x_i+\xi_i) \quad (i = 2, \ldots, n).$$

The remaining work can be done by computing both a_1 and b_1 from a system of linear equations for a_1 and b_1 that is analogous to system (8).

For the sake of completeness finally we consider:

Case 6b: b and d are of the form (6), i.e.,

$$b(y) = rq(-\varepsilon y) \quad \text{and} \quad d(y) = -sq(-\varepsilon y)$$

for appropriately choosen $\varepsilon\in F_0$ and arbitrary $r, s\in F$, where p and q are suitable

5) Since b and d do not depend on e we write b and d instead of $b(e, \cdot)$ and $d(e, \cdot)$.

6) For the case $\operatorname{grad}\psi=0$ a special treatment is necessary. It can be shown, however, that this case is impossible.

inverses of the standard quadratic function, i.e., $z \mapsto z^2$. This solution requires $2\lambda=1$ and therefore Char $F \neq 2$. A similar analysis as in case 6a yields

$$q(u+v)+q(\bar{u}+\bar{v}) = q(\bar{u}+v)+q(u+\bar{v}), \tag{28}$$

$$\left(u, \bar{u} \in U := -\varepsilon \sum_{i=2}^{k} X_i,\ v, \bar{v} \in V := -\varepsilon \sum_{i=k+1}^{n} X_i\right).$$

After squaring out equation (28) twice in a suitable way we obtain

$$4(u-\bar{u})(\bar{v}-v) = 0 \quad (u, \bar{u} \in U, v, \bar{v} \in V)$$

which, because of Char $F \neq 2$, is a contradiction. Therefore it is shown that case 6a holds which proves the remainder. By now, we are ready to consider the general case, i.e.,

$$\frac{f_1(x_1)}{\sum_{i=1}^{n} f_i(x_i)} = \lambda \frac{g_1(x_1)}{\sum_{i=1}^{n} g_i(x_i)} + (1-\lambda) \frac{h_1(x_1)}{\sum_{i=1}^{n} h_i(x_i)} \tag{1}$$

for $n \geqq 2$, $\lambda \in F_0 \setminus \{1\}$ and $f_i, g_i, h_i\colon X_i \to F$, $i=1, \ldots, n$, where the X_i's are arbitrary sets and $0 \notin f_1(X_1) \cup f_2(X_2)+\ldots+f_n(X_n)$. Of course, we continue to require all denominators to be different from zero for all $x_i \in X_i$, $i=1, \ldots, n$.

The distinction between $n=2$ and $n \geqq 3$, again, is mode for simplicity of illustration.

Theorem 3. *Let* $n=2$, $|f_1(X_1)| \geqq 4$, $|f_2(X_2)| \geqq 4$ *and*

$$\prod_{z \in f_1(X_1)} z \neq \prod_{z \in f_2(X_2)} z$$

for $|f_1(X_1)| = |f_2(X_2)| = 4$. *Then all solutions of equation* (1) *either can be written in the form*

$$\begin{cases} g_1(x_1) = r(\lambda f_1(x_1) - \varepsilon), \\ g_2(x_2) = r(\lambda f_2(x_2) + \varepsilon), \\ h_1(x_1) = s(\mu f_1(x_1) + \varepsilon), \\ h_2(x_2) = s(\mu f_2(x_2) - \varepsilon) \\ \text{with } \mu = 1-\lambda,\ r, s \in F_0 \text{ and } \varepsilon \in F \end{cases} \tag{29}$$

or in the form

$$\begin{cases} g_1(x_1) = \dfrac{r f_1(x_1)}{\lambda - \varepsilon f_1(x_1)}, \\ g_2(x_2) = \dfrac{r f_2(x_2)}{\lambda + \varepsilon f_2(x_2)}, \\ h_1(x_1) = \dfrac{s f_1(x_1)}{\mu + \varepsilon f_1(x_1)}, \\ h_2(x_2) = \dfrac{s f_2(x_2)}{\mu - \varepsilon f_2(x_2)} \\ \textit{with } \mu = 1-\lambda,\ r, s \in F_0 \textit{ and } \varepsilon \in F \textit{ such that} \\ 0 \notin (\lambda - \varepsilon f_1(X_1)) \cup (\lambda + \varepsilon f_2(X_2)) \cup (\mu + \varepsilon f_1(X_1)) \cup (\mu - \varepsilon f_2(X_2)) \end{cases} \tag{30}$$

or for $2\lambda=1$ *in the form*

$$(31)\qquad \begin{cases} g_1(x_1) = rp(\varepsilon f_1(x_1), x_1), \\ g_2(x_2) = rq(-\varepsilon f_2(x_2), x_2), \\ h_1(x_1) = sp(\varepsilon f_1(x_1), x_1), \\ h_2(x_2) = -sq(-\varepsilon f_2(x_2), x_2) \\ \text{for appropriately choosen } \varepsilon\in F_0 \text{ and, } r, s\in F_0 \text{ and} \\ p(\cdot, x_1) \text{ and } q(\cdot, x_2) \text{ are suitably choosen root} \\ \text{functions.} \end{cases}$$

Conversely, all functions given above solutions of equation (1).

PROOF. The second assertion is obvious. To show the remaining assertion in the first part of this proof we restrict X_i to X_i^* such that f_i is injective on X_1^* and $f_i(X_i)=f_i(X_i^*)$, $i=1, 2$. Now the assertion of Theorem 3 can easily be proven by inversion of the f_i's, application of Theorem 1 and reinversion.

It remains to be shown that the assertion also holds for every $\bar{x}_i\in X_i\setminus X_i^*$ with both the same functional form and the same coefficients as before. We define

$$X_i^{**} := \big(X_i^*\setminus\{x_i\in X_i | f_i(x_i) = f_i(\bar{x}_i)\}\big)\cup\{\bar{x}_i\}$$

and in the same way as before we obtain a pair of solutions (g_i, h_i) of the form (29), (30) or (31). Three common points of X_i^* and X_i^{**} imply that the functional form and the corresponding coefficients are the same.

Theorem 4. *Let* $n\geqq 3$ *and* $|f_i(X_i)|\geqq 3$, $i=1, \dots, n$. *Then all solutions of equation* (1) *can be written in the form*

$$(32)\qquad \left.\begin{cases} g_i(x_i) = r\big(\lambda f_i(x_i)-\zeta_i\big), \\ h_i(x_i) = s\big(\mu f_i(x_i)-\xi_i\big) \end{cases}\right\} \quad (i = 1, \dots, n)$$

with $\mu=1-\lambda$, *arbitrary* $r, s\in F_0$ *and arbitrary* $\zeta_i, \xi_i\in F$, $i=1, \dots, n$, *such that* $\zeta_1=\xi_1$ *and*

$$(33)\qquad \sum_{i=1}^{n}\zeta_i = \sum_{i=1}^{n}\xi_i = 0$$

hold.

PROOF. The proof is similar to the one given for Theorem 3. It has to be mentioned, however, that now two common points of X_i^* and X_i^{**} are sufficient to extend the solutions.

3. Consequences for an Aggregation Problem for Turnover and Expenditure Functions

We have solved functional equation (1) in order to solve equation

$$\frac{\sum_{k=1}^{m} b_k d_j(p_j, a_j, b_1, \ldots, b_m)}{\sum_{i=1}^{n} d_i(p_i, a_i, b_1, \ldots, b_m)} = \sum_{k=1}^{m} e_{jk}(p_1, \ldots, p_n, a_1, \ldots, a_n, b_k) \tag{2}$$

for $n \geqq 3$ and $m \geqq 2$. The economical interpretation was sketched in Section 0. For solving equation (2) we assume that an individual budget constraint holds for every consumer. This implies that the expenditures of a consumer are equal zero if his budget is equal zero. For employing equation (1) we let all budgets b_k equal zero except one of them. This yields

$$e_{jk}(p, a, b_k) = b_k \frac{d_{jk}(p_j, a_j, b_k)}{\sum_{i=1}^{n} d_{ik}(p_i, a_i, b_k)} \quad (j = 1, \ldots, \mathrm{n};\ k = 1, \ldots, m)$$

where

$$d_{jk}(p_j, a_j, b_k) := d_j(p_j, a_j, 0, \ldots, 0, b_k, 0, \ldots, 0).$$

Plugging this result into equation (2) for fixed budgets we obtain indeed a functional equation of the form (1) that we have solved already.

Theorem 5. *Let* $m \geqq 2,\ 0 \in B_k,\ k = 1, \ldots, m,$ *and* $B := B_1 \times \ldots \times B_m \setminus \{0\},\ B \neq \emptyset.$ *Let* $|d_j(P_j, A_j, b)| \geqq 3,\ b \in B, j = 1, \ldots, n.$ *Then there exist* $n(m+1)+1$ *functions, namely*

$$\gamma\colon B \to \mathbf{R}_{++},$$

$$\varphi_i\colon P_i \times A_i \to \mathbf{R}_{++} \quad (i = 1, \ldots, n),$$

$$\zeta_{ik}\colon B_k \to \mathbf{R} \quad (k = 1, \ldots, m)$$

satisfying

$$\begin{cases} \sum_{i=1}^{n} \zeta_{ik}(b_k) = 0 \quad (k = 1, \ldots, m) \\ \zeta_{ik}(b_k) < \varphi_i(p_i, a_i) \ (b_k \in B_k,\ p_i \in P_i,\ a_i \in A_i,\ i = 1, \ldots, n) \end{cases} \tag{34}$$

such that each solution of equation (2) *can be represented as follows*

$$d_i(p_i, a_i, b) = \gamma(b)\varphi_i(p_i, a_i) - \frac{\sum_{k=1}^{m} b_k \zeta_{ik}(b_k)}{\sum_{k=1}^{m} b_k} \quad (i = 1, \ldots, n), \tag{35}$$

$$e_{ik}(p, a, b_k) = b_k \frac{\varphi_i(p_i, a_i) - \zeta_{ik}(b_k)}{\sum_{j=1}^{n} \varphi_j(p_j, a_j)} \quad (i = 1, \ldots, n;\ k = 1, \ldots, m). \tag{36}$$

Conversely, functions of the forms (35) *and* (36) *satisfying equation* (34) *are solutions of equation* (2).

For the PROOF see FUNKE (1982, Theorem 2).

As discussed in that paper already the resulting turnover and expenditure functions at first sight seem to be acceptable from an economic point of view. But if we employ one and only one of the following properties which economically are reasonable then there arise severe objections against the use of such turnover functions.

(37) *Arbitrary reduction of consumer k's expenditures:*

For every commodity $i=1, ..., n$ and all $b_k \in B_k$ there exists a sequence

$$(p_i^k, a_i^k)_{k \in \mathbf{N}} \subset P_i \times A_i$$

such that the expenditure of consumer k converges to 0:

$$\lim_{k \to \infty} e_{ik}(p, a, b_k) = 0 \quad ((p_i, a_i) = (p_i^k, a_i^k)).$$

Except for the dependence on a_i property (37) states a "concept of prohibitive price". Economically spoken, consumer k's expenditure for commodity i becomes arbitrarily small if the price of this commodity sufficiently increases. We say, consumer k behaves in line with real market conditions.

(38) *Absence of consumer k's money illusion:*

If all prices and consumer k's budget are multiplied by the same factor consumer k does not change his demand:

$$\frac{e_{ik}(\lambda p, a, \lambda b_k)}{\lambda p_i} = \frac{e_{ik}(p, a, b_k)}{p_i} \quad (p \in P, a \in A, b_k \in B_k, \ i = 1, ..., n)$$

and $\lambda \in \mathbf{R}_{++}$ such that $\lambda p \in P$ and $\lambda b_k \in B_k$.[7])

Property (38) states that changes of money units do not alter a consumer's behaviour.

For both economical and technical[8]) reasons we assume the following two properties:

(39) There exists an arbitrary $i \in \{1, ..., n\}$ and an arbitrary small open interval $P_i^* \subset P_i$ where consumer k's expenditure for commodity i is decreasing.

(40) There exists an arbitrarily small open interval $P_i^{**} \subset P_i$, $i=1, ..., n$, such that

7) For the sake of simplicity some domains are required to be intervals, that is,

$$B_k := (0, \bar{b}_k) \quad (\bar{b}_k > 0, \ k = 1, ..., m),$$

$$P := \mathop{\times}_{i=1}^{n} (p_i^1, p_i^u) \quad (0 \leqq p_i^1 < p_i^u \leqq \infty, \ i = 1, ..., n).$$

8) Homogeneity yields the following functional equation for restricted domain: $f(\lambda x) = \alpha(\lambda)g(x) + \beta(\lambda)$. For this equation we need some regularity condition as was shown in Funke (1982, Theorem 4).

consumer k's expenditure share is restricted as follows

$$\frac{e_{ik}(p, a, b_k)}{b_k} \leqq \chi_i < 1,$$

where $p_i \in P_i^{**}$ and all other variables are suitably choosen and constant.

Theorem 6. *Let consumer k's expenditure function be a solution of the aggregation problem* (2) *as represented in Theorem 5 by functions of the form* (36) *with respect to condition* (34). *There remains no individuality, that is,*

$$\zeta_{ik}(b_k) \equiv 0 \quad (i = 1, ..., n),$$

if either property (37) *is safisfied or the properties* (38), (39) *and* (40) *are satisfied simultaneously.*

The PROOF is given in FUNKE (1982, Theorem 3).

We have shown that all consumers behave equally if at least one of the following holds: Firstly the consumers have no money illusion or they reduce their expenditure shares sufficiently strong for those commodities whose prices are increasing more than others. The consequence of this is, that consumers reacting reasonably on changing prices and budgets do not have any individual demand. Moreover, the "individual" Engel-curves

$$\begin{cases} \left\{x \in \mathbf{R}_+^n \,|\, x_i = \dfrac{e_{ik}(p, a, b_k)}{p_i},\ b_k \in B_k,\ i = 1, ..., n\right\} \\ (a_i \in A_i,\ p_i \in P_i,\ k = 1, ..., m) \end{cases}$$

are straight lines. The concept of inferiority and superiority for commodities is ruled out by this fact. Roughly spoken, the commodities have no individuality, too. However, this does not seem reasonable: We would expect a consumer whose budget is increasing to raise his expenditure shares for luxury goods and to reduce those for basic goods.9) Obviously, such a behaviour is impossible for straight Engel-curves.

On the one hand, these disadvantages are very strong objections against using such demand and turnover functions. On the other hand, such functions have a number of advantages compared to the linear demand functions usually employed. Firstly, they satisfy the budget constraint, secondly, they never have negative values, and thirdly, the functional class is much larger than the one of the linear functions. Of course, the last property guarantees much better properties for approximation than was possible for linear demand functions.

Is it possible to escape from this dilemna? If the "absence of money illusion" or "arbitrary reduction of expenditures" is to be maintained one has to drop out another property. If we do not require "independence of irrelevant information" we loose the nice functional form on the left hand side of equation (2), also. A reasonable way out of the dilemna is to weaken the aggregation problem. This can be done in two ways at least:

9) This does not mean that the demand for such commodities has to be reduced necessarily.

Firstly, until now we have required our equation of aggregation (2) to hold for $b \in B$, i.e., we have to allow that a consumer has no money to live on. The suitable way would be to introduce

$$B_k := (b_k^1, b_k^u) \quad \text{with} \quad b_k^u > b_k^1 \geqq b^* > 0,$$

where we may call b^* a subsistence level.

Secondly, we have assumed that the demand of a consumer does not depend on other consumers' budgets.

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(Received June 1114, 1984)