BV-solutions of a linear functional equation

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Abstract. BV-solutions of the linear functional equation $\varphi(x)=g(x)\varphi[f(x)]+h(x)$ are considered and, under some general hypotheses, it is proved that this equation has a unique solution $\varphi\in BV\langle a, b\rangle$ which is given by a series convergent in the sense of the norm in the space $BV\langle a, b\rangle$.

In this paper we consider the solutions of bounded variation (BV-solutions) of the linear functional equation

$$\varphi(x) = g(x)\varphi[f(x)]+h(x), \tag{1}$$

where f, g and h are given and φ is an unknown function.
By BV $\langle a, b\rangle$ we denote Banach 's functions space $\varphi:\langle a, b\rangle\to R$ with the norm defined by the formula

$$\|\varphi\| = |\varphi(a)|+\operatorname*{Var}_{\langle a,b\rangle}\varphi, \tag{2}$$

where, as usual,

$$\operatorname*{Var}_{\langle a,b\rangle}\varphi := \sup\sum_{i=1}^{s}|\varphi(x_i)-\varphi(x_{i-1})|,$$

and the supremum is taken over all partitions of the interval $\langle a, b\rangle$.

In the case when $g(x)\equiv$const. BV-solutions of the equation (1) have been considered in the paper [1]. Even in this special case our main result says more. It appears that the series (3) converges in the sense of the BV-norm to the solution of the equation (1), whereas in the paper [1] only pointwise convergence is obtained.

We shall prove the following theorem, which generalizes Theorem 1 from the paper [1] above mentioned:

Theorem. *If the following conditions are fulfilled:*

(i) f: $\langle a, b\rangle\to\langle a, b\rangle$ *is continuous, strictly increasing and* $f(a)=a$,

(ii) $g, h\in BV\langle a, b\rangle$,

(iii) $\sup\limits_{x\in\langle a,b\rangle}|g(x)|+\operatorname*{Var}\limits_{\langle a,b\rangle}g<1$,

then the equation (1) *has in the interval* $\langle a, b\rangle$ *a unique solution* $\varphi\in BV\langle a, b\rangle$. *This* *solution is given by the series*

$$\varphi(x) = \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} g[f^i(x)]\, h[f^k(x)] + h(x), \quad x \in \langle a, b\rangle \tag{3}$$

which converges in the sense of the BV-norm.

PROOF. In the space $BV\langle a, b\rangle$ we consider the transform

$$T[\varphi](x) = g(x)\varphi[f(x)] + h(x), \quad x \in \langle a, b\rangle. \tag{4}$$

On account of hypotheses (i) and (ii) the transform (4) maps, of course, the space $BV\langle a, b\rangle$ into itself. We shall prove that the transform (4) is a contraction map.

Let $P\langle a, b\rangle$ be the set of all partitions of the form $a = x_0 < x_1 < \ldots < x_{s-1} < x_s = b$ of the interval $\langle a, b\rangle$.

Take an arbitrary $\varphi_1, \varphi_2 \in BV\langle a, b\rangle$ and estimate the expression $\|T[\varphi_1] - T[\varphi_2]\|$. Putting $\varphi := \varphi_1 - \varphi_2$ and taking into account (4), (2) and (i), we get

$$\|T[\varphi_1] - T[\varphi_2]\| = \|g\varphi_1[f] + h - g\varphi_2[f] - h\| =$$

$$= \|g(\varphi_1[f] - \varphi_2[f])\| = \|g\varphi[f]\| =$$

$$= |g(a)\,\varphi(a)| + \sup_{P\langle a,b\rangle} \sum_{i=1}^{s} |g(x_i)\,\varphi[f(x_i)] - g(x_{i-1})\,\varphi[f(x_{i-1})]| =$$

$$= |g(a)|\,|\varphi(a)| + \sup_{P\langle a,b\rangle} \sum_{i=1}^{s} \big|g(x_i)\big(\varphi[f(x_i)] - \varphi[f(x_{i-1})]\big) +$$

$$+ \varphi[f(x_{i-1})]\big(g(x_i) - g(x_{i-1})\big)\big| \leqq$$

$$\leqq |g(a)|\,|\varphi(a)| + \sup_{x \in \langle a,b\rangle} |g(x)| \sup_{P\langle a,b\rangle} \sum_{i=1}^{s} |\varphi[f(x_i)] - \varphi[f(x_{i-1})]| +$$

$$+ \sup_{x \in \langle a,b\rangle} |\varphi[f(x)]| \sup_{P\langle a,b\rangle} \sum_{i=1}^{s} |g(x_i) - g(x_{i-1})| =$$

$$= |g(a)|\,|\varphi(a)| + \sup_{x \in \langle a,b\rangle} |g(x)| \operatorname*{Var}_{\langle a,b\rangle} \varphi[f] + \sup_{x \in \langle a,b\rangle} |\varphi[f(x)]| \operatorname*{Var}_{\langle a,b\rangle} g.$$

Since, of course,

$$\operatorname*{Var}_{\langle a,b\rangle} \varphi[f] \leqq \operatorname*{Var}_{\langle a,b\rangle} \varphi$$

and

$$\sup_{x \in \langle a,b\rangle} |\varphi[f(x)]| \leqq \sup_{x \in \langle a,b\rangle} |\varphi(x)| \leqq |\varphi(a)| + \operatorname*{Var}_{\langle a,b\rangle} \varphi = \|\varphi\|,$$

we have further

$$\|T[\varphi_1]-T[\varphi_2]\| \leqq \sup_{x\in\langle a,b\rangle} |g(x)|\,|\varphi(a)|+\sup_{x\in\langle a,b\rangle} |g(x)| \operatorname*{Var}_{\langle a,b\rangle} \varphi +$$

$$+\|\varphi\| \operatorname*{Var}_{\langle a,b\rangle} g = \sup_{x\in\langle a,b\rangle} |g(x)|\,\|\varphi\|+\|\varphi\| \operatorname*{Var}_{\langle a,b\rangle} g =$$

$$=\big(\sup_{x\in\langle a,b\rangle} |g(x)|+\operatorname*{Var}_{\langle a,b\rangle} g\big)\,\|\varphi\| = \big(\sup_{x\in\langle a,b\rangle} |g(x)|+\operatorname*{Var}_{\langle a,b\rangle} g\big)\|\,\varphi_1-\varphi_2\|$$

and so, by hypothesis (iii), the transform (4) is a contraction map.

Then, in virtue of Banach's fixpoint theorem, there exists a unique fixpoint of the transform (4) in the space $BV\langle a, b\rangle$ given as the limit of the sequence of successive approximations $\{\varphi_n\}_{n=0,1,\dots}$ convergent in the sense of the norm (2). In other words, there exists a unique function $\varphi\in BV\langle a, b\rangle$, which fulfils equation (1) in the interval $\langle a, b\rangle$. In particular, taking $\varphi_0\equiv 0$ in $\langle a, b\rangle$ we obtain formula (3).

References

[1] J. MATKOWSKI, M. C. ZDUN, Solutions of bounded variation of a linear functional equation, *Aequationes Mathematicae*, **10** (1974), 223—235.

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