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Symmetric units in integral group rings

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Abstract. In this paper, we study the question of when the symmetric units in an integral group ring $\mathbb{ZG}$ form a multiplicative group. When G is periodic, necessary and sufficient conditions are given for this to occur.

1. Introduction

Let $U(KG)$ be the group of units of the group ring KG of the group G over a commutative ring K. The anti-automorphism $g \to g^{-1}$ of G extends linearly to an anti-automorphism $a \to a^*$ of KG. Let $S_*(KG) = \{x \in U(KG) \mid x^* = x\}$ be the set of all symmetric units of $U(KG)$.

The subgroup $U_*(KG) = \{x \in U(KG) \mid xx^* = 1\}$ is called the *unitary* subgroup of $U(KG)$. It is easy to see ([4], Proposition 1.3) that if $K = \mathbb{Z}$ then $U_*(\mathbb{Z}G)$ is trivial, i.e. $U_*(\mathbb{Z}G) = \pm G$. If $U(\mathbb{Z}G) \neq \pm G$, then in $U(\mathbb{Z}G)$ there always exist nontrivial symmetric units, for example xx^* where x is a nontrivial unit in $U(\mathbb{Z}G)$.

In this paper we answer the question: for which groups G do the symmetric units of the integral group ring $\mathbb{Z}G$ form a multiplicative group? If K is a commutative ring of characteristic p and G is a locally finite p-group this question for KG was described in [2].

Lemma (see [2]). *Let K be a commutative ring and G be an arbitrary group. If $S_*(KG)$ is a subgroup in $U(KG)$ then $S_*(KG)$ is abelian and normal in $U(KG)$.*

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Theorem. *If $S_*(\mathbb{Z}G)$ is a subgroup in $U(\mathbb{Z}G)$, then the set $t(G)$ of elements of G of finite order is a subgroup in G, every subgroup of $t(G)$ is normal in G and $t(G)$ is either abelian or a hamiltonian 2-group. Conversely, suppose that the group G satisfies the above conditions and $G/t(G)$ is a right ordered group. Then $S_*(\mathbb{Z}G)$ is a subgroup in $U(\mathbb{Z}G)$.*

2. Proof of the theorem

If the subgroup $t(G)$ of the group G has the given properties and the quotient group $G/t(G)$ is right ordered, then by Theorem 5.2 [1]

$$V(\mathbb{Z}G) = G \cdot V(\mathbb{Z}t(G)).$$

Hence, every element $u \in S_*(\mathbb{Z}G)$ can be written as bw, where b is an element of G and $w \in U(\mathbb{Z}t(G))$. Suppose that b is of infinite order and $w = \alpha_1 g_1 + \ldots + \alpha_s g_s$. Then $bw = w^* b^{-1}$ and $\mathrm{Supp}(bwb) = \{bg_1b, \ldots, bg_sb\} = \{g_1^{-1}, \ldots, g_s^{-1}\}$. Thus $bg_1b = g_i^{-1}$ and $(bg_1)^2 = g_i^{-1}g_1$ is an element of finite order, which is a contradiction.

We conclude that $S_*(\mathbb{Z}G) \subseteq U(\mathbb{Z}t(G))$. If $t(G)$ is abelian then $S_*(\mathbb{Z}G)$ is a subgroup. On the other hand, if $t(G)$ is a hamiltonian 2-group then by Corollary 2.3 in [4], $V(\mathbb{Z}t(G)) = t(G)$ and so $S_*(\mathbb{Z}G)$ coincides with the centre of $t(G)$ and is again a subgroup.

So now we assume that $S_*(\mathbb{Z}G)$ is a subgroup in $U(\mathbb{Z}G)$. We first show that any subgroup of $t(G)$ is normal in G (this also proves that $t(G)$ is a subgroup of G). If not, then there exist $x \in t(G), y \in G$ with $y^{-1}xy \notin \langle x \rangle$. But then $u = 1 + (1 - x)y\hat{x}$ is a nontrivial bicyclic unit in $\mathbb{Z}G$ (where $\hat{x} = 1 + x + \ldots + x^{n-1}$, $n = o(x)$), and Marciniak and Sehgal proved in [3] that $\langle u, u^* \rangle$ is a nonabelian free subgroup of $U(\mathbb{Z}G)$. In particular, this means that $uu^* \neq u^*u$ and that uu^*, u^*u do not commute with each other. Since uu^* and u^*u are in $S_*(\mathbb{Z}G)$, this contradicts the lemma.

We now have that $t(G)$ is either abelian or hamiltonian. To finish the proof, we need only to show that if $Q = \langle a, b \mid a^4 = 1, a^2 = b^2, ba = a^3b \rangle$ is the usual quaternion group and g is of odd prime order p, then $Q \times \langle g \rangle$ contains a pair of noncommuting symmetric units.

Recall ([4], p. 34) that if x is of order n in G and $(i, n) = (j, n) = 1$, and $ik \equiv 1 \pmod{n}$, then

$$u = (1 + x^j + \ldots + x^{j(i-1)})(1 + x^i + \ldots + x^{i(k-1)}) + \frac{1 - ik}{n}\hat{x}$$

is a (Hoechsmann) unit in $\mathbb{Z}G$.

First assume $p \neq 3$. Then ag and bg are of order $4p$, and setting $i = j = 3$ (and $3k \equiv 1 \pmod{4p}$) we obtain units

$$u = (1 + (ag)^3 + (ag)^6)(1 + (ag)^3 + \ldots + (ag)^{3(k-1)}) + \frac{1-3k}{4p}\widehat{ag}$$

$$v = (1 + (bg)^3 + (bg)^6)(1 + (bg)^3 + \ldots + (bg)^{3(k-1)}) + \frac{1-3k}{4p}\widehat{bg}\,.$$

Now $u_1 = (ag)^{-2}u$ and $v_1 = (bg)^{-2}v$ are symmetric units. We claim that u_1 and v_1 do not commute. Since $(ag)^{-2}$ and $(bg)^{-2}$ are central, this is equivalent to showing that u and v do not commute.

Since $\frac{1-3k}{4p}\widehat{ag}$ and $\frac{1-3k}{4p}\widehat{bg}$ are central, this is equivalent to showing that u_2 and v_2 do not commute where

$$\begin{aligned} u_2 &= (1 + (ag)^3 + (ag)^6)(1 + (ag)^3 + \cdots + (ag)^{3(k-1)}) \\ &= 1 + 2(ag)^3 + 3(ag)^6 + \ldots + 3(ag)^{3(k-1)} + 2(ag)^{3k} + (ag)^{3(k+1)} \end{aligned}$$

$$v_2 = 1 + 2(bg)^3 + 3(bg)^6 + \ldots + 3(bg)^{3(k-1)} + 2(bg)^{3k} + (bg)^{3(k+1)}\,.$$

Since all terms with even exponents are central, this is equivalent to showing that u_3 and v_3 do not commute where

$$u_3 = 2(ag)^3 + 3(ag)^9 + \ldots + 3(ag)^{3(k-2)} + 2(ag)^{3k}$$

$$v_3 = 2(bg)^3 + 3(bg)^9 + \ldots + 3(bg)^{3(k-2)} + 2(bg)^{3k}.$$

But in u_3v_3 only 4 products are not divisible by 3. Since $3k \equiv 1 \pmod{4p}$, these reduce to $4abg^6 + 8a^3bg^4 + 4abg^2$. In v_3u_3, the same products reduce to $4a^3bg^6 + 8abg^4 + 4a^3bg^2$. Because all other products are divisible by 3, we see $u_3v_3 \neq v_3u_3$.

If $p = 3$, the same argument works with $i = j = k = 5$. In this case, direct calculation shows that if u and v are defined as before, the symmetric units $(ag)^4u$ and $(bg)^4v$ do not commute. □

Note that when G is periodic, the theorem shows that $S_*(\mathbb{Z}G)$ is a subgroup only in the obvious cases – namely when G is either abelian or a hamiltonian 2-group.

We remark that it is possible to avoid using the result from [3] and to prove that every subgroup of $t(G)$ is normal in G by a direct argument instead. We have decided to use [3] in order to indicate how useful the Marciniak–Sehgal result can be.

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