On Lie algebras connected with associative *PI*-algebras

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1. Let $\mathfrak{X}$ be a nontrivial variety of associative algebras over a field k of characteristic zero. Following [1] denote by $\vec{\mathfrak{X}}$ the class of Lie algebras which possess enveloping algebras belonging to $\mathfrak{X}$ (such Lie algebras were called special or *SPI*-algebras in [2]). It is easy to see that $\vec{\mathfrak{X}}$ is a quasi-variety of Lie algebras (cf. [1]). A question was posed in [2] whether a homomorphic image of an *SPI*-algebra is an *SPI*-algebra itself. In other words: does the variety of Lie algebras generated by $\vec{\mathfrak{X}}$ belong to $\vec{\mathfrak{Y}}$ for some nontrivial variety of associative algebras $\mathfrak{Y}$. In this note we prove that:

(1) a quasi-variety $\vec{\mathfrak{X}}$ need not be a variety;

(2) all homogeneous algebras of a variety generated by an *SPI*-algebra are *SPI*-algebras (see also [8]).

2. We recall some notations and definitions from [3].

Let $U(\varrho)$ be a universal enveloping algebra of a Lie algebra ϱ. A homomorphism $\varphi\colon \varrho\to\mathfrak{h}$ induces a homomorphism $\varphi_*\colon U(\varrho)\to U(\mathfrak{h})$. The ideal $\operatorname{Ker}\varphi_*$ is generated by $\operatorname{Ker}\varphi$ and will be denoted by $\omega\operatorname{Ker}\varphi$. Let $\eta\colon S(\varrho)\to U(\varrho)$ be the linear isomorphism of the Poincaré—Birkhoff—Witt theorem. For $a_1, a_2, \dots, a_n\in\varrho$ let $\eta(a_1a_2\dots a_n)=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} a_{\sigma(1)}a_{\sigma(2)}\dots a_{\sigma(n)}$, where $\mathfrak{S}_n$ is the symmetric group of the set $\{1, 2, \dots, n\}$. $U(\varrho)=\bigoplus_{i=0}^{\infty} U^i(\varrho)$, where $U^i(\varrho)$ is an image of the subspace $S^i(\varrho)\subset S(\varrho)$ of homogeneous polynomials of degree i. If $\varphi\colon \varrho\to\mathfrak{h}$ is a homomorphism of Lie algebras then $\varphi_*(U^n(\varrho))\subset U^n(\mathfrak{h})$.

A free associative algebra $A(X)$ with a (countable) set of free generators $X=\{x_1, x_2, \dots\}$ is a universal enveloping algebra of its Lie subalgebra $L(X)$ generated by X (the commutator brackets are used as usual: $[a, b]=ab-ba$). $L(X)$ is a free Lie algebra freely generated by X. $A(X)$ is naturally graded by the degrees of its elements which are (noncommutative) polynomials in free variables. An ideal $H\subset L(X)$ (or $A(X)$) is called homogeneous if it is generated by homogeneous elements with respect to this grading. A Lie algebra $L(X)/H$ is called homogeneous if H is a homogeneous ideal in $L(X)$. An ideal $\omega H\subset A(X)$ is (obviously) homogeneous if H is homogeneous.

Let $A^+(X)\subset A(X)$ be a set of polynomials with zero constant terms. A linear mapping $\pi\colon A^+(X)\to L(X)$ is defined on monomials by the formula $(x_{i_1}x_{i_2}\dots x_{i_n})\pi=$

$=(\mathrm{ad}\,(x_{i_1})\circ\mathrm{ad}\,(x_{i_2})\circ\ldots\circ\mathrm{ad}\,(x_{i_n}))\cdot(x_{i_{n-1}})$. If $v\in L(X)$ is homogeneous of degree m then $v\pi=mv$. The following lemma is obvious (it implies that π can be defined on universal enveloping algebras of homogeneous Lie algebras).

Lemma 1. *If H is a homogeneous ideal in $L(X)$ then* $(\omega H)\pi=H$.

3. We need also some basic facts about verbal ideals in universal enveloping algebras (cf. [4]). An ideal $I\subset A(X)$ is said to be verbal if it is invariant relative to all the endomorphisms of $A(X)$ induced by the endomorphisms of $L(X)$. One can easily verify that verbal ideals in $A(X)$ are homogeneous (linearization). It is also clear from the definition that T-ideals are verbal. Note also, that an intersection of a verbal ideal of $A(X)$ with $L(X)$ is a verbal ideal of $L(X)$ in the usual sense. Now, let $U(\varrho)$ be a universal enveloping algebra of a Lie algebra ϱ. An ideal in $U(\varrho)$ is said to be verbal if it consists of the elements of the form v^μ, where v runs over a certain verbal ideal $I\subset A(X)$ and μ runs over all the homomorphisms $A(X)\to U(\varrho)$ induced by the homomorphisms $L(X)\to\varrho$. Denote by I_ϱ a verbal ideal in $U(\varrho)$ which corresponds to a verbal ideal $I\subset A(X)$. It is easy to see that if $\psi\colon\varrho\to\mathfrak{h}$ is an epimorphism then $\psi_*(I_\varrho)=I_\mathfrak{h}$. Finally, suppose that I is a T-ideal of identities which are satisfied by a variety of associative algebras $\mathfrak{X}$. Another obvious fact is

Lemma 2. $\varrho\in\vec{\mathfrak{X}}$ *if and only if* $\varrho\cap I_\varrho=\{0\}$.

4. Proposition 1. *There exist varieties of associative algebras $\mathfrak{X}$ for which quasi-varieties $\vec{\mathfrak{X}}$ are not varieties of Lie algebras.*

PROOF. Let $\varrho\in\vec{\mathfrak{X}}$. By lemma 2 $I_\varrho\cap\varrho=\{0\}$. Let $\varphi\colon\varrho\to\mathfrak{h}$ be an epimorphism. Suppose that we can choose $u=v+w\in I_\varrho$ with $v\in\varrho$ and $w\in\bigoplus_{i\geq 2}U^i(\varrho)$ such that $\varphi(v)\neq 0$ and $\varphi_*(w)=0$. Then $0\neq\varphi(v)=\varphi_*(u)\in I_\mathfrak{h}\cap\mathfrak{h}$ and $\mathfrak{h}\notin\vec{\mathfrak{X}}$. We see that it is sufficient to find a variety of associative algebras $\mathfrak{X}$, a Lie algebra $\varrho\in\vec{\mathfrak{X}}$, an ideal $\varrho_1\in\varrho$ and an element $v_1+v_2\in I_\varrho$ such that $v_1\in\varrho$, $v_2\in\bigoplus_{i\geq 2}U^i(\varrho)$, $v_2\in\omega\varrho_1$ and $v_1\notin\varrho_1$. Let us begin with the free algebra $A(X)=U(L(X))$. The set P_n of multilinear polynomials in n variables $x_1, x_2, \ldots, x_n$ is a left and right $k\mathfrak{S}_n$-module. Let $P_n^i=U^i(L(X))\cap P_n$, $i=1, 2, \ldots, n$. Clearly, the P_n^i are left (but not right) $k\mathfrak{S}_n$-modules. The mapping π restricted to P_n acts as a right multiplication by an element $\pi_n\in k\mathfrak{S}_n$ [5]. Let $I\subset A(X)$ be the T-ideal generated by the element $u(1+\alpha\pi_n)$, where $u\in P_n^2$ and $\alpha\in k\mathfrak{S}_n$. Let also $L_0=L(X)\cap I$.

Lemma 3. $\omega L_0\cap P_n=\{0\}$.

PROOF. First, observe that $\omega L_0\cap P_n=L_0\cap P_n$, as, clearly, $L_0\cap P_i=\{0\}$ for $i<n$. Now, $I\cap P_n=k\mathfrak{S}_n u(1+\alpha\pi_n)$ and if $\beta u+\beta u\alpha\pi_n\in L(X)$ for some $\beta\in k\mathfrak{S}_n$ then $\beta u=0$ and $\beta u\alpha\pi_n=0$.

Lemma 2 and the remarks that precede it show that $L_1=L(X)/L_0\in\vec{\mathfrak{X}}$. In view of lemma 3, there remains to find a homogeneous ideal $H\in L(X)$ such that $u\alpha\pi_n\notin H$ and $u\in\omega H$. An example can be found for $n=4$. Let $u=[[x_1, x_2], x_3]x_4+x_4[[x_1, x_2], x_3]$. Let the ideal $H\subset L(X)$ be generated by $[[x_1, x_2], x_3]$ and let

$\alpha \in \mathfrak{S}_4$ be the cycle (4321). A straightforward calculation shows that

$$u\alpha\pi_4 \equiv -2\,[[x_1, x_2,], [x_3, x_4]] + [[x_1, x_3], [x_2, x]] +$$
$$+ [[x_1, x_4], [x_2, x_3]] - 2\,[x_3, [x_2, [x_1, x_4]]] \pmod{H}.$$

Using Hall's basis in $L(X)$ (see [3]) one can easily verify that $u\alpha\pi_4 \notin H$. This proves proposition 1.

5. The following lemma may be interesting on its own.

Lemma 4. *For any associative PI-algebra A with a unity there exists a nontrivial variety of associative algebras* $\mathfrak{Y}$ *such that if* $u \in A(X)$ *is an identity of* $\mathfrak{Y}$ *then* $u\pi$ *is an identity of A.*

Proof. Let A^0 be the algebra opposite to A and let $B = A^0 \otimes A$. The vector space $\tilde{A}$ of A can be endowed with a $B-B$-bimodule structure if we define $(a \otimes b) \cdot c = acb$ and $c \cdot (a \otimes b) = 0$ for all $a, b, c \in A$. Convert $\tilde{A}$ into the algebra with trivial multiplication and take the semidirect sum $C = \tilde{A} \oplus B$ (the multiplication in C is given by the rule $(a_1 + b_1)(a_2 + b_2) = b_1 a_2 + b_1 b_2$, where $a_1, a_2 \in \tilde{A}$; $b_1, b_2 \in B$). A theorem of A. Regev [6] shows that C is a *PI*-algebra. Let $c_i = a_i + b_i$, $a_i \in \tilde{A}$, $b_i \in B$; $i = 1, 2, \ldots, n$. One easily obtains that $c_1 c_2 \ldots c_n = b_1 b_2 \ldots b_{n-1} a_n + b_1 b_2 \ldots b_n$. It is clear now that if $u = u(x_1, x_2, \ldots, x_n) = \sum_{i_1, i_2, \ldots, i_m} \lambda_{i_1 i_2 \ldots i_m} X_{i_1} X_{i_2} \ldots X_{i_m} \in A(X)$ then $u(c_1, c_2, \ldots, c_n) = \sum_{i_1, i_2, \ldots, i_m} \lambda_{i_1 i_2 \ldots i_m} b_{i_1} b_{i_2} \ldots b_{i_{m-1}} a_{i_m} + u(b_1, b_2, \ldots, b_n)$. If u is an identity of C then $u(b_1, b_2, \ldots, b_n) = 0$ and $u(c_1, c_2, \ldots, c_n) = \sum_{i_1, i_2, \ldots, i_m} \lambda_{i_1 i_2 \ldots i_m} b_{i_1} b_{i_2} \ldots$ $\ldots b_{i_{m-1}} a_{i_m}$ for all $c_i \in C$. Set $b_j = 1 \otimes a_j - a_j \otimes 1$, $j = 1, 2, \ldots, n$. Obviously, $0 = u(c_1, c_2, \ldots, c_n) = \sum_{i_1, i_2, \ldots, i_m} \lambda_{i_1 i_2 \ldots i_m} (\mathrm{ad}(a_{i_1}) \circ \ldots \circ \mathrm{ad}(a_{i_{m-1}}))(a_{i_m}) = (u\pi)(a_1, a_2, \ldots, a_n)$, where the right hand side is calculated in A.

Proposition 2. *Any homogeneous Lie algebra from a variety generated by a special Lie algebra is itself special.*

Proof. Let A be an associative enveloping *PI*-algebra of a special Lie algebra L. One can assume that A possesses a unit element. We will show that a homogeneous algebra L_1 which belongs to the variety generated by L is contained also in $\bar{\mathfrak{Y}}$, where $\mathfrak{Y}$ is the variety obtained from A with the use of the construction of Lemma 4. Let $L_1 = L(X)/H$, where H is a homogeneous ideal in $L(X)$. Note, that $H \supset$ {identities of L}. Suppose that $L_1 \notin \bar{\mathfrak{Y}}$. This means that some homogeneous element $f \in L(X) \setminus H$ equals $u + a$ in $A(X)$, where u is an identity of $\mathfrak{Y}$ and $a \in \omega H$. One obtains from this equality that $nf = u\pi + a\pi$, where $n = \deg f \neq 0$. But $u\pi \in H$ by lemma 4 and $a\pi \in H$ by lemma 1, QED.

Corollary. *A relatively free Lie algebra which belongs to a variety generated by a special Lie algebra is itself special.*

Concluding remarks. Proposition 1 suggests an interesting problem of characterizing varieties of associative algebras $\mathfrak{X}$ for which $\bar{\mathfrak{X}}$ is a variety. On the other hand, it is unknown to the author whether proposition 2 can be extended to cover the case of nonhomogeneous algebras. Some examples of *SPI*-algebras can be found in [2] and [7].

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