Report of the International Conference on Generalized Functions Debrecen, Hungary, 1984

An International Conference on Generalized Functions was held in Debrecen, Hungary, from November 4 to November 9, 1984. The conference was organized by the Regional Committee of the Hungarian Academy of Sciences, the János Bolyai Mathematical Society and the Mathematical Institute of the Lajos Kossuth University.

The Organizing Committee for the conference consisted of B. SZŐKEFALVI-NAGY (chairman), E. GESZTELYI (organizing chairman), G. FAZEKAS (secretary), I. FENYŐ, L. MÁTÉ and Á. SZÁZ. The 40 participants came from Austria, Bulgaria, England, France, FRG, GDR, Hungary, India, Nigeria, Poland, USA, USSR and Yugoslavia.

The conference was opened at 11 a.m. on Monday, November 5 at the House of the Academy. In the name of the organizers, Professor GESZTELYI welcomed the participants and expressed that this conference was intended to be a continuation of the succesful conferences organized previously in Poland, Yugoslavia, GDR, Bulgaria, FRG, and USSR. The opening speech was held by Professor SZŐKEFALVI-NAGY who made some comments about the historical aspects of the subject and stressed the increasing role of generalized functions in mathematics and mathematical physics.

The scientific program of the conference started immediately after the opening ceremony with the lectures of Professors MIKUSIŃSKI, VLADIMIROV and SZŐKEFALVI-NAGY. The regular sessions of the conference contained 35 scientific talks mostly followed by discussions. Moreover, at the end of the conference, there was a special session devoted to open problems and free discussions. The chairmen of the sessions were P. ANTOSIK, I. FENYŐ, B. FISHER, B. FUCHSSTEINER, E. GESZTELYI, G. L. KRABBE, H. S. SCHULTZ, B. SZŐKEFALVI-NAGY, B. STANKOVIĆ, V. S. VLADIMIROV and L. ZSIDÓ.

The social program of the conference consisted of a reception given on Tuesday evening at the House of the Academy and a bus excursion on Wednesday which included a visit to the famous college and the castle of *Sárospatak* and wine tasting in an old winery in *Tokaj.*

The conference was closed by Professor GESZTELYI at noon on Friday, November 9. In the name of the participants, Professor VLADIMIROV thanked the work of the organizers and gave a brief account of the main topics presented at the conference. Plans for a future meeting were announced by Professor STANKOVIC.

Participants

ANTOSIK, P. (Inst. Math. Polish Acad., 40-013 Katowice, Poland)
BITTNER, R. (ul. Slupska 11A m 15, 80-392 Gdańsk-Oliwa, Poland)
BRYCHKOV, YU. A. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)
BURZYK, J. (Inst. Math. Polish Acad., 40-013 Katowice, Poland)
COLOMBEAU, J. F. (Univ. Bordeaux 1, 33405 Talence, France)
DIEROLF, P. (Univ. Trier, 5500 Trier, BRD)
DIMOVSKI, I. (Inst. Math., 1090 Sofia, Bulgaria)
FAZEKAS, G. (L. Kossuth Univ., 4010 Debrecen, Hungary)
FEICHTINGER, H. G. (Univ. Wien, 1090 Wien, Austria)
FENYŐ, I. (Techn. Univ. Budapest, 1111 Budapest, Hungary)
FISHER, B. (Univ. Leicester, LE 1 7RU, England, U.K.)
FUCHSSTEINER, B. (Univ.-Gesamthochschule Paderborn, 4790 Paderborn, BRD)
GESZTELYI, E. (L. Kossuth Univ., 4010 Debrecen, Hungary)
GLAESKE, H. J. (Friedrich-Schiller-Univ., 6900 Jena, DDR)
GORBACHUK, M. L. (Inst. Mat., 25216 Kiev, USSR)
KOPRINSKY, S. (Hipodruma bl. 119-3, 1612 Sofia, Bulgaria)
KRABBE, G. L. (Purdue Univ., West Lafayette, IN 47907, USA)
KUCERA, J. (Washington State Univ. Pullman, Washington 99164, USA)
MÁTÉ, L. (Techn. Univ. Budapest, 1111 Budapest, Hungary)
MIELOSZYK, E. (ul. Leszczyńskich 3/62, 80-464 Gdańsk, Poland)
MIKUSIŃSKI, J. (ul. Jaworowa 1, 40—6550 Katowice, Poland)
ORTNER, N. (Univ. Innsbruck, 6020 Innsbruck, Austria)
PAP, E. (Univ. Novi Sad, 2100 Novi Sad, Yugoslavia)
POPOOLA, O. (Univ. of Lagos, Lagos, Nigeria)
PREUSS, W. (Ingenieur-Hochschule Wismar, 2400 Wismar, DDR)
RAO, G. L. N. (Jameshedpur Cooperative College, Ranchi Univ., Jamshedpur 831001, India)
SCHULTZ, H. S. (California State Univ., Fullerton, CA 92634 USA)
SIMON, L. (L. Eötvös Univ., 1088 Budapest, Hungary)
SKORNIK, K. (Inst. Math. Polish Acad., 40-013 Katowice, Poland)
STANKOVIĆ, B. (Univ. Novi Sad, 2100 Novi Sad, Yugoslavia)
SZÁZ, Á. (L. Kossuth Univ., 4010 Debrecen, Hungary)
SZÉKELYHIDI, L. (L. Kossuth Univ., 4010 Debrecen, Hungary)
SZŐKEFALVI-NAGY, B. (J. Bolyai Inst., 6720 Szeged, Hungary)
TAKACI, Á. (Univ. Novi Sad, 2100 Novi Sad, Yugoslavia)
TASCHE, M. (Wilhelm-Pieck-Univ., 2500—Rostock, DDR)
VLADIMIROV, V. S. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)
WAWAK, R. (Warsaw Univ., PKiN p. IX, Warszawa, Poland)
ZAYED, A. I. (California Polytechnic State Univ., San Luis Obispo, California 93407, USA)
ZHARINOV, V. V. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)
ZSIDÓ, L. (Univ. Stuttgart, 7000 Stuttgart 80, BRD)

Non-participating contributors

AHUJA, G. (Feroze Gandhi College, RaeBareli (U.P.), India)
AL-JANABI, A. S. A. R. (Al Mustansiriyah Univ., Baghdad, Iraq)
BLEYER, A. (Techn. Univ. Budapest, 1111 Budapest, Hungary)
BOEHME, T. K. (Univ. of California, Santa Barbara, California, USA)
CREUTZBURG, R. (Wilhelm-Pieck-Univ., 2500 Rostock, DDR)
DIETZEL, E. (Friedrich-Schiller-Univ., 6900 Jena, DDR)
FINOL, C. (UCV, Caracas, Venezuela)
GORBACHUK, V. I. (Inst. Math., 252167 Kiev, USSR)
KADLUBOWSKA, E. (Warsaw Univ., PKiN p. IX, Warszawa, Poland)
KOBUS, Z. (ul. Dywizjonu 303 nr 5E m 26, 80-486 Gdaúsk, Poland)
MARICHEV, O. I. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)
MITROVIĆ, D. (Faculty of Technology, 41000 Zagreb, Yugoslavia)
PANDEY, G. S. (Vikram Univ., Ujjain, India)
PILIPOVIĆ, S. (Univ. Novi Sad, 2100 Novi Sad, Yugoslavia)
RJABCEV, I. I. (ul. Slavy 7, kv. 3, 440600 Penza, USSR)
SZIGETI, F. (L. Eötvös Univ., 1088 Budapest, Hungary)
VOLOVICH, I. V. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)
YAKUBOVICH, S. B. (Steklov Math. Inst., 117966, GSP-1, Moscow, USSR)

Abstracts of contributions

Ahuja, G.: **On certain products of distributions**

The product of two distributions f and g was given by FISHER [1, p. 291] as the limit of the sequence $\{f_n g_n\}$ and then modified by him in a recent paper [2] as follows:

The product fg exists and is equal to a distribution h on an open interval (a, b) if and only if

$$\lim_{n\to\infty} \langle fg_n, \emptyset\rangle = \lim_{n\to\infty} \langle f, g_n\emptyset\rangle = \langle h, \emptyset\rangle \tag{1}$$

for all testing functions $\emptyset \in K$ with compact support contained in (a, b).

By the application of this definition we have tried to establish the Leibnitz's theorem for the r^{th} derivative of the product of two distributions and the distributive law for generalized functions.

Theorem (1). *If f and g are distributions such that their product is well defined and their derivatives of r^{th} order exist on (a, b), then for $\emptyset \in K$*

$$\left\langle \frac{d^r}{dx^r}(fg), \emptyset \right\rangle = \left\langle \sum_{i=0}^{r} r_{c_i} f^{(r-i)} g^{(i)}, \emptyset \right\rangle$$

where $r_{c_i} = \dfrac{i!}{(r-i)!\,i!}$.

Theorem (2). *If f, g and h are arbitrary distributions such that*

$$f = F^{(r)} \quad with \quad F \in L^p(a, b), \quad g = G^{(r)} \quad with \quad G \in L^p(a, b)$$

and $h^{(r)} \in L^{(q)}(a, b)$, *where* $F^{(r)}$ *denotes the* r^{th} *derivative of the summable function* F *and* $\frac{1}{p}+\frac{1}{q}=1$, *then for all the products exist on* (a, b)

$$(f+g)h = fh+gh, \quad h(f+g) = hf+hg.$$

Equation (1) also helps us in defining the product of three distributions in the following way:

Suppose f, g and h are arbitrary distributions and let

$$g_n = g * \delta_n \quad \text{and} \quad h_n = h * \delta_n$$

then the product fgh exists on (a, b) if and only if

(2)
$$\langle fgh, \emptyset\rangle = \lim_{n\to\infty} \langle fgh_n, \emptyset\rangle = \\ = \lim_{n\to\infty} \langle fg_n, h_n\emptyset\rangle = \\ = \lim_{n\to\infty} \langle f(g_n h_n), \emptyset\rangle$$

for all testing functions $\emptyset$ with compact support in (a, b).

By making use of this definition we have established the following theorem and given a counter example:

Theorem (3). *If* f, g *and* h *are three distributions defined on an open interval* (a, b) *such that*

$$f = F^{(r)} \quad \text{with} \quad F \in L^{p_1}(a, b),$$

$$g^{(r)} \in L^{p_2}(a, b) \quad \text{and} \quad h^{(r)} \in L^{p_3}(a, b)$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *then the product* fgh *exists and*

$$fgh = \sum_{i=0}^{r} \sum_{j=0}^{i} r_{c_i} i_{c_i} (-1)^i [Fg^{(i-j)} h^{(j)}]^{(r-i)}.$$

Example: If $\operatorname{Re}(\lambda, \mu, \nu) < 0$ then

$$(x+io)^\lambda (x+io)^\mu (x+io)^\nu = (x+io)^{\lambda+\mu+\nu}$$

where

$$(x+io)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda,$$

$$x_+^\lambda = \begin{cases} x^\lambda & \text{for } x > 0, \\ 0 & \text{for } x \leqq 0 \end{cases} \quad \operatorname{Re}\lambda > -1,$$

$$x_-^\lambda = \begin{cases} |x|^\lambda & \text{for } x < 0 \\ 0 & \text{for } x \geqq 0, \end{cases} \quad \operatorname{Re}\lambda > -1.$$

References

[1] B. Fisher, (1971). The product of distributions. *Quart. Jour. Math. (Oxford) (2)* **22**, pp. 291—98.

[2] B. Fisher, (1980). On defining the product of distributions. *Math. Nachr.* **99**, pp. 239—49.

Al-Janabi, A. S. A. R.: **On a generalized Riesz theorem**

F. Riesz has proved the following characterization of the functions belonging to the space $W_p^1[a, b]$ (see [1]). A function f is absolutely continuous and its derivative belongs to $L_p[a, b]$ iff there exists a constant K, such that for each system $\{[a_i, b_i]\subset[a, b]: i\in I\}$ of nonoverlapping intervals the following inequality holds:

$$\sum_{i\in I}\frac{|f(b_i)-f(a_i)|^p}{|b_i-a_i|^{p-1}}\leqq K.$$

The following generalization of this theorem is used without proof in a paper on the characterization of the spaces $W_p^k\,\Omega$ (see [2]). Let (X, S, λ) be a σ-finite measure space. A signed measure μ over the measurable space (X, S) is absolutely continuous with respect to λ and its Radon—Nikodym derivative $d\mu/d\lambda$ belongs to $L_p(X, \lambda)$ iff there exists a constant K, such that for each system $\{E_i\in S: i\in I\}$ of nonoverlapping measurable sets of finite measure $\lambda(E_i)$ the following inequality holds:

$$\sum_{i\in I}\frac{|\mu(E_i)|^p}{\lambda(E_i)^{p-1}}\leqq K.$$

Thus the proof of [2] will be made complete.

References

[1] F. Riesz, Untersuchungen über systeme integrierbarer Funktionen, *Math. Ann.* **69** (1910), 1449—1497.

[2] F. Szigeti, On a generalization of a Riesz theorem, *Publ. Math. (Debrecen)*.

Antosik, P.: **A lemma on matrices and its applications**

1. N denotes the set of all positive integers. A quasi-norm on an abelian group X is a functional satisfying the conditions:

$$|0| = 0,\quad |-x| = |x|,\quad |x+y|\leqq|x|+|y|.$$

We write

$$x_n\to x \quad\text{iff}\quad |x_n-x|\to 0.$$

Lemma. *Assume that X is a quasi-normed group and $x_{ij}\in X$ for $i, j\in N$. If for each increasing sequence of positive integers m_i there exists a subsequence $\{n_i\}$ of $\{m_i\}$ such that*

(i) $$x_{n_i n_j}\to 0 \text{ as } i\to\infty \text{ for } j\in N$$

and

(ii) $$\sum_{j=1}^{\infty} x_{n_i n_j}\to 0,$$

then

$$x_{ii}\to 0.$$

PROOF. If the Lemma is not true, we may assume that $|x_{ii}|>\varepsilon>0$ for $i\in N$, $x_{ij}\to 0$ as $j\to\infty$ for $i\in N$ and $x_{ij}\to 0$ as $i\to\infty$ for $j\in N$. Otherwise we would take a submatrix. By induction (see, [2]) we select a sequence $\{p_i\}$ such that $A_i\to 0$ with

$$A_i=\sum_{j=1}^{i-1}|x_{p_i p_j}|+\sum_{j=i+1}^{\infty}|x_{p_i p_j}|.$$

Let $\{q_i\}$ be a subsequence of $\{p_i\}$ such that $B_i\to 0$ with

$$B_i=\sum_{j=1}^{\infty}x_{q_i q_j}.$$

Hence and from the inequality

$$|x_{q_i q_i}|\leqq A_i+B_i$$

we get $|x_{q_i q_i}|\to 0$. On the other hand $|x_{q_i q_i}|>\varepsilon$. This contradiction proves the Lemma.

2. Applications of the Lemma.

Theorem 1. *(Joint continuity.) Assume that X is a quasi-normed group and at the same time X is a linear space. If the multiplication αx is separately continuous, then it is jointly continuous.*

PROOF. Assume that $\alpha_n\to 0$ and $x_n\to 0$. Due to the completeness of the scalar field, the matrix $\alpha_j x_i$ satisfies the conditions of the Lemma. Consequently, $|\alpha_i x_i|\to 0$.

Theorem 2. *(Continuous convergence.) Assume that X is a complete quasi-normed group, Y is a quasi-normed group, f_n for $n\in N$ are continuous and additive mappings from X to Y and $x_n\to 0$ in X. If $f_n(x)\to 0$ for each $x\in X$, then $f_n(x_n)\to 0$.*

PROOF. Consider matrix $f_i(x_j)$. Let $\{m_i\}$ be an increasing sequence of positive integers. Since X is complete there exists a subsequence $\{n_i\}$ of $\{m_i\}$ such that $x_{n_1}+$ $+x_{n_2}+\ldots=x$ for some x in X. We note that $\{f_{n_i}(x_{n_j})\}$ satisfies (i) and (ii). Consequently, $f_n(x_n)\to 0$.

Theorem 3. *(Banach—Steinhaus.) Assume that $\{f_n\}$ and $\{x_n\}$ for $n\in N$ are as in Theorem 1 and Y is a quasi-normed linear space. If $\{f_n(x)\}$ is a bounded sequence for each $x\in X$, then $f_n(x_n)\to 0$.*

PROOF. Let $\{m_n\}$ be a sequence of integers such that $m_n\to\infty$ and $m_n x_n\to 0$. We note that

$$f_n(x_n)=m_n^{-1}f_n(m_n x_n)$$

and $\{m_i^{-1}f_i(m_j x_j)\}$ satisfies the conditions of the Lemma. Consequently $f_i(x_i)\to 0$.

Theorem 4. *(Nikodym.) Assume that Σ is a σ-ring of sets, Y is a quasi-normed group, μ_n for $n\in N$ are countably additive set functions from Σ to Y and $\{E_n\}$ is a pairwise disjoint sequence in Σ. If $\mu_n(E)\to 0$ for $E\in\Sigma$, then $\mu_n(E_n)\to 0$.*

PROOF. Note that $\{\mu_i(E_j)\}$ satisfies the conditions of the Lemma. Consequently, $\mu_i(E_i)\to 0$.

Theorem 5. *(Nikodym.) Assume that* μ_n *and* $\{E_n\}$ *are as in Theorem* 3 *and* Y *is a quasi-normed linear space. If* $\{\mu_n(E)\}$ *is a bounded sequence for each* $E\in\Sigma$, *then the sequence* $\{\mu_n(E_n)\}$ *is bounded.*

PROOF. Suppose that $\{\alpha_n\}$ is a scalar sequence and $\alpha_n\to 0$. Then the matrix $\{\alpha_i\mu_i(E_j)\}$ satisfies the conditions of the Lemma. Consequently, $\alpha_i\mu_i E_i\to 0$ which proves the Theorem.

References

[1] ANTOSIK, P., On the Mikusiński Diagonal Theorem, *Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys.* **13** (1971), 305—310.
[2] ANTOSIK, P., A diagonal theorem for nonnegative matrices, *ibid*, **24** (1976), 855—858.

Bittner, R.: **On the foundations of a non-linear operational calculus**

There can be given four principles of Non-Linear Operational Calculus concerning

1^0 any (non-linear or linear) non-contradictory given Equation $S(x)=v$ called Differential Equation.
2^0 Surjection S from X onto V called Derivative.
3^0 Element $x\in X$ called Solution.
4^0 Element $v\in V$ called Known.

Principle I. There can be given a full description of an orbit $\mathrm{Orb}_S\, v$ i.e. a set of solutions x for any known v by the limit condition of solutions. This set has a structure of a group.

Principle II. There can be defined an Exponential Flow. i.e. a solution of an (in general non-linear) Equation $S(x)=v(x)$ with right side depending in x by logarithms v of the Equation and given limit condition x_0.

Principle III. There can be defined a superposition x of elements $x_1, \ldots, x_m$.

Principle IV. There can be defined a superpositions of derivatives $S_1, \ldots, S_m$ and a method of succesive iterations ξ_n for the Exponential Flow of the non-contradictions Equation $S(x)=v(x)$. One can define a nonhomogeneous metric by which the sequence ξ_n is convergent in the set of nonsingular elements, i.e. for with limit conditions different from 0. There are given examples in differential calculus, in two valued logic, in physics.

The Non-Linear Operational Calculus have given is consistent with linear one. The bijections of groups-orbits (which are fundamental in the Non-Linear Calculus) in a Linear Operational Calculus are additions. Homomorphisms of orbits in the Non-Linear Calculus are linear operations in a Linear Operational Calculus. We see that a Non-Linear Calculus is a well posed full generalization of a Linear Operational Calculus.

Bittner, R. and Kobus, Z.: **About eigensolutions of abstract differential equations with mixed conditions**

We consider linear differential equation of the order n with a given derivative $S\colon X \to X$

$$(1) \qquad a_n S^n x + a_{n-1} S^{n-1} x + \ldots + a_1 Sx + a_0 x = \lambda x, \quad x \in X$$

where λ and a_i are endomorphisms of X.

Onto solutions x we impose conditions on the mixed type

$$(2) \qquad \sum_{j=1}^{N} \sum_{\beta=0}^{r_j} a_{\beta j}^{(\gamma)} x_{\beta j} = 0, \quad \gamma = 1, 2, \ldots, w; \; a_{\beta j}^{(\gamma)} \in \operatorname{Ker} S, \; x_{\beta j} = \sigma(t_j) S^\beta x$$

($\sigma(t_j)$-limit condition).

Using formula

$$(3) \qquad S^n x = p^n x - p^n x_0 - p^{n-1} x_1 \ldots p x_{n-1}, \quad x_i = s(q) S^i x, \quad i = 0, 1, \ldots, n+1$$

with Heaviside's operator p to (1) and applying the Second Heaviside's Theorem we get solutions

$$(4) \qquad x = \sum_{i=0}^{n-1} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i)} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} + \sum_{k=1}^{l} \sum_{r=1}^{l_k} A_{kr}^{(i)} \frac{t^{\alpha_k+1-r}}{(\alpha_k+1-r)!} e^{R_k t(q)} \right] x_i^0,$$

where R_k are the roots of the equation

$$W(p, \lambda) = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0 - \lambda = 0$$

with multiplicaties α_k, $e^{R_k t(q)} x_i$-corresponding exponential functions.

Acting with operation σ onto general solution (4) and using (2) we get the system of homogeneous w equations with unknown $x_0^0, x_1^0, \ldots, x_{n-1}^0$.

If the order of the matrix of this system is smaller than n then equation (1) with conditions (2) has eigensolutions. In particular we consider the case where polynomial $W(p, \lambda)$ has non-zero roots $R_1, R_2, \ldots, R_n$ with multiplication one while condition (2) is of the form

$$\sum_{\beta=0}^{r_1} A_\beta^{(\gamma)} x_{\beta 1} = \sum_{\beta=0}^{r_2} B_\beta^{(r)} x_{\beta 2}.$$

We consider also a system of differential equation

$$(5) \qquad \begin{aligned} Sx_1 + a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= \lambda_1 x_1 \\ Sx_2 + a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= \lambda_2 x_2, \\ Sx_n + a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n &= \lambda_n x_n \end{aligned}$$

where coefficients of this system and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are endomorphisms commutative with each other.

To obtain eigensolutions on the system (5) we impose conditions of the mixed type

$$\sum_{k=1}^{N}\sum_{p=1}^{n}\sum_{\varrho=1}^{\varrho_{pk}} \alpha_{w\varrho}^{(p,k)}\,\sigma(t_k)\,S^{\varrho-1}x_p = 0, \quad w = 1, 2, \dots, m.$$

Similarly as in case of linear differential equation of the order n we get necessary and sufficient condition for existence eigensolutions on the system (5).

There are example of equation of second order with difference derivative and of the system two equations with derivative $S=\frac{d}{dt}$.

Bleyer, A. and Preuss, W.: **On representation theorems for derivations and their applications**

The contribution was dealing with sequentially closed derivations in rings and algebras and sequentially continuous derivations in fields. Several representation theorems
(published in:

[1] Bleyer, A. and Preuss, W.: A note to general notions of the derivation and its applications. *Periodica Math. Hung.* 11(1)/1980, 1—6.

[2] Preuss, W. and Bleyer, A.: On derivations in rings and algebras provided with convergence structures. *Abbhandlungen der Akademie der Wiss. der DDR Abt. Math.-Nat.-Technik* 1979, *Nr. 4 N*, 157—159)

were applied in order to give representations for derivations in c^*-algebras and in the Mikusiński operator field.

Brychkov, Yu. A., Marichev, O. I. and Yakubovich, S. B.: **Factorization of integral transformations**

It is proved that integral transformations of convolution type such as fractional integration, Hankel, Stieltjes, Hilbert, Meijer, ${}_1F_1$, ${}_2F_1$, G and other transformations can be represented by compositions of Laplace transformations.

The same is valid for integral transformations with respect to parameters of special functions in kernel, for instance, for Kontorovich—Lebedev, Mehler—Fock, Olevsky, Wimp and some other transformations.

Burzyk, J.: **On Fourier transform of distributions**

Colombeau, J. F.: **New generalized functions**

In "New Generalized Functions and Multiplication of Distributions" [North Holland Math. Studies 84, 1984] (refered below as "NGF") we introduced "new generalized functions" to explain heuristic computations of Physics and to give a meaning to any finite product of distributions.

There we explained how we were led to define these generalized functions with Distribution Theory as starting point. Here we present these generalized functions without any previous knowledge of locally convex spaces and distributions. We define the distributions as those particular generalized functions which are locally derivatives of continuous functions (any partial derivative of a generalized function — in our sense — is still a generalized function so that the above definition of distributions is quite natural).

Then we consider tempered generalized functions, i.e. generalized functions which are in some sense growing at infinity not faster than a polynomial (as well as all their partial derivatives). Any tempered distribution, and more generally any finite product of tempered distributions, is a tempered generalized function. In this setting the Fourier transform and the convolution product behave very well. We apply these concepts to explain heuristic calculations of Physics (the Hamiltonian formalism of the free fields and the removal of divergences in Perturbation Theory).

In our setting, partial differential equations have new solutions. As a particular case we obtain global solutions on $\mathbf{R}^4$ of nonlinear wave equations with Cauchy data distributions on $\mathbf{R}^3$ (the study of these equations is justified by the fact that they are scalar models of interacting field equations). These results show that our setting is perfectly adapted to the study of nonlinear partial differential equations and indicates some new perspectives in this field.

Creutzburg, R. and Tasche, M.: **Construction of moduli for complex number-theoretic transforms**

Introduction. With the rapid advances in large scale integration, a growing number of digital signal processing applications becomes attractive. The number-theoretic transform (NTT) was introduced as a generalization of the discrete Fourier transform (DFT) over residue class rings of integers and allows fast convolutions without round-off errors.[1,2] Its main drawback is a rigid relationship between word length and obtainable transform length and a limited choice of possible word length. In order to enlarge the transform length of conventional NTT's, complex number-theoretic transforms (CNT) were introduced. However, it is not easy to find convenient moduli m that are large enough to avoid overflow, and to find primitive N-th roots of unity modulo m with small binary weight for transform lengths N that are highly factorizable and large enough for practical applications.[5] In a recent paper[6], the authors have presented a solution of this problem in the ring Z of integers by studying cyclotomic polynomials. In this letter we present simple constructive methods for the finding of all convenient moduli m for CNT's under the assumption that a special transform length N and a special element $\alpha \in Z[i]$ with small binary weight are given.

Complex number-theoretic transforms. Let $Z[i]$ be the ring of the Gaussian integers $\xi=\xi_1+i\xi_2\,(\xi_1, \xi_2 \in Z)$ where $N(\xi)=\xi_1^2+\xi_2^2$ denotes the norm of ξ. Furthermore let $m>1$ be an odd integer with the prime factorization

$$m = p_1^{u_1} \dots p_s^{u_s} q_1^{v_1} \dots q_t^{v_t} \tag{1}$$

where $p_j \equiv 1 \mod 4$, $(j=1, \ldots, s)$ and $q_k \equiv 3 \mod 4$, $(k=1, \ldots, t)$. Then $\alpha \in Z[i]$ is called a primitive N-th root of unity modulo m if

$$\alpha^N \equiv 1 \mod m, \tag{2}$$

$$(\alpha^n - 1, m) = 1 \text{ for every } n = 1, \ldots, N-1.$$

Note that by definition (2) the integer $m>1$ isaso-called primitive divisor of $\alpha^N - 1$.[7] The following theorem[12] states necessary and sufficient conditions that $\alpha \in Z[i]$ is a primitive N-th root of unity modulo m. Let χ_N be the N-th cyclotomic polynomial.

Theorem 1. *Let $m>1$ be an odd integer. An element $\alpha \in Z[i]$ is a primitive N-th root of unity modulo m if and only if one of the following conditions holds:*

1) $\chi_N(\alpha) \equiv 0 \mod m$, $(N, m)=1$; (3)
2) $\alpha^N \equiv 1 \mod m$, $(\alpha^d - 1, m) = 1$ *for every divisor d of N with N/d prime;*
3) $\alpha^N \equiv 1 \mod m$, $(N(\alpha^d - 1), m) = 1$ *for every divisor d of N with N/d prime.*

A necessary and sufficient condition for the existence of such primitive N-the roots of unity modulo m^4 is

$$N | GCD(p_1 - 1, \ldots, p_s - 1, q_1^2 - 1, \ldots, q_t^2 - 1). \tag{4}$$

Now let $\underline{x} = (x_0, \ldots, x_{N-1})$ and $\underline{y} = (y_0, \ldots, y_{N-1})$ be two N-point integer sequences. Note that the equality of such sequences $\underline{x}$ and $\underline{y}$ is explained by $x_k \equiv y_k \mod m$ $(k=0, \ldots, N-1)$. The CNT of length N with $\alpha \in Z[1]$ as a primitive N-th root of unity modulo m and its inverse are defined as the following mappings between N-point integer sequences:

$$X_n \equiv \sum_{k=0}^{N-1} x_k \alpha^{nk} \mod m, \quad (n = 0, \ldots, N-1), \tag{5}$$

$$x_k \equiv N' \sum_{n=0}^{N-1} X_n \alpha^{-nk} \mod m, \quad (k = 0, \ldots, N-1),$$

where $NN' \equiv 1 \mod m$. The CNT has a similar structure and properties like the DFT, particularly the cyclic convolution property. For given transform length N and given element $\alpha \in Z[i]$ one has to choose the modulus m by (3) as a divisor of $\chi_N(\alpha)$. But in general the prime factorization of $\chi_N(\alpha)$ is difficult to find in $Z[i]$. Hence we consider some special cases in which this prime factorization is easy to perform in Z by the help of the properties of cyclotomic polynomials. The following example shows the application of known properties[11] of $\chi_N(x)$ for the Winograd-number[4] $N = 840 = 2^3 \times 3 \times 5 \times 7$:

$$\chi_{840}(x) = \chi_{2\times 3\times 5\times 7}(x^4) = \chi_{3\times 5\times 7}(-x^4) = \frac{\chi_3(-x^{140})\,\chi_3(-x^4)}{\chi_3(-x^{20})\,\chi_3(-x^{28})}.$$

If one chooses the element $\alpha = 1 + i \in Z[i]$ then the calculation gives

$$\chi_{840}(1+i) = \chi_{105}(4) = \chi_{105}(2)\,\chi_{210}(2) =$$
$$= 211 \times 29\,191 \times 106\,681 \times 152\,041 \times 664\,441 \times 1\,564\,921.$$

For the prime factorization of such large numbers the reader is referred to.[7,8] Note that the calculation of $\chi_{840}(1+i)$ was carried out by only real operations.

Construction of moduli for CNT's: Theorem 1 states a necessary and sufficient condition for moduli m, so that an element $\alpha \in Z[i]$ is a primitive N-th root of unity modulo m. But in some important cases for practical applications one can specify theorem 1 in the following way.

Theorem 2. *Let $N=2^n N_1$ ($N_1 \geqq 1$ odd, $n \geqq 3$) and $\alpha = \pm 2^k(1+i)$, ($k \geqq 0$) be given. Then one has to choose as a modulus m for a* CNT *a divisor of*

$$\chi_N(\pm 2^k(1+i)) = \begin{cases} \chi_{N/8}(2^{4k+2}) = \chi_{N/4}(2^{2k+1})\,\chi_{N/8}(2^{2k+1}) & \text{for} \quad n=3, \\ \chi_{N/4}(2^{2k+1}) & \text{for} \quad n>3 \end{cases}$$

with $(m, N)=1$. In other words, m is a primitive divisor of $2^{(2k+1)N/4}-1$, if $n=3$ and $2^{(2k+1)N/2}-1$ if $n>3$, respectively.

Note that the primitive divisors of 2^s-1 are listed.[7,8]

Corollary 1. If $\alpha=1+i$ is a primitive N-th root of unity modulo m with $4|N$, then $\alpha^2=2i$ is a primitive $N/2$-th root of unity modulo m.

Corollary 2. Let $N=8p$ ($p>3$ prime) and $\alpha=1+i\in Z[i]$ be given. Then one has to choose as a modulus m a divisor of

$$\chi_{8p}(1+i) = \chi_p(2)\,\chi_{2p}(2) = (2^p+1)(2^p-1)/3. \tag{6}$$

The related CNT is called complex Pseudo—Mersenne transform.

Corollary 3. Let $N=8p^2$ ($p>3$ prime) and $\alpha=1+i\in Z[i]$ be given. Then one has to choose as a modulus m for a CNT a divisor of

$$\chi_{8p^2}(1+i) = \frac{(2^{p^2}+1)}{(2^p+1)} \times \frac{(2^{p^2}-1)}{(2^p-1)} = \frac{2^{2p^2}-1}{2^{2p}-1}. \tag{7}$$

In other words, m is a primitive divisor of $2^{2p^2}-1$.

Corollary 4. Let p, q be primes with $2<q<p$ and $2^q \not\equiv \pm 1 \bmod p$. Then for $N=8pq$ and $\alpha=1+i$ one has to choose as a modulus m for a CNT a divisor of

$$\chi_{8pq}(1+i) = \frac{3(2^{pq}+1)}{(2^p+1)(2^q+1)} \times \frac{(2^{pq}-1)}{(2^p-1)(2^q-1)}. \tag{8}$$

In other words, m is a primitive divisor of $2^{2pq}-1$.

Note that the first factor in (7)—(8) is a Pseudo—Fermat number and the second factor is a Pseudo—Mersenne number. In this direction our theorem 2 and the above corollaries generalize known results of recent works on Pseudo—Fermat- and Pseudo—Mersenne transforms.[5,9,10]

References

[1] J. M. POLLARD, The fast Fourier transform in a finite field, *Math. Comput.*, 1971, **25**, pp. 365—374.

[2] R. C. AGARWAL and C. S. BURRUS, Number theoretic transform to implement fast digital convolution, *Proc. IEEE*, 1975, **63**, pp. 550—560.

[3] I. S. REED and T. K. TRUONG, Complex integer convolutions over a direct sum of Galois fields, *IEEE Trans.*, 1975, *IT-21*, pp. 169—175.

[4] H. J. NUSSBAUMER, Fast Fourier transform and convolution algorithms *(Springer, Berlin—Heidelberg—New York*, 1981).

[5] E. DUBOIS and A. N. VENETSANOPOULOS, The generalized Fourier transform in rings of algebraic integers, *IEEE Trans.*, 1980, *ASSP-28*, pp. 169—175.

[6] R. CREUTZBURG and M. TASCHE, Construction of moduli for number-theoretic transforms via cyclotomic polynomials, *Preprint* WPU *Rostock, Section Mathematik*, 1983.

[7] M. KRAITCHIK, Introduction a la Theorie des Nombres, *(Gauthier-Villars, Paris,* 1952*)*.

[8] J. BRILLHART, D. H. LEHMER and J. L. SELFRIDGE, New primality criteria and factorizations of $2^m \pm 1$, *Math. Comput.*, 1975, **29**, pp. 620—647.

[9] H. J. NUSSBAUMER, Digital filtering using complex Mersenne transforms, *IBM J. Res. Dev.*, 1976, **20**, pp. 498—504.

[10] H. J. NUSSBAUMER, Digital filtering using pseudo Fermat number transforms, *IEEE Trans.*, 1977, *ASSP-25*, pp. 79—83.

[11] H. LÜNEBURG, Galoisfelder, Kreisteilungskörper und Schieberegisterfolgen, *(Bibliograph. Institut, Mannheim—Wien—Zürich,* 1979*)*.

[12] R. CREUTZBURG and M. TASCHE, *F*-Transformation und Faltung in kommutativen Ringen, *Elektron.* Informationsverarb. Kybernet. *(in print)*.

[13] J. BRILLHART, D. H. LEHMER, J. L. SELFRIDGE, B. TUCKERMAN and S. S. WAGSTAFF JR., Factorizations of $b^n \pm 1$ up to High Powers, *Contemporary Mathematics* **22**, AMS, *Providence*, 1983.

Dierolf, P.: **On $\Lambda-\Omega$-extendable distributions**

Let $\Lambda \subset \Omega$ be open subsets of $\mathbf{R}^n$ and denote by $T(\Omega)$ the standard *LF*-space topology of the space $D(\Omega)$ of testfunctions on Ω. The relative topology $T(\Omega) \cap \cap D(\Lambda)$ is coarser than the standard topology $T(\Lambda)$ of $D(\Lambda)$, and by the HAHN—BANACH-theorem, the dual $(D(\Lambda), T(\Omega) \cap D(\Lambda))'$ consists exactly of those distributions R on Λ which can be extended to a distribution T on Ω, $R = T|\Lambda$.

We show that the completion $P(\Lambda, \Omega) = \widetilde{(D(\Lambda), T(\Omega) \cap D(\Lambda))}$ is a limit-subspace of $(D(\Omega), T(\Omega))$. With the help of this *LF*-spacerepresentation of $P(\Lambda, \Omega)$ we calculate the space $O_M(P(\Lambda, \Omega))$ of multipliers of $P(\Lambda, \Omega)$, and the subspace $P(\Lambda, \Omega)_{ar}$ of absolutely regular distributions in $P(\Lambda, \Omega)'$.

References

P. DIEROLF, Some locally convex spaces of regular distributions. *To appear* in Studia Math.

P. DIEROLF, J. VOIGT, Calculation of the bidual of some function spaces. Integrable distributions. *Math. Annalen* **253** (1980), 63—87.

P. DIEROLF, Multiplication and convolution operators between spaces of distributions. In: *Functional Analysis: Surveys and Recent Results III*, K. D. BIERSTEDT and B. FUCHSSTEINER (eds.), *Elsevier Science Publishers* B. V. (North-Holland) (1984), 305—330.

S. LOJASEWICZ, Sur le problème de la division, *Studia Math.* **18** (1959), 87—136.

Dietzel, E.: **On the Meijer-transform**

I. H. Dimovski has given the following definition of the MEIJER-transform:

$$K_\nu\{f(t), p\} = \frac{1}{\Gamma(\nu+1)} L\left\{t^{\nu+1}\int_0^1 (1-t_1)^{\nu-1/2} t_1^{-\nu/2} f(t\sqrt{t_1})\,dt_1, p\right\}$$

for functions

$$f\in K = \{g/g(t) = g_1(t)\,t^{\nu-2},\ g_1\in C[O,\infty),\ \nu\geqq 0\ \ g(t) = O(e^{ct}) \text{ for } t\to\infty\},$$

where $L\{h(t), p\}$ denotes the usual LAPLACE-transform. It is easy to verify, that it holds the following

Theorem. *Let* $f\in K$, *fhen*

$$K_\nu\{f(t), p\} = L\{\lambda^{\nu-1} L\{f(\sqrt{t})\,t^{-\nu/2}, 1/\lambda\}, p^2/4\}/(2\sqrt{\pi}).$$

This relation allows us to use tables for the LAPLACE-transform and to prove theorems for the MEIJER-transform applying the well-known properties of the LAPLACE-transform. *For example:*

1. Let $f\in K$ and $f(t)\sim At^\alpha$ for $t\to\infty$ ($\alpha>\nu-2$, A complex). Then $K_\nu\{f,p\}$ exists for $\operatorname{Re} p>0$ and if $A\neq 0$ $K_\nu\{f,p\}$ has a singularity in $p=0$ and

$$K_\nu\{f,p\}\sim A2^{\alpha+\nu+1}\Gamma(\alpha/2-\nu/2+1)\,\Gamma(\alpha/2+\nu/2+1)/(p^{\alpha+\nu+2}\sqrt{\pi})$$

for $p\to 0$ and $|\arg p|\leqq\varphi<\pi/4$.

2. Let $F(p)=K_\nu\{f,p\}$ convergent for real $p>0$ $(p>p_0>0)$ and f a real function of K. Further let $f(t)\,t^{-\nu}$ be an increasing function and $f(t)\geqq Bt^{\gamma-\nu-2}$ if $t>0$. Then follows from $F(p)\sim C/p^\gamma$ $(\gamma>2\nu)$ for $p\to 0$ $(p\to\infty)$ on the real axis that

$$f(t)\sim C\sqrt{\pi}\,2^{-j+1}t^{\gamma-\nu-2}/[\Gamma(\gamma/2)\,\Gamma(\gamma/2-\nu)]\quad\text{for } t\to\infty.$$

Dimovski, I. H.: **Operational calculi for non-local boundary value problems in several variables**

The author's approach to operational calculi for local and non-local boundary value problems for ordinary linear differential operators [1] is extended to linear PDO. The corresponding operational calculi are build using multiplier quotients constructions for product-convolutions. A typical example of product-convolution is the following: Let P and Q be linear functionals on $C([0,1])$. Then the operation

$$(f*g)(x,t) = P_\xi Q_\tau\left\{\frac{1}{2}\int_\xi^x\int_\tau^t f(x+\xi-\eta, t+\tau-\sigma)\,g(\eta,\sigma)\,d\eta\,d\sigma - \right.$$
$$\left. -\frac{1}{2}\int_{-\xi}^x\int_\tau^t f(|x-\xi-\eta|, t+\tau-\sigma)\,g(|\eta|,\sigma)\operatorname{sgn}(x-\xi-\eta)\,\eta\,d\eta\,d\sigma\right\}$$

is such a product-convolution.

Till to now, all attempts for biulding operational calculi for PDO are intended for Cauchy problems only (see [2] and [3]) where the technique of the distribution theory is far more effective. Instead, for common boundary value problems in finite domains the only effective operational method up to now is that of the finite integral transforms. The method proposed has two advantages with respect to the method of the finite integral transforms: it avoids expanding of the boundary value functions in series, and the final summing of the solution obtained. The convolutional representations of the solutions of the non-local boundary value problems (the Duhamel-type representations) can be used for numerical calculation of the solutions as an alternative of the difference methods, used now. Some non-local b.-v. problems for the heat equation $u_t=u_{xx}+f(x,t)$ with Ionkin's and Dezin's conditions are considered in detail.

References

[1] I. DIMOVSKI, Convolutional calculus. *Sofia*, 1982.
[2] M. GUTTERMAN, On operational method in partial differential equations. *SIAM J. Appl. Math.* **17** (1969), 468—493.
[3] H. S. SHULTZ, Operational calculus for functions in two variables. *SIAM J. Math. Anal.* **9** (1978), 660—666.

Fazekas, G.: **Tchebycheff—Markoff problems and linear programming**

Let $u_1, u_2, \ldots, u_m$, ω be continuous real valued functions on the finite closed interval $[\alpha, \beta]$ and let $b=(b_1, b_2, \ldots, b_m)\in\mathbf{R}^m$. A Tchebycheff—Markoff problem is to determine functions σ in the set

$$V(\underline{b}) = \left\{\sigma\in NBV_+[\alpha,\beta] \,\Big|\, \int_\alpha^\beta u_i(t)\,d\sigma(t) = b_i,\ i=1,2,\ldots,m\right\}$$

for which the extremal values of

$$E(\sigma) = \int_\alpha^\beta \omega(t)\,d\sigma(t)$$

are attained.

This problem can be regarded as an infinite dimensional generalization of the linear optimization problem of form "find the extremal values of $\langle \underline{c}, \underline{x}\rangle$ subject to the constraints $A_{\underline{x}}=\underline{b}$, $\underline{x}\geqq\underline{0}$ where A is an $m\times n$ matrix and $\underline{c}\in\mathbf{R}^n$".

Our investigations were based on this fact. We characterized the extreme points of $V(\underline{b})$ and showed that the Tchebycheff—Markoff problem can be solved approximately by help of the simplex method.

Feichtinger, H. G.: **Minimal Banach spaces and atomic representations**

Atomic representations, i.e. the characterization of arbitrary elements in a given Banach space as convergent sums of elements of a particularly simple form (the "atoms") play an important role in the modern treatment of Banach spaces of functions (distributions), in particular in the theory of real Hardy spaces as promoted by R. Coifman and G. Weiss. A new atomic representation for the elements of the Segal algebra $S_0(G)$ (introduced earlier by the author) on a locally compact abelian group G will be given: For any nonzero $f_0 \in S_0(G)$ (e.g. $f_0 \in L^1(G)$ with compactely supported Fourier transform) one has:

$$S_0(G) = \{f | f(x) = \sum_{n \geqq 1}^{\infty} a_n \chi_n(x) f_0(x - y_n),$$

where the sequences $(y_n)_{n \geqq 1}$ in G and $(\chi_n)_{n \geqq 1}$ in $\hat{G}$ are choosen arbitrary, and the complex sequence $(a_n)_{n \geqq 1}$ satisfies $\sum_{n \geqq 1} |a_n| < \infty\}$. This space of test functions (which is a Banach space with a suitable norm) and its Banach dual have many interesting properties (allowing a kernel theorem, or the representation of multipliers), making them very useful tools for the treatment of questions of abstract harmonic analysis. In the case $G = \mathbf{R}^m$ most of these properties can be easily derived from the above characterization using well-known properties of the Gauss—Weierstraß kernel $(:= f_0)$.

The same device which leads to the above characterization may also be applied, more generally, to other so-called Wiener type spaces, [using their minimality within certain classes of Banach spaces of continuous functions]. Since the Schwartz space $\mathfrak{S}(\mathbf{R}^m)$ of rapidely decreasing functions is the intersection of such spaces atomic decompositions for the elements of $\mathfrak{S}(\mathbf{R}^m)$ are also available. Finally, it will be mentioned that a similar technique applies to give atomic characterizations for homogeneous Besov spaces of order zero which may directly be compared with the above mentioned Hardy spaces.

Fenyő, I.: **On the Hankel-transformation of Schwartz distributions**

A definition for the Hankel-transformation of a distribution is given which is based on an analogous statement to the Paley—Wiener Theorem.

$D_+(a)$ denotes the subspace of Schwartz-testing functions with support on $(0, a)$ $(a > 0)$, $H_n(a)$ (n is a fixed nonnegative entire) is the space of functions with following properties: $\psi \in H_n(a)$ iff

a) $\psi(s)$ is an entire function $(s \in \mathbf{C})$

b) $|s^k \psi(s)| \leqq C_k \exp(a\,|\mathrm{Im}\, s|)$ $(k = 0, 1, 2, \ldots)$

c$_n$) $\psi(-s) = (-1)\psi(s)$ $(s \in \mathbf{C})$

d_n) $\int_0^\infty s^{k-1} \psi(s)\, ds = 0, \quad k = n, n+2, n+4, \ldots$

e$_n$) $|\psi(s)| = 0(|s|^n)$ $(s \to 0)$.

For $a \to \infty$ denote by D_+ the inductive limit of $D(a)$ and by H_n that of $H_n(a)$. We introduce in D_+ and H_n convenient topologies. (We remark that H_n is a subspace of the testing-functions space Z of ultradistributions.) It is proved that the usual Hankel-transformation yields an algebraic and topological isomorphism between the spaces D_+ and H_n. On account this we define the Hankel-transform $H_n(u)$ for the distribution u as an element of the space H_n by the following relation

$$\langle H_n(u), s\psi\rangle = \langle u, tH_n(\psi)\rangle$$

for all $\psi \in H_n$. Now it can be proved in an easy way, that all formal properties are valid for the Hankel-transforms of distributions which are well known for the Hankel-transforms of functions.

Finol, C. E.: **On dilation functions and some applications to Orlicz spaces**

The aim of this talk to study some properties of the so called dilation functions (see [2]) and applications of these to questions on Orlicz spaces and linear bounded operators on them. A necessary condition for the existence of a translation invariant operator $T\colon L_{\Phi_1}(\mathbf{R}^n, \mu) \to L_{\Phi_2}(\mathbf{R}^n, \lambda)$ will be given. For a submultiplicative function Φ_1 a sufficient condition for every linear bounded operator $T\colon l_{\Phi_1} \to l_{\Phi_2}$ to be strictly singular, will be given in this talk.

References

[1] C. E. Finol, Linear transformations intertwining with group representations, *Ph. D. Thesis, Chelsea College of Science and Technology, Univ. of London,* 1978.
[2] S. G. Krein, Ju. I. Petunin, E. M. Semenov, Interpolation of linear operators, *Amer. Math. Soc. Transl.* **54**, *Prov. Rhode Island,* 1982.

Fisher, B.: **On defining the change of variable in distributions**

Let ϱ be a fixed infinitely differentiable function having the properties

(i) $\varrho(x) = 0$ for $|x| \geqq 1$,

(ii) $\varrho(x) \geqq 0$,

(iii) $\varrho(x) = \varrho(-x)$,

(iv) $\int_{-1}^{1} \varrho(x)\,dx = 1.$

The function δ_n is defined by

$$\delta_n(x) = n\varrho(nx)$$

for $n=1, 2, \dots$.

Now let F be a distribution and let f be an ordinary summable function. The distribution $F(f(x))$ is defined to exist and be equal to h on the open interval (a, b) if

$$N-\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\,\varphi(x)\,dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in (a, b), where

$$F_n(x) = F(x) * \delta_n(x)$$

for $n=1, 2, \dots$ and N is the neutrix having domain $N'=\{1, 2, \dots, n, \dots\}$ and range the real numbers with negligible functions $n^{\lambda} \ln^{r-1} n$, $\ln^r n$ for $\lambda>0$ and $r=1, 2, \dots$ and all functions which converge to zero as n tends to infinity.

Fuchssteiner, B.: **Distribution algebras and elementary shock wave analysis**

Describing the evolution of a dynamical system from its infinitesimal viewpoint is certainly in many respects superior to any other description. For example, finding for a completely integrable nonlinear flow on some infinite dimensional manifold the symmetry group explicitely seems to be an impossible task, whereas finding the infinitesimal generators of one-parameter symmetry groups is a routine matter nowadays.

This would not be possible if we were not able to describe the flow by its infinitesimal behaviour. In fact, the strength and the beauty of areas like theoretical mechanics, with all the impact it had on the development of pure mathematics, is based on the infinitesimal aspects of the systems under consideration.

To include also noncontinuous solutions into this framework is one of the reasons which led to the invention of distribution theory. But alas, distributions do not constitute an algebra and may of the relevant flows are nonlinear, at least when interaction is involved. So it seems as if noncontinuous solutions (for example shock waves) of nonlinear systems cannot be treated from the infinitesimal viewpoint thus forbidding the application of the heavy machinery of classical mechanics to these systems. To show that this is not necessarily so, is the content of this talk.

Moreover, we demonstrate that the usual computational concept for shock waves imposes a canonical algebraic structure on a suitable subspace of distributions.

We review the main features and the structure of this algebra of almost-bounded distributions and it is shown that this algebra is the canonical extension of a well known construction in shock wave analysis. All results are discussed in context of the evolution eqvation of shallow water wave theory (lowest order). The impact of distribution solutions on the Hamiltonian structure and the existence of symmetry groups and conservation laws is discussed.

Gesztelyi, E.: **Tchebycheff systems and color recognition**

Let $K \subset V$ be a convex closed cone of the topological vector space V. Let $A\colon V \to \mathbf{R}^m$ be a linear transformation of finite rank. The element $x_0 \in K$ is said to be recognizable by A if

(1) $$\forall x \in K\colon\ x \neq x_0 \Rightarrow A(x) \neq A(x_0).$$

We investigate the recognizability of elements by A, when $K=S(\alpha,\beta)=$ $=\{x=s\xi|\xi\in NBV_+(\alpha,\beta)\}$ and A may be written in the form

$$(2)\qquad A(x)=\int_{\alpha}^{\beta}\underline{u}(t)\,d\xi(t)\quad (s\xi=x,\ \xi\in NBV_+(\alpha,\beta)),$$

where $\underline{u}(t)=(u_1(t),\ldots,u_m(t))$ is a continuous vector valued function such that $\underline{u}(\alpha)=\underline{u}(\beta)=0$. This case plays an important role in the color recognition.

A transformation A of the form (2) is said to be normal if exactly the spectral colors (i.e. elements of the form $pe^{-\lambda s}$, $\lambda\in(\alpha,\beta)$, $p\geqq 0$) are recognizable by A. We prove the following theorem:

Theorem. *A transformation of the form* (2) *is normal iff* $m=3$ *and the functions* u_1, u_2, u_3 *form a Tchebycheff system on* (α,β).

Glaeske, H. J.: **Some remarks on asymptotics in the theory of integral transforms of distributions**

The first investigations in question the asymptotic behaviour of integral transformations of distributions are connected with the names of Lighthill, Jones, Lavoine, Mangad and Zemanian. They considered only semiregular distributions. To include singular distributions too, Lavoine introduced in 1975 the definition of equivalence of distributions in the origin and in 1973 resp. 1977 Drožinov and Zavjalov have published the concept of quasiasymptotics in the origin and for $x\to+\infty$ respectively.

Here we want to present another definition, which is fit for the transfer of many well-known Abelian and Tauberian theorems from integral transformations of functions to such of distributions (see also D. Müller: „Abelsche und Taubersche Sätze für einige Integraltransformationen von Distributionen", dissertation, Jena, 1982).

As usual let $\mathscr{D}'_+$ resp. $\mathscr{D}'_{[a,b]}$ be the spaces of distributions form $\mathscr{D}'(R_1)$ with support in $[0,\infty)$ resp. $[a,b]$. Furthermore let $C^+_{[a,b]}$ be the space of functions on R with support in $[0,\infty)$, which are continuous in $[a,b]$. $\mathbf{N}_0$ is the set of nonnegative integers.

Definition. Two distributions $f,g\in\mathscr{D}'_+$ are called similar for $x\to0+$, $(f\simeq g$, $x\to0+)$, if the following conditions are fulfilled:

(a) It exists an interval $[a,b]$, $a<0<b$, functions $f_0,g_0\in C^+_{[a,b]}$ and an integer $k\in\mathbf{N}_0$ so, that

$$f=D^kf_0,\quad g=D^kg_0.$$

(b) It exists a real valued function $m_0\in C^+_{[a,b]}$, not changing its sign in $[a,b]$, so that $f_0(x)$ and $g_0(x)$ are asymptotic to $m_0(x)$, if x tends to $0+$.

The similarity of two distributions is symmetrically and transitively. Moreover, if two continuous real valued functions are asymptotic if x tends to $0+$, then they are similar considered as regular distributions. The converse is not true.

Proposition 1. A distribution $f \in \mathcal{D}'_+$ is similar to the delta-distribution iff there exist an interval $[a, b]$, $a<0<b$, a function $f_0 \in C^+_{[a,b]}$ and a natural number $k \geqq 2$ so, that $f = D^k f_0$ in $\mathcal{D}'_{[a,b]}$, and $f_0(x)$ is asymptotic to $x^{k-1}/(k-1)!$ if x tends to $0+$.

Using the concept of similarity one can prove as well Abelian theorems of real and nonreal kind as Tauberian theorems for the Laplace transform of distributions. For example we have

Proposition 2. If two Laplace transformable distributions $f, g \in \mathcal{D}'_+$ are similar for $x \to 0+$, then their Laplace transforms $F(s)$, $G(s)$ are asymptotic for $s \to +\infty$.

Proposition 3. If $f \in \mathcal{D}'_+$ is Laplace transformable, and if its Laplace transform $F(s)$ is asymptotic to s^q, $q > -1$ if $|s| \to +\infty$ in a half-plane $\operatorname{Re}(s) > \alpha$, then for $x \to 0+$ we have

$$f \simeq D^{q+3}(x_+^{2-\{q\}}/\Gamma[3-\{q\}]),$$

where $q = [q] + \{q\}$, $q \in N_0$, $0 \leqq \{q\} < 1$.

Analogous one can define the similarity if x tends to $+\infty$. These definitions are fit for the proof of a lot of Abelian and Tauberian theorems as well for the Laplace transform as for the Mellin- and the Stieltjes transform.

О сходимости рядов фурье в различных классах обобщённых функций

В. И. Горбачук, М. Л. Горбачук:

Пусть A — самосопряжённый полуограниченный снизу оператор с дискретным спектром в гильбертовом пространстве $\mathfrak{H}$ ($(\cdot, \cdot)$ — скалярное произведение, $\|\cdot\|$ — норма) и $\{e_k\}_{k=1}^{\infty}$ — ортонормированный базис в нём, соответствующий последовательности $\{\lambda_k\}_{k=1}^{\infty}$ собственных чисел, расположенных в порядке возрастания; при этом предполагается, что существует $p > 0$ такое, что $\sum_{\lambda_k > 1} \lambda_k^{-p} < \infty$. Положим

$$\mathfrak{H}_\infty = \bigcap_{n=1}^{\infty} D(A^n), \quad \mathfrak{G}_\beta^\alpha = \{x \in \mathfrak{H}_\infty : \|A^n x\| < C\alpha^n n^{n\beta}, \quad \alpha > 0, \beta \geqq 0\}$$

($D(A)$ — область определения оператора A). Множество $\mathfrak{G}_\beta^\alpha$ образует банахово пространство относительно $\|x\|_{\mathfrak{G}_\beta^\alpha} = \sup_n (\|A^n x\| \alpha^{-n} n^{-n\beta})$ и $\mathfrak{G}_\beta^{\alpha'} \subset \mathfrak{G}_\beta^{\alpha''}$ при $\alpha' < \alpha''$.

Обозначим

$$\mathfrak{G}_{\{\beta\}} = \lim_{\alpha \to \infty} \operatorname{ind} \mathfrak{G}_\beta^\alpha, \quad \mathfrak{G}_{(\beta)} = \lim_{\alpha \to 0} pr\, \mathfrak{G}_\beta^\alpha.$$

В пространстве $\mathfrak{H}_\infty$ также введём топологию проективного предела гильбертовых пространств

$$\mathfrak{H}_n = D(A^n), \quad \|x\|_{\mathfrak{H}_n} = (\|x\|^2 + \|A^n x\|^2)^{1/2}.$$

Сопоставим $x \in \mathfrak{H}$ ряд Фурье $\sum x^k e_k$, $x^k = (x, e_k)$.

Теорема 1. *Имеют место равенства*

$$\mathfrak{G}_{\{0\}} = \{x \in \mathfrak{H} \mid \exists N = N(x): x^n = 0, \ n > N\},$$

$$\mathfrak{G}_{\{\beta\}} = \{x \in \mathfrak{H} \mid \exists \alpha > 0, \exists c > 0: |x^k| < c \exp(-\alpha \lambda_k^{1/\beta})\},$$

$$\mathfrak{G}_{(\beta)} = \{x \in \mathfrak{H} \mid \forall \alpha > 0 \exists c > 0: |x^k| < c \exp(-\alpha \lambda_k^{1/\beta})\},$$

$$\mathfrak{H}_\infty = \{x \in \mathfrak{H} \mid \forall \alpha > 0 \exists c > 0: |x^k| < c \lambda_k^{-\alpha}\}$$

(k достаточно большие); ряд Фурье элемента какого-либо из указанных пространств сходится к нему в этом же пространстве.

Сопряжённые с $\mathfrak{H}_\infty$, $\mathfrak{G}_{\{\beta\}}$ $(\beta \geqq 0)$ и $\mathfrak{G}_{(\beta)}$ $(\beta > 0)$ пространства обозначим через $\mathfrak{H}_{-\infty}$, $\mathfrak{G}'_{\{\beta\}}$ и $\mathfrak{G}'_{(\beta)}$ соответственно. В силу плотности вложений $\mathfrak{G}_{\{0\}} \subset \subset \mathfrak{G}_{(\beta)} \subset \mathfrak{G}_{\{\beta\}} \subset \mathfrak{H}_\infty \subset \mathfrak{H}$ получаем цепочку

$$\mathfrak{G}_{\{0\}} \subset \mathfrak{G}_{(\beta)} \subset \mathfrak{G}_{\{\beta\}} \subset \mathfrak{H}_\infty \subset \mathfrak{H} \subset \mathfrak{H}_{-\infty} \subset \mathfrak{G}'_{\{\beta\}} \subset \mathfrak{G}'_{(\beta)} \subset \mathfrak{G}'_{\{0\}}$$

плотно и непрерывно вложенных друг в друга пространств. Пространство $\mathfrak{G}_{\{0\}}$, состоящее из конечных сумм вида $\sum x^k e_k$, назовём основным, а $\mathfrak{G}'_{\{0\}}$ — пространством обобщённых элементов. Если $x' \in \mathfrak{G}'_{\{0\}}$, а $x \in \mathfrak{G}_{\{0\}}$, то под (x', x) понимается расширение скалярного произведения $(\cdot, \cdot)$ до билинейной формы на $\mathfrak{G}'_{\{0\}} \times \mathfrak{G}_{\{0\}}$.

Пусть C^∞ — пространство всех последовательностей $\{c^k\}_{k=1}^\infty$ комплексных чисел с покоординатной сходимостью. Отображение

$$\mathfrak{G}'_{\{0\}} \ni x' \to \{x'^k\}_{k=1}^\infty \in C^\infty, \quad x'^k = (x', e_k),$$

взаимно однозначно и взаимно непрерывно. С его помощью $\mathfrak{G}'_{\{0\}}$ отождествляется с пространством формальных рядов Фурье, т. е. для произвольного элемента $x' \in \mathfrak{G}'_{\{0\}}$ его ряд Фурье $\sum x'^k e_k$ сходится к x' в $\mathfrak{G}'_{\{0\}}$; обратно, любой ряд $\sum c^k e_k$ сходится в $\mathfrak{G}'_{\{0\}}$ к некоторому элементу x', причём $x'^k = c^k$.

Теорема 2. *Если* $x' \in \mathfrak{G}'_{\{0\}}$, *то*

$$(x' \in \mathfrak{G}'_{\{\beta\}}) \Leftrightarrow (\forall \alpha > 0 \exists c > 0: |x'^k| < c \exp(\alpha \lambda_k^{1/\beta})),$$

$$(x' \in \mathfrak{G}'_{(\beta)}) \Leftrightarrow (\exists \alpha > 0, \exists c > 0: |x'^k| < c \exp(\alpha \lambda_k^{1/\beta})),$$

$$(x' \in \mathfrak{H}_{-\infty}) \Leftrightarrow (\exists \alpha > 0, \exists c > 0: |x'^k| < c \lambda_k^{\alpha}).$$

Назовём преобразованием Гаусса—Вейерштрасса ряда Фурье $\sum x'^k e_k$ вектор-функцию

$$(\Gamma x')(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} x^k e_k.$$

При $x' \in \mathfrak{G}'_{\{1\}}$ и $t > 0$ она бесконечно дифференцируема в пространстве $\mathfrak{H}$ (даже в $\mathfrak{G}_{\{1\}}$) и $(\Gamma x')(t) \to x'$, $t \to 0$, в $\mathfrak{G}'_{\{1\}}$.

Рассмотрим банахово пространство $\mathfrak{B}$ такое, что $\mathfrak{G}_{\{1\}} \subset \mathfrak{B} \subset \mathfrak{G}'_{\{1\}}$, причём вложения плотны и непрерывны. Будем говорить, что метод суммирования (1) $\mathfrak{B}$-регулярен, если $(\Gamma x')(t) \to x'$, $t \to 0$, в пространстве $\mathfrak{B}$. Обозначим через $\hat{A}$ расширение оператора A, действующее в $\mathfrak{G}'_{\{0\}}$ как $\hat{A}f' = \sum \lambda_k f'^k$.

Теорема 3. *Метод суммирования* (1) *$\mathfrak{B}$-регулярен тогда и только тогда, когда оператор $-\hat{A}|\mathfrak{B}$ генерирует полугруппу класса C_0 в пространстве $\mathfrak{B}$.* Здесь $-\hat{A}|\mathfrak{B}$ обозначает сужение оператора $-\hat{A}$ на $\mathfrak{B}$.

Kadlubowska, E. and Wawak, R.: **Local order functions and regularity of the product of distributions**

W. Ambrose introduced in [1] the notion of the local order functions for distributions, which he used for defining a product of distributions. The product turned out to be a generalization of Sato—Hörmander product connected with the notion of the wave front of distribution. The local order function considered by Ambrose is bound with the class L^2. In this paper we define other local order functions, connected with L^p. Let $(x, l) \in R^n \times S^{n-1}$.

Definition. $O^p_U(x, l) = \{\alpha \in R \mid$ there exist neighbourhoods Q of x and L of l such that for all $\omega \in D(Q)$:

$$(\omega U)^\wedge (y)(1+y^2)^{\alpha/2} \in L^p(\Gamma_L)$$

where $\Gamma_L = \{rx | x \in L,\ r \in R^+\}$ and "$\wedge$" denotes the Fourier transform.

Lemma. *Let* $1 \leqq p, q \leqq +\infty$ *and* $\frac{1}{p} + \frac{1}{q} \geqq 1$. *Let* $U, V \in D'(R^n)$ *and* $x \in R^n$. *If for all* $l \in S^{n-1} O^p_U(x, 1) + O^p_V(x, -l) \supset O^+$, *where* $O^+ = (-\infty, 0]$ *then there exists a neighbourhood* Q *of* x *such that for all* $\omega, \psi \in D(Q)(\omega U)^\wedge (\psi V)^\vee \in L^1(R^n)$ *and we can define in a natural way the product* UV.

We give now an estimation of the order function of product of distributions, depending on the values of the order functions of its factors. It is a generalization of the well known inclusion:

$$WF(UV) \subset WF(U) \cup WF(V) \cup (WF(U) \oplus WF(V)).$$

First we define, for $k \in S^{n-1} k^+ = \{(l, m) \in S^{n-1} \times S^{n-1} \mid$ there exist $a, b \in R^+$ such that $al + bm = k\}$, and $[k]$ = the closure of k^+.

Theorem. *Let* p, q, r *be such that* $1 \leqq p, q, r \leqq +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. *Let* $U, V \in D'(R^n)$ *be such that*

$$O^p_U(x, l) + O^q_V(x, -l) \supset O^+ \quad \text{for all} \quad (x, l) \in R^n \times S^{n-1}.$$

Then:

$$O^r_{UV}(x, k) \supset \min\left(\min_{(1, m) \in [k]} (O^p_U(x, 1) + O^q_V(x, m)),\ O^p_U(x, k),\ O^q_V(x, k)\right)$$

for $k \in S^{n-1}$, $x \in R^n$.

Now we give a sufficient condition for associativity of product in terms of local order functions. Let:

$$S^+ = \{(l, m, k)\in(S^{n-1})^3 \mid \text{there exist } a, b, c\in R^+ \text{ such that } al+bm+ck = 0\}$$

and $[S]$- the closure of S^+.

Theorem. *Let* $1\leqq p, q, s\leqq +\infty$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{s}\geqq 2$. *Let U, V, W be distributions. If $(U, V)\in M_{p,q}(n)$, $(V, W)\in M_{q,s}(n)$, $(U, W)\in M_{p,s}(n)$ and U, V, W satisfy the following condition:*

$$O_U^p(x, l)+O_V^q(x, m)+O_W^s(x, k)\supset O^+$$

for all $(l, m, k)\in[S]$, $x\in R^n$ then the product U, V, W exists and is associative (The space $M_{p,q}(n)$ consists of all couples (U, V) satisfying the assumptions of the Lemma.)

References

[1] W. Ambrose, Product of distributions with values in distributions, *Journal für die reine und angewandte Mathematik* **315** (1980), 73—91.

Krabbe, G. L.: **Some definite integrals of generalized functions**

A *"test-function"* is a $C^\infty(\mathbf{R})$ function whose every derivative vanishes at the origin. If $F(t)\in L^{\text{loc}}(\mathbf{R})$, then $\{F(t)\}$ is the operator which assigns to each test-function $\varphi(t)$ the test-function $\varphi_F(\tau)$ such that

$$\varphi_F(\tau) = -\int_\tau^0 F(\tau-\sigma)\varphi(\sigma)\,d\sigma \quad \text{for} \quad \tau\in\mathbf{R} = (-\infty, \infty).$$

A *"D-function"* is a piecewise-continuous function whose domain consists of the points in $\mathbf{R}$ where it is continuous. Let $D_m(\mathbf{R})$ be the family of all the D-functions $G(t)$ having no removable discontinuities and whose derivatives $G^{(m)}(t), \ldots, G^{(1)}(t)$ are D-functions such that $0=G^{(m-1)}(0^-)=\ldots=G^{(1)}(0^-)=G(0^-)$. An *"operand"* is the composition $\{G(t)\}\circ s^m$, where $G(t)\in L^{\text{loc}}(\mathbf{R})$ and where s^m is the operator which assigns to each test-function $\varphi(t)$ its derivative $\varphi^{(m)}(t)$. Thus, if f is an operand, then $f=\{G(t)\}\circ s^m$ for some $G(t)\in L^{\text{loc}}(\mathbf{R})$ and some integer $m\geqq 1$, if $G(t)$ is a member of the family $D_m(\mathbf{R})$ and if $m\geqq 1$, then $f(t) \overset{\text{def}}{=} G^{(m)}(t)$ and $Df\overset{\text{def}}{=} s\circ f - f(0^-)s^0$. If $g=\{H(t)\}\circ s^n$, where $H(t)$ is a member of the family $D_n(\mathbf{R})$ and $n\geqq 1$, then g is called a *"D-operand"* and $g(t)=H^{(n)}(t)$.

Operands form a commutative subalgebra of the algebra of linear operators which assign test-functions to test-functions; since $f\circ s^0=f$ for every operand f, the operand s^0 is the multiplicative unit.

Given $x\in\mathbf{R}$, a D-operand g gives rise to a D-operand $e^{-xD}g$ such that $e^{-xD}g(t)=g(t-x)$; if $h=\{H(t)\}$ for some D-function $H(t)$ having no removable discontinuities, then $h(t)=H(t)$; such D-operands h can be identified with the D-functions $h(t)$ they give rise to. Let y be a D-operand: the operand Dy is a D-operand if *(and only if)* the usual derivative $dy(t)/dt$ is a D-function; if so, then $Dy(t)=$

$=dy(t)/dt$. Let $U(\tau)=0$ for $\tau<0$ and $U(\tau)=1$ for $\tau>0$: if $x\in\mathbf{R}$, then $\delta_x= =D\{U(t-x)\}$ (by definition): unlike the operand s^0, the operand δ_x is a D-operand; also, $\delta_x(t)=dU(t-x)/dt=\delta'_x(t)$, where $\delta'_x=D\delta_x$ (by definition); moreover, $\delta_x= =e^{-xD}\delta_0$.

Let f be a D-operand. There is a unique D-operand y such that $s\circ y=f$; it turns out that $Dy=f$; if α and β are real numbers, then

$$\int_{\alpha^-}^{\beta^+} f\underline{(t)\,dt} \stackrel{\text{def}}{=} y(\beta^+)-y(\alpha^-).$$

Theorem. *If*

$$H(x)=\int_{\alpha^-}^{x^+} f\underline{(t)\,dt} \quad \textit{for all} \quad x\in\mathbf{R},$$

then $H^{(1)}(\sigma)=f(\sigma)$ *at each point* σ *where* $f(\sigma)$ *is defined. If* g *and* Dg *are* D*-operands, then*

$$g(\beta^+)-g(\alpha^-)=\int_{\alpha^-}^{\beta^+} Dg\underline{(t)\,dt}.$$

Let $F(t)$ and $G(t)$ be D-functions such that $F(t)H(t)$ is also a D-function. If $h=\{H(t)\}$, then

$$F(t)\,h \stackrel{\text{def}}{=} \{F(t)\,H(t)\} \quad \text{and} \quad \int_{\alpha^-}^{\beta^+} F(t)\,h\underline{(t)\,dt}=\int_{\alpha}^{\beta} F(\sigma)\,h(\sigma)\,d\sigma;$$

if $F^{(k)}(t)H^{(r)}(t)$ is a D-function when $0\leqq k, r\leqq m$, then the D-operand $F(t)D^m h$ is easily defined; in particular, if $F(t)H(t)$ is continuous except at a point x, then

$$\int_{\alpha^-}^{\beta^+} F(t)\,Dh\underline{(t)\,dt}=\int_{\alpha}^{\beta} F(\sigma)\,H^{(1)}(\sigma)\,d\sigma+[F(x^+)\,H(x^+)-C]\int_{\alpha^-}^{\beta^+} \delta_x\underline{(t)\,dt},$$

where $C=F(x^-)H(x^-)$. Equally obvious is the consequence

$$F(t)\,D\delta_x=F(x)\,\delta'_x-F^{(1)}(x)\,\delta_x,$$

which implies the equations

$$\int_{\alpha^-}^{\beta^+} F(t)\,\delta'_x\underline{(t)\,dt}=-F^{(1)}(x)\int_{\alpha^-}^{\beta^+} \delta_x\underline{(t)\,dt}=\begin{cases}-F^{(1)}(x) & \text{for} \quad \alpha\leqq x\leqq\beta\\ 0\ldots & \text{otherwise}\end{cases}$$

— provided that $F^{(1)}(t)$ is continuous at the point x.

To each $x\in\mathbf{R}$ let there correspond a D-function $H_x(t)$. If m is an integer $\geqq 0$, then

$$\int_{-\infty}^{\infty} s^m\circ\{H_x(t)\}\,dx \stackrel{\text{def}}{=} D^m\Big\{\int_{-\infty}^{\infty} H_x(t)\,dx\Big\}$$

— provided that the right-hand side is a D-operand. If $F(t)$ is a D-function, then

$$\{F(t)\}=\int_{-\infty}^{\infty} F(x)\,\delta_x\,dx=\int_{-\infty}^{\infty} F(x)\,e^{-sx}\,dx.$$

Kucera, J.: **Convolution of temperate distributions**

Laurent Schwartz' space $\mathcal{O}_c'$ of convolution operators consists of all distributions which convolve with every $f \in \mathfrak{S}'$, where $\mathfrak{S}'$ is the space of temperate distributions. However, for each particular $f \in \mathfrak{S}'$, the space of distributions which convolve with f is strictly larger than $\mathcal{O}_c'$. We study such "enlarged" convolution spaces.

Denote by $\|\cdot\|_0$ the norm in the space $L^2(R^n)$ of square integrable functions and put $L_k = \{\varphi\colon R^n \to C;\ \|\varphi\|_k^2 = \sum_{|\alpha|+|\beta| \leq k} \|x^\alpha D^\beta \varphi\|_0^2 < +\infty\}$, $k \in N$. Each space L_k, and its dual L_{-k}, is Hilbert, $\mathfrak{S} = \operatorname{proj\,lim} L_k$, and $\mathfrak{S}' = \operatorname{ind\,lim} L_{-k}$. For each $q \in N$, let $\mathcal{O}_q^*$ be the space of all $f' \in \mathfrak{S}'$ for which the convolution $f * g$ makes sense for any $g \in F_{-q}$ and the mapping $g \mapsto f * g\colon L_{-q} \to \mathfrak{S}'$ is continuous. Then $\mathcal{O}_q^* \subset \subset \mathfrak{L}(L_{-q}, \mathfrak{S}')$ and we may equip $\mathcal{O}_q^*$ with the bounded topology of $\mathfrak{L}(L_{-q}, \mathfrak{S}')$. This makes $\mathcal{O}_q^*$ a reflexive, complete, and bornological space. The convolution on each product $\mathcal{O}_q^* \times L_{-q}$, $q \in N$, is continuous while it is not continuous on $\mathcal{O}_c' \times \mathfrak{S}'$.

If $W(x) = (1+|x|)^{1/2}$, $x \in R^n$, we can characterize the elements of $\mathcal{O}_q^*$ as follows: $f \in \mathcal{O}_q^*$ iff $W^q(f * \varphi) \in L^2(R^n)$ for any $\varphi \in \mathfrak{S}$. It turns out that each map $p\varphi\colon f \mapsto \|f * \varphi\|_q$, $\varphi \in \mathfrak{S}$, is a continuous seminorm on $\mathcal{O}_q^*$ and the topology of $\mathcal{O}_q^*$ is generated by the family $\{p\varphi; \varphi \in \mathfrak{S}\}$. Schwartz proves that $f \in \mathcal{O}_c'$ iff $f * \varphi \in \mathfrak{S}$ for every $\varphi \in D$. Hence $\mathcal{O}_c' = \bigcap \{\mathcal{O}_q^*, q \in N\}$. Moreover $\mathcal{O}_c' = \operatorname{proj\,lim} \mathcal{O}_q^*$.

We have another characterization: $f \in \mathcal{O}_q^*$ iff $f = \sum_{\alpha \in A} D^\alpha (W^{-q} f_\alpha)$, where $A \subset N^n$ is a finite set and each $f_\alpha \in L^2(R^n)$. This implies $\mathcal{O}_q^* = \operatorname{ind\,lim}_{p \to \infty} (1-\Delta)^p L_q$, where Δ is the Laplace operator and the topology in $(1-\Delta)^p L_p$ is chosen so that the mapping $f \mapsto (1-\Delta)^p f\colon L_q \to (1-\Delta)^p L_q$ is an isomorphism. As another consequence, the identity map: $\mathcal{O}_q^* \to \mathfrak{S}'$ is continuous.

Máté, L.: **Remark to the Figa—Talamanca theorem**

Translation-invariance and commuting with convolution are the same things for function spaces which are $L^1(G)$-modules. Hence seeking translation-invariant linear operators in function spaces has its counterpart in seeking linear operators, commuting with the module operation in a module. Moreover, the results on such operators in Banach modules resp. Banach algebras have frequently a non-trivial meaning in the theory of translation-invariant operators.

A dual space representation (i.e. a representation by a Banach space of bounded linear functionals) for translation-invariant linear operators of $L^p(G)$ $1 \leq p < \infty$ was given by A. Figa—Talamanca in 1965 for Abelian G and thus a connection also had been given between the operators of convolution by a "good" function from $C_c(G)$ and *all* translation-invariant linear operators. Figa—Talamanca's results were generalized by P. Eymard for any amenable G in 1971. We remark, that in this case every translation-invariant linear operator is ("automatically") bounded.

In this lecture we give the counterpart of Figa—Talamanca's construction for a wide class of reflexive Banach spaces which are modules and it will be shown that the constructed functionals "are" operators commuting with the module operation.

What are the conditions for the representability of all such operators by this construction?

We express the amenability of G with the properties of $C_c(G)$ resp. $M_{bd}(G)$ considered as convolution algebras and we show that for reflexive Banach spaces which are modules possesing this "amenable" property, the Figa—Talamanca type construction is a dual space representation for *all* module homomorphisms.

Mieloszyk, E.: **Operational calculus in algebras**

Let an operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$ be given, where $L^1 \subset L^0$, L^1, L^0 are commutative algebras with unity **1** over the field of real numbers and with the multiplication such that for $f, g \in L^1$

$$S(f \cdot g) = (Sf) \cdot g + f \cdot (Sg), \tag{1}$$

$$s(q)(f \cdot g) = (s(q)f) \cdot (s(q)g). \tag{2}$$

Definition 1. We will say that there exists an element $u \overset{\mathrm{df}}{=} E_1^{T(q)p}$ if and only if $E_1^{T(q)p}$ is a solution of the abstract differential equation

$$Su = p \cdot u \tag{3}$$

with condition

$$s(q)u = \mathbf{1}, \quad \text{where} \quad u \in L^1,\ p \in L^0 \quad \text{and} \quad E_1^{T(q)p} \in \mathrm{Inv}. \tag{4}$$

Theorem 1. *If there exists an element $E_1^{T(q)p}$ then three operations:*

$$S_p u \overset{\mathrm{df}}{=} Su + p \cdot u, \tag{5}$$

$$T_p(q) f \overset{\mathrm{df}}{=} [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p}, \tag{6}$$

$$s_p(q)\, u \overset{\mathrm{df}}{=} (s(q)u) \cdot E_1^{-T(q)p} \tag{7}$$

satisfy axioms of operational calculus, where $u \in L^1$, $f \in L^0$.

Operation S_p is a derivative, operation $T_p(q)$ is an integral, operation $s_p(q)$ is a limit condition.

Theorem 2. *If $a_1, a_2 \in L^0$, $a_1 \in \mathrm{Inv}$ and if there exists an element $E_1^{T(q)a_1^{-1}a_2}$ then the abstract differential equation*

$$a_1 \cdot Su + a_2 \cdot u = f \tag{8}$$

with condition

$$s(q)u = u_0, \quad \textit{where} \quad u \in L^1, \quad f \in L^0, \quad u_0 \in \mathrm{Ker}\, S \tag{9}$$

has only one solution defined by formula

$$u = [T(q)(a_1^{-1} \cdot f \cdot E_1^{T(q)a_1^{-1}a_2})] \cdot E_1^{-T(q)a_1^{-1}a_2} + u_0 \cdot E_1^{-T(q)a_1^{-1}a_2}. \tag{10}$$

Definition 2. If there exists an element $E_1^{T(q)p}$ then for the elements $x, y \in L^0$ we will define the multiplication $x \circ y$ by the formula

$$x \circ y \overset{\text{df}}{=} E_1^{T(q)p} \cdot x \cdot y. \tag{11}$$

The multiplication 0 satisfies condition (1) for the derivative S_p and condition (2) for the limit condition $s_p(q)$.

Example. In case of the operational calculus with the derivative

$$S\{u(x_1, x_2, \ldots, x_n)\} \overset{\text{df}}{=} \left\{ \sum_{i=1}^{n} b_i \frac{\partial u(x_1, \ldots, x_n)}{\partial x_i} \right\}$$

where $u \in L^1 \overset{\text{df}}{=} C^2(R^{n-1} \times \langle x_n^1, x_n^2 \rangle;\ R)$,

$p \in L^0 \overset{\text{df}}{=} C^1(R^{n-1} \times \langle x_n^1, x_n^2 \rangle, R)\, x_n^0 \in \langle x_n^1, x_n^2 \rangle,\ b_i \in R$ for $i = 1, 2, \ldots, n,\ b_n \neq 0$

the multiplication $\circ$ has the following form

$$x \circ y = \{x(x_1, x_2, \ldots, x_n)\} \circ \{y(x_1, x_2, \ldots, x_n)\} =$$

$$= \left\{ e^{\frac{1}{b_n} \int_{x_n}^{x_n} p_0 \left(x_1 - \frac{b_1}{b_n}(x_n - \tau), \ldots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau \right) d\tau} x(x_1, x_2, \ldots, x_n)\, y(x_1, x_2, \ldots, x_n) \right\}.$$

Mikusiński, J.: **On the logarithm of the differential operator**

Mitrović, D.: **A new proof of some distributional relations of the Plemelj type**

It is well known that the distributions

$$\delta_x^+ := \lim_{\varepsilon \to +0} -\frac{1}{2\pi i(x + i\varepsilon)}, \quad \delta_x^- := \lim_{\varepsilon \to +0} -\frac{1}{2\pi i(x - i\varepsilon)} \tag{1}$$

satisfy the Plemelj relations

$$\delta_x^+ = \frac{\delta_x}{2} - \frac{1}{2\pi i}\, vp\, \frac{1}{x}, \tag{2}$$

$$\delta_x^- = -\frac{\delta_x}{2} - \frac{1}{2\pi i}\, vp\, \frac{1}{x}, \tag{3}$$

where the limits are taken in the $D' = D'(R)$ topology (weak).

Developing the theory of the distributional spaces $\mathcal{O}'_\alpha = \mathcal{O}'_\alpha(R)$ Bremermann has proved ([1]) the relations (2)—(3) taking the limits (1) in the $\mathcal{O}'_\alpha$ topology $(\alpha < 0)$. His proof here will be considerably simplified by

Proposition 1. *The relations* (2)—(3) *with the convergence in D' are equivalent to relations* (2)—(3) *with the convergence in $\mathcal{O}'_\alpha$* $(\alpha < 0)$.

Also, the similar statement holds for the first and second part of Theorem 2 ([2]):

Proposition 2. *Let* $T \in \mathcal{O}'_\alpha$ *for* $-1 \leqq \alpha < 0$,

$$\hat{T}(z) = \frac{1}{2\pi i}\left\langle T_t, \frac{1}{t-z}\right\rangle \quad \textit{and suppose} \quad \hat{T}(z) = O\left(\frac{1}{|z|}\right)$$

as $|z| \to \infty$. *Then the relations*

$$\lim_{\varepsilon\to+0} \hat{T}(x+i\varepsilon) := \hat{T}_x^+ = \frac{T_x}{2} - \frac{1}{2\pi i}\left(T * vp\,\frac{1}{x}\right), \tag{4}$$

$$\lim_{\varepsilon\to+0} \hat{T}(x-i\varepsilon) := \hat{T}_x^- = -\frac{T_x}{2} - \frac{1}{2\pi i}\left(T * vp\,\frac{1}{x}\right) \tag{5}$$

with the convergence in D′ are equivalent to relations (4)—(5) *with the convergence in* $\mathcal{O}'_\alpha$ *(for the same* α*).*

Proofs (based on some results in [3]), details and complements will appear at a later date.

References

[1] H. J. Bremermann, Distributions, complex variables, and Fourier transform, *Addison-Wesley, 1965.*

[2] D. Mitrović, The Plemelj distributional formulas, *Bull. Amer. Math. Soc.* **77** (1971), 562—563.

[3] D. Mitrović, A distributional representation of strip analytic functions, *Intern. J. Math. and Math. Sci.* **1** (1982), 1—9.

Ortner, N.: **A fundamental solution of the product of Laplace operators**

From the beginnings of the theory of elasticity and electrodynamics, one was looking for fundamental solutions of the operators

$$\Delta_2(\partial_1^2 + a\partial_2^2) \quad \text{(or equivalently } \partial_1^4 + 2(1-2\varepsilon^2)\partial_1^2\partial_2^2 + \partial_2^2) \quad \text{and} \quad \Delta_3(\Delta_2 + a\partial_3^2),$$

Δ_n denoting the n-dimensional Laplace operator (cf. Brioullin 1918, Bureau 1947, 1948, Garnir 1952, Stein 1959). This fact inspired G. Herglotz in 1926—1928 to work out, more generally, an explicit expression for a fundamental solution of a product of Laplacean's, i.e.

$$(\Delta_{n-1} + a_1\partial_n^2) \dots (\Delta_{n-1} + a_m\partial_n^2), \tag{1}$$

$$a_1, \dots, a_m > 0, \quad m \in \mathbf{N},\ n \in \mathbf{N},\ m \geqq 2.$$

In contrast to his results concerning the fundamental solution of a product of wave operators (compare: G. Herglotz, Gesammelte Schriften, p. 556, *Vandenhoek & Ruprecht, Göttingen* 1979; N. Ortner, Die Fundamentallösung von Produkten hyperbolischer Operatoren. I. *Preprint, Innsbruck,* 1983), his construction of a fundamental solution of (1) is complicated and does not yield explicit expressions (p. 560 and 561). Even for $n=2$ and $m=2$ I did not succeed to derive an expression for a fundamental solution of the operator of the anisotropic orthogonal plate using his method.

Therefore, to give an explicit expression for a fundamental solution of (1), I apply the new method of "parameter-integration", developed by P. WAGNER in his thesis (Parameterintegration zur Berechnung von Fundamentallösungen. Dissertationes Math. CCXXX, *Warszawa,* 1984).

If the numbers a_j are all distinct, proposition 3 (p. C3) gives a fundamental solution E of (1), namely,

$$E=(m-1)\sum_{j=1}^{m} c_j \int_{a_0}^{a_j} (a_j-\lambda)^{m-2} E_\lambda \, d\lambda. \tag{2}$$

Here, E_λ denotes a fundamental solution of

$$(\Delta_{n-1}+\lambda\partial_n^2)^m, \quad \text{and } c_j \text{ is the product} \quad c_j=\prod_{\substack{k=1\\ k\neq j}}^{m}(a_j-a_k)^{-1},$$

and $a_0>0$ is an arbitrary constant. Furthermore, E_λ has to be chosen such that

(i) $(1+|x|^2)^{-(2m-n+1)/2}E_\lambda\in B'$ (= space of distributions vanishing at infinity),

(ii) $\lambda\to E_\lambda$ is continuous on the interval determined by the greatest and the smallest of the numbers $a_0, a_1, \ldots, a_m$. These requirements are fulfilled by

$$E_\lambda = A_{m,n}\frac{1}{\sqrt{\lambda}}\left(\varrho^2+\frac{z^2}{\lambda}\right)^{m-n/2} \tag{3}$$

with

$$\varrho_2=x_1^2+\ldots+x_{n-1}^2, \quad z^2=x_n^2, \quad A_{m,n}=\frac{(-1)^m\Gamma(n/2-m)}{2^{2m}\pi^{n/2}(m-1)!},$$

n odd or n even and $2m<n$.

For n even and $2m>n$ we can take

$$E_\lambda = B_{m,n}\frac{1}{\sqrt{\lambda}}\left(\varrho^2+\frac{z^2}{\lambda}\right)^{m-n/2}\log\left(\varrho^2+\frac{z^2}{\lambda}\right) \tag{4}$$

with

$$B_{m,n}=\frac{(-1)^{n/2+1}}{2^{2m}\pi^{n/2}(m-1)!\,(m-n/2)!}.$$

So the construction of a fundamental solution of (1) reduces to the evaluation of definite integral of type (2) with the expressions (3) or (4) for E_λ.

One can show that a_0 disappears in the final results.

Theorem. *A fundamental solution of*

$$\prod_{j=1}^{m}(\Delta_{n-1}+a_j\partial_n^2) \quad (m\geqq 2,\ a_j>0 \text{ pairwise distinct})$$

is given

(1.) for $n=2l$ even and $2m<n$, by:

$$\frac{(-1)^{l-1}}{2^{2m-1}\pi^l(m-2)!}\left(\frac{\partial}{\partial(\varrho^2)}\right)^{l-m-1}\varrho^{1-2m}\sum_{j=1}^{m}c_j\left\{\sum_{k=1}^{m-2}\left[\binom{m-2}{k}(-1)^k\cdot\right.\right.$$

$$\cdot(a_j\varrho^2+z^2)^{m-2-k}\sum_{r=0}^{k-1}\binom{k-1}{r}z^{2k-2r-2}\frac{(\sqrt{a_j}\varrho)^{2r+3}}{2r+3}\Bigg]+$$

$$\left.\left.+(a_j\varrho^2+z^2)^{m-2}\left[\varrho\sqrt{a_j}-z\operatorname{arc\,tg}\left(\frac{\varrho}{z}\sqrt{a_j}\right)\right]\right]\right\}.$$

(2.) for $n=2l-1$ odd and $2m<n$, by:

$$\frac{(-1)^{l-1}\Gamma(l-m-1/2)}{2^{2m}\pi^{l-1}\Gamma(l-m)\Gamma(m-1/2)}\left(\frac{\partial}{\partial(\varrho^2)}\right)^{l-m-1}\varrho^{2-2m}\sum_{j=1}^{m}c_j(a_j\varrho^2+z^2)^{m-3/2}.$$

(3.) for $n=2l-1$ odd and $2m\geqq n$, by:

$$\frac{(-1)^l\Gamma(l-m-1/2)}{2^{2m}\pi^{l-1/2}(m-2)!(m-1)!}\varrho^{4-2l}\lim_{c\to1}\frac{\partial^{m-1}}{\partial c^{m-1}}\left\{c^{2-m}\sum_{j=1}^{m}c_j\left[\left[\sum_{k=1}^{m-2}\binom{m-2}{k}(-1)^k\cdot\right.\right.\right.$$

$$\cdot[a_jc\varrho^2+(c-1)z^2]^{m-2-k}\sum_{r=0}^{k-1}\left[\binom{k-1}{r}c^r(-z^2)^{k-1-r}\frac{(a_j\varrho^2+z^2)^{r+3/2}}{r+3/2}\right]+$$

$$\left.\left.\left.+[a_jc\varrho^2+(c-1)z^2]^{m-2}\left[\frac{2}{c}\sqrt{a_j\varrho^2+z^2}+\frac{z}{c^{3/2}}\log\frac{-z+\sqrt{c(a_j\varrho^2+z^2)}}{z+\sqrt{c(a_j\varrho^2+z^2)}}\right]\right]\right]\right\}.$$

(4.) for $n=2l$ even and $2m\geqq n$ in[1]).

Remark. As in the well-known formula for the fundamental solution of the wave equation in $2l-1$ ($l\geqq2$) space variables the expressions **(1.)** and **(2.)** arise by means of differentiations with respect to $\varrho^2=x_1^2+\ldots+x_{2l-4}^2$.

Pap, E.: **Semigroups of operators on the space of generalized functions** Exp $\mathscr{A}'$

Pandey, G. S.: **On the initial value problem for a distributional Meijer—Laplace transformation**

Let $\Phi(t)$ be a right- sided locally integrable function satisfying the conditions:
i) $\Phi(t)$ $(t)=0$ for $-\infty<t<T$.
ii) There exists a real number such that the function

$$e^{-\omega st}G_{pq}^{hu}\left(2st\,\middle|\,{}^{a_v}_{b_v}\right)\Phi(t)$$

[1]) In a talk on this subject I gave a final formula and a sketch of the proof of this formula in the case n even, $2m\geqq n$. An abstract was published: N. O., Fundamentallösung von Laplace-operatoren. Oberwolfach, Tagungsbericht, 10, Partielle Differentialgleichungen, 1983, p. 14

is absolutely integrable over $-\infty < t < \infty$, where

$$G_{pq}^{hu}(2st \mid {}^{a_v}_{b_v}) \equiv G_{pq}^{hu}(2st \mid {}^{a_1, a_2, \ldots, a_p}_{b_1, b_2, \ldots, b_q})$$

is Merijer's G-function, $P+q<2(h+u)$ and w is a fixed positive number.

The Meijer—Laplace transformation of a distribution is an operation L which assigns a distribution $F(S)$ to each locally integrable $\Phi(t)$ such that

$$L\Phi(t) \stackrel{\Delta}{=} F(s) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} e^{-\omega st} G_{pq}^{hu}(2st \mid {}^{a_v}_{b_v}) \Phi(t)\, dt.$$

The function $F(S)$ is called Meijer—Laplace-transform of $\Phi(t)$. Our assumption that

$$\Phi(t) = 0 \quad \text{for} \quad -\infty < t < T \text{ implies}$$

that

$$L\Phi(t) \stackrel{\Delta}{=} F(s) \stackrel{\Delta}{=} \int_{T}^{\infty} e^{-\omega st} G_{pq}^{hu}(2st \mid {}^{a_p}_{b_q}) \Phi(t)\, dt.$$

We write $\beta = \min \operatorname{Re}(b_1, b_2, \ldots, b_q)$.

The object of the present paper is to establish a theorem which relates the initial value of a distribution to the final value of its Meijer—Laplace transform. Precisely, we shall prove the following:

Theorem. *Let $\Phi(t)$ be a Meijer—Laplace transformable distribution with the support in $0 \leqq t < \infty$ and let $\Phi(t)$ be a regular distribution corresponding to a Lebesgue integrable function $\psi(t)$ in a small neighbourhood of the origin such that*

$$\int_0^{\xi} \frac{|\psi(t)|}{t^{\gamma}}\, dt = \alpha \quad as \quad \xi \to 0,$$

then

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\gamma+1} F(\sigma)}{G(2/w)} = \alpha,$$

where

$$G(2/w) = G_{p+1,q}^{h,u+1}\left(\frac{2}{w} \,\middle|\, {}^{-\gamma, a_1, \ldots, a_p}_{b_1, \ldots, b_q}\right),$$

$$R(b_j + \gamma + 1) > 0; \quad j = 1, 2, \ldots, h$$

and

$$\beta + \gamma > 0.$$

A corresponding result for the final value of Meijer—Laplace transform also holds.

Pilipovic, S.: **Class of spaces of periodic ultradistributions**

We introduce a new class of spaces of periodic ultradistributions. Elements of these spaces can be uniquely expanded into a Fourier series. We give the characterization of elements of these spaces by the growth rate of their coefficients. We also give a structural theorem for elements of these spaces. The obtained results are compared with the known results.

Rao, G. L. N.: **The space H'_{ab} and a generalized L—H transform**

V. K. Kapoor and S. Masood discussed an integral transform whose kernel is a Meijer's G-function. This reduces to generalization of both Laplace and Hankel transforms as particular cases. Further, these authors proved a complex inversion formula for the same transformation in the classical sense. The purpose of this paper is to develop the generalized Laplace—Hankel or simply the generalized L—H transform in the distributional setting and extend the complex inversion formula to distributions interpreting convergence in the weak distributional sense. A week version of a uniqueness theorem also would be given.

In 1983 January, the author presented some results on the distributional complex inversion theory for Varma's second generalization of Laplace transform at the Conference of Indian Science Congress Association. A brief mention about these results would be also made in the present paper.

Rjabcev, I. I.: **To the abstract perfect operators**

The method of perfect operators modifies, unites and in an abstract form also generalizes the theory of generalized functions (distributions) and the algebraic operational calculus.

The general scheme of the method is such. The generalized objects are introduced not as functionals but as Weston's perfect operators — the operators a: $a\Phi \subseteq \Phi$ on the commutative algebra Φ without zero divisors, commutative with every operator of multiplication by an algebra element. The operational calculus characterizing algebraization of the initial linear system L_ω with respect to some non-zero linear operator S: $SL_\omega = M_\omega \subseteq L_\omega$ is achieved with enlargement of L_ω not to the quotient field but to the ring $P = P[\Phi]$ of all perfect operators on the algebra $\Phi = \bigcap_{n=1}^{\infty} S^n M_\omega$, $\Phi \subseteq M_\omega \subseteq L_\omega \subseteq P$. For this purpose for some pairs of elements of L_ω the operation $*$ is introduced of internal multiplication of the type of Dimovski's convolution — bilinear, commutative, associative and commutative with the operator S. It is supposed that the zero divisors are absent, $L_\omega * M_\omega \subseteq L_\omega$, $M_\omega * M_\omega \subseteq M_\omega$ and $\Phi \neq \{0\}$.

The ring $P = P[\Phi]$ is not a field (if we except the trivial case $P = \Phi$), P is isomorphic to the part of quotient field $\bar{\Phi}$ for the ring Φ. But this does not restrict the possibilities of the method, in so far as both the algebraic and the infinitesimal calculus of perfect operators may be constructed without turning to the notion of quotient field.

The ring $P=P[\Phi]$ enlarges the initial system just as much as it may for tracing the initial system inherent structure with respect to the operator $D=S^{-1}$, just the representativeness of all elements (in the sense of some specific convergence) in the form $\lim_{\nu\to+\infty} D^{n_\nu} f_\nu$ $(n_\nu \in \{0, 1, 2, \ldots\}, f_\nu \in M_\omega)$. On the concrete model [1, 2] this signifies the distinction of the class of all such objects which still may be interpreted as the (finite at left) generalized functions with the corresponding local properties. In the general case [3] also the analogy with generalized functions is observed.

The following two theorems, in difference from the corresponding Mikusiński theorems in his quotient field, are proved without the help of prolonging on all the axis, that permits here also to manage with the apparatus of rings without the use of the notion of quotient field.

Theorem 1. *There exists not more than one function $u(\lambda)$ with values in $P=P[\Phi]$, satisfying, on the bounded set $\Lambda \subseteqq R$, the equation $a_0 * u'(\lambda)=a * u(\lambda)+f(\lambda)$ and the condition $u(\lambda_0)=u_0$ $(a_0, a, u_0 \in P, a_0 \neq 0, f(\Lambda) \subseteqq P)$.*

Theorem 2. *There exists not more than one function $u(\lambda)$ with values in $P=P[\Phi]$, satisfying, on the bounded set $\Lambda \subseteqq R$, the equation $a_0 * u''(\lambda)=a * u(\lambda)+f(\lambda)$ and the conditions $u(\lambda_0)=u_0$, $u'(\lambda_0)=u_0'$$(a_0, a, u_0, u_0' \in P, a_0 \neq 0, f(\Lambda) \subseteqq P)$.*

These theorems permit to introduce, together with the notion of exponential function as the solution of the corresponding Cauchy problem on all the axis, the notion of right (left) exponential function as the solution of the same problem on the right (left) semi-axis. On the concrete model no difficulties arise in applications of these functions to the solutions of problems for the equations in partial derivatives.

Here some principal questions of abstract perfect operator theory are touched that are elaborated in detail in a monograph prepared by the author (the summary-article has been adopted for publication).

Literature

[1] И. И. Рябцев, Об операционном исчислении в терминах совершенных операторов. — *Изв. вузов. Матем.*, 1978, № 5, — p. 103—113.

[2] И. И. Рябцев, Метод совершенных операторов. — *Изд. Саратовского университета*, 1978, 103 p.

[3] И. И. Рябцев, К теории совершенных операторов. — In the book: Труды Международной конференции «Обобщенные функции и их применения в математической физике», *Москва*, 24—28 ноября 1980 г. — М.: ВЦ АН СССР, 1981, p. 456—461.

Shultz, H. S.: **Two-variable operational calculus**

For $0<a\leqq\infty$ and $0<b\leqq\infty$ define L to be the set of locally integrable complex-valued functions of two variables on the subset $J=[0, a)\times[0, b)$ of the plane. These functions are extended to be zero for $x<0$ and $y<0$. For f and g in L

we define the convolution

$$f*g(x,y)=\int_0^y\int_0^x f(x-u,y-v)\,g(u,v)\,du\,dv \quad (x,\ y \text{ in } J).$$

This operation is commutative and associative.

Let Q be the subset of L consisting of those functions which are infinitely differentiable and which, along with all partial derivatives, vanish on $\{(x,y)\in J:\ x=0 \text{ or } y=0\}$. A mapping A from Q into Q is said to be "perfect" if $A(p*q)=Ap*q$ for all p and q in Q. By defining $\{f\}q=f*q$ we inject L into the commutative algebra P of perfect operators. Since this injection maps convolution into multiplication it serves as a generalization (there are no growth restrictions) of the two-dimensional Laplace transform.

Suppose g is a locally integrable function of one variable on $(-\infty, a)$ which vanishes on $(-\infty, 0)$. We define

$$\{g(x)\}\,q(x,y)=\int_0^x g(x-u)\,q(u,y)\,du$$

and define $\{g(y)\}$ similarly. The operators $\{g(x)\}$ and $\{g(y)\}$ are perfect.

The algebra P contains the two operators of partial differentiation and admits operational formulas such as

$$\{f_x\}=D_x\{f\}-\{f(0,y)\},$$

$$\{g'(x)\}=D_x\{g(x)\}-g(0),$$

$$\frac{\{f\}}{D_x+D_y}=\Big\{\int_0^\infty f(x-t,\,y-t)\,dt\Big\}$$

and

$$\frac{\{g(x)\}}{D_x+D_y}=\{g(x-y)\}.$$

There is an algebraic isomorphism between P and the space of distributions on $(-\infty, a]\times(-\infty, b]$ having support in J; this isomorphism yields a natural definition of the convolution of two such distributions.

Simon, L.: **On generalized solutions of nonlinear elliptic equations on unbounded domains**

Consider an elliptic differential operator of the form

$$u\mapsto A(u)+B(u)=\sum_{|\alpha|\leqq m}(-1)^{|\alpha|}D^\alpha f_\alpha(x,u,\dots,D^\beta u,\dots)+$$

$$+\sum_{|\alpha|\leqq l}(-1)^\alpha D^\alpha g_\alpha(x,u,\dots,D^\beta u,\dots),\quad u\in V$$

where $D^\alpha=\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$, $|\alpha|=\sum_{j=1}^n\alpha_j$, $|\beta|\leqq m$, $p>1$, $p-1<q\leqq p$, $0<l<m-$

$-\frac{n}{p}(1-p+q)$; V is a closed subspace of the Sobolev space $W_p^m(\Omega)$, $\Omega\subset R^n$ is a possibly unbounded domain. Assume that f_α satisfies some conditions formulated by F. E. BROWDER such that A is a pseudomonotone operator; further g_α (essentially) satisfies

$$g_\alpha(x, \xi_0, \ldots, \xi_\beta, \ldots)\xi_\alpha \geqq 0$$

and

$$|g_\alpha(x, \xi)| \leqq C(|\eta|)(1+|\zeta|^q)$$

where $\xi=(\xi_0, \ldots, \xi_\beta, \ldots)=(\eta, \zeta)$ and η contains those coordinates ξ_β of ξ for which $|\beta|<m-\frac{n}{p}$, C is an arbitrary fixed continuous function.

Then for any linear continuous functional F on V there exists $u\in V$ such that

$$(1)\qquad \sum_{|\alpha|\leqq m}\int_\Omega f_\alpha(x, u, \ldots, D^\beta u, \ldots)D^\alpha v\,dx+ \sum_{|\alpha|\leqq l}\int_\Omega g_\alpha(x, u, \ldots, D^\beta u, \ldots)D^\alpha v\,dx = \langle F, v\rangle$$

for any $v\in V$.

Similar existence theorems were proved by L. BOCCARDO, F. MURAT, J. P. PUEL for second order equations on bounded domains, in the case $q=p$, $V=W_p^{m,0}(\Omega)$.

The proof is based on the fact that $A+B_j$ is a pseudomonotone operator where B_j is defined by

$$B_j(u) = \sum_{|\alpha|\leqq l}(-1)^{|\alpha|}D^\alpha g_{\alpha,j}(x, u, \ldots, D^\alpha u, \ldots),$$

$$g_{\alpha,j}(x,\xi) = \begin{cases} g_\alpha(x,\xi) & \text{if } |x|\leqq j, \quad |g_\alpha(x,\xi)|\leqq j,\\ j\dfrac{g_\alpha(x,\xi)}{|g_\alpha(x,\xi)|} & \text{if } |x|\leqq j, \quad |g_j(x,\xi)|>j,\\ 0 \text{ if } |x|>j. \end{cases}$$

Thus there exists $u_j\in V$ such that

$$(A+B_j)(u_j) = F.$$

By using an argument of J. L. LIONS it can be shown that (u_j) contains a subsequence which converges to a solution of (1) in $W_p^m(\omega)$ for any $\omega\subset\Omega$.

Stankovic, B.: **Abelian and Tauberian theorems for Stieltjes transform of distributions**

J. LAVOINE and O. P. MISRA gave a definition of the Stieltjes transform of distributions which belong to a subspace of tempered distributions with the support in $[0, \infty)$. This definition was used by many authors, especially by those engaged in proving Abelian theorems for this transform.

S. Pilipović and me, we changed slightly this definition of the Stieltjes transform in such a way that it was available for the whole space

$$S'_+ = \{f\in S'(R^n),\ \operatorname{supp} f \subset \bar{R}^n_+\}$$

$(S'(R^n)$ is the space of tempered distributions in n dimension). In case $n=1$, this definition includes the mentioned one.

Using the notion of the quasiasymptotic, introduced by V. S. Vladimirov and his pupils, we proved a theorem of the Abelian type which is more general than those we know.

In the second part we proved a relation between the Stieltjes transform of a tempered distribution and the itereted Laplace transform of a numerical function. This put us in the way to use the well elaborated theory of classical Laplace transform.

In the third part of our results we deal with the Tauberian type theorems for the Stieltjes transform of tempered distributions.

Száz, Á.: **Continuities in relator spaces**

By a relator space, we mean an ordered pair $X(\mathscr{R})=(X, \mathscr{R})$ consisting of a set X and a nonvoid family $\mathscr{R}$ of reflexive relations on X which we call a relator on X.

We define a function f from one relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$ to be continuous, or more precisely $(\mathscr{R}, \mathscr{S})$-continuous if $f^{-1}\circ S\circ f\in\mathscr{R}$ for all $S\in\mathscr{S}$, that is, $f^{-1}\circ\mathscr{S}\circ f\subset\mathscr{R}$.

By introducing appropriate operations on relators, we can get all the important continuity properties of a function as particular cases of the above definition.

For instance, to obtain uniform, proximal, resp. topological continuity of f, we have to consider the following refinements of $\mathscr{R}$.

$$\mathscr{R}^* = \{S\subset X\times X:\ \exists R\in\mathscr{R}:\ R\subset S\},$$

$$\mathscr{R}^\# = \{S\subset X\times X:\ \forall A\subset X:\ \exists R\in\mathscr{R}:\ R(A)\subset S(A)\},$$

$$\hat{\mathscr{R}} = \{S\subset X\times X:\ \forall x\in X:\ \exists R\in\mathscr{R}:\ R(x)\subset S(x)\}.$$

In this general, unifying framework, we have proved generalized forms of several standard theorems about continuities. For instance, we quote here the next two theorems:

Theorem 1. *A function f from a directed relator space $X(\mathscr{R})$ into an arbitrary one $Y(\mathscr{S})$ is*

(i) *uniformly continuous iff* $y_\alpha\in\operatorname{Lim}_{\mathscr{R}}\limits_{\alpha} x_\alpha$ *implies* $f(y_\alpha)\in\operatorname{Lim}_{\mathscr{S}}\limits_{\alpha} f(x_\alpha)$;

(ii) *proximally continuous iff* $B\in Cl_{\mathscr{R}}(A)$ *implies* $f(B)\in Cl_{\mathscr{S}}(f(A))$;

(iii) *topologically continuous iff* $x\in\lim_{\mathscr{R}}\limits_{\alpha} x_\alpha$ *implies* $f(x)\in\lim_{\mathscr{S}}\limits_{\alpha} f(x_\alpha)$, *or equivalently* $x\in cl_{\mathscr{R}}(A)$ *implies* $f(x)\in cl_{\mathscr{S}}(f(A))$.

Theorem 2. *A function f from a directed relator space $X(\mathscr{R})$ into another directed one $Y(\mathscr{S})$ is uniformly continuous if any one of the following conditions holds:*

(i) *f is topologically continuous, $\mathcal{R}$ is locally uniform and compact and $\mathcal{S}$ has the neighbourhood property;*

(ii) *f is proximally continuous and $\mathcal{S}$ is symmetric, uniform and precompact;*

(iii) *f is proximally continuous, $\mathcal{R}$ is linearly ordered and $\mathcal{S}$ is symmetric and uniform.*

Because of the inadequancy of closed (or open) sets in non-topological relator spaces, we call a locally directed relator $\mathcal{R}$ on X compact if

$$\operatorname{adh}_{\mathcal{R}}\, \underset{\alpha\in A}{x_\alpha} = \bigcap_{R\in\mathcal{R}} \bigcap_{\alpha\in A} \bigcup_{\beta\geqq\alpha} R^{-1}(x_\beta) \neq \emptyset$$

for any net $(x_\alpha)_{\alpha\in A}$ in X. The local directedness of $\mathcal{R}$ is equivalent to the directedness of $\hat{\mathcal{R}}$.

Moreover, we remark that by applying our basic definitions to relations instead of functions, we get some new kinds of continuities of relations which are much weaker than the corresponding upper and lower semicontinuities. However, by using a straightforward notion of a hyperspace of a relator space, these mild continuities can also be reduced to continuities of the induced set-valued functions.

Concerning the historical development and the bibliography of the subject the interested reader is refered to our extensive paper *"Relator spaces"* to appear in Acta Math. Acad. Sci. Hungar.

Székelyhidi, L.: **The Fourier transform of exponential polynomials**

The Fourier transform of exponential polynomials on topological Abelian groups is introduced by defining a polynomial-valued linear operator M on the space of all exponential polynomials, which is homogeneous with respect to the ring of polynomials and commutes with all translations. If f is an exponential polynomial, then its Fourier transform $\hat{f}$ is a polynomial-valued function on the set of all exponentials defined by $\hat{f}(m)=M(f\cdot\check{m})$, where m is an exponential and $\check{m}(x)=m(-x)$.

The fundamental properties of the map $f\to\hat{f}$ are investigated. This Fourier transform can be used to determine all exponential polynomial solutions of some functional equations, linear differential and difference equations with polynomial coefficients, some types of partial differential equations, etc.

Examples for the possible applications are also given.

Szigeti, F.: **New characterization of certain Sobolev spaces by generalized Riesz theorems**

The following Riesz' theorem characterizes the Sobolev space $W_p^1(a, b)$. An absolutely continuous function f: $[a, b]\to\mathbf{R}$ (or $\mathbf{C}$) has its derivative $f'\in L_p(a, b)$ if and only if, there exists a real number $K\geqq 0$, such that for any system $]a_i, b_i[\subset$

$\subset[a, b]$ $(i\in I)$ of nonoverlapping bounded subintervals the inequality

$$\sum_i \frac{|f(b_i)-f(a_i)|^p}{|b_i-a_i|^{p-1}} \leqq K$$

holds. This theorem can be generalized for a pair of measures. From this generalization we can characterize the spaces $W_p^k(\Omega)$ when k is integer, and $\Omega\in\mathbf{R}^n$ satisfies some regularity condition. This result is much more complicate then the original version of the Riesz' theorem. Thus, in this talk we show another characterization for the Sobolev space $W_p^s(\Omega)$ where $s>1+\frac{n-1}{p}$. A corollary of this theorem is a direct generalization of the original Riesz' theorem with interesting consequences in the study of the composition law of functions belonging to certain Sobolev spaces.

Reference

F. Szigeti, Necessary conditions for certain Sobolev spaces, *Acta Math. Hung.*

Szőkefalvi-Nagy, B.: **Geometric characterization of the set of positive functions in L^2**

Takaci, Á.: **On the distributional Stieltjes transform**

We prove Abelian theorems for the distributional Stieltjes transform, provided that the observed distribution has quasi-asymptotic behaviour at infinity related to a regularly varying function $\varrho(t)=t^\alpha\cdot L(t)$. Here $\alpha\in R$ and $L(t)$ is a slowly varying function. The cases $\alpha\notin\mathbf{Z}_-$ and $\alpha\in\mathbf{Z}_-$ are analysed separately.

Vladimirov, V. S. and Volovich, I. V.: **The Wiener—Hopf equation, the Riemann—Hilbert problem and orthogonal polynomials**

The discrete Wiener—Hopf equation with

$$a_{|k-j|}=\frac{1}{\pi}\int_0^\pi \cos[(k-j)\theta]f(\theta)\,d\theta, \quad f\geqq 0, \quad f\in\mathfrak{B}_1, \quad \ln f\in\mathfrak{B}_1$$

is reduced to a generalized Riemann—Hilbert problem for the unit circle which is solved in the Nevanlinna—Smirnov's algebra $\mathcal{N}_*$. The results are applied to some problems in statistical physics. Details are in *Theoretical and Mathematical Physics,* 1983, JAN.

Zayed, A. I.: **Generalized functions and the problem of uniqueness of the boundary values of analytic functions in the unit disk**

Let $\mathscr{D}$ be the unit disk, $\partial\mathscr{D}$ be its boundary; the unit circle and $\mathscr{H}$ be the class of all analytic functions in $\mathscr{D}$. Let H^1 denote the Hardy class of index 1 and N denote the Nevanlinna class of analytic functions in $\mathscr{D}$, i.e.,

$$H^1=\left\{f\colon f\in\mathscr{H},\ \lim_{r\to 1^-}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta<\infty\right\},$$

$$N=\left\{f\colon f\in\mathscr{H},\ \lim_{r\to 1^-}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta<\infty\right\}.$$

It is known that if $f\in N$, then $f(z)=O\left(\exp o\left(\frac{1}{1-|z|}\right)\right)$ as $|z|\to 1$. We define the class $\mathscr{H}(\alpha)$ $(0<\alpha<1)$ as the class of all analytic functions $f(z)\in\mathscr{H}$ such that $f(z)=O\left(\exp o((1-|z|)^{-\beta})\right)$ as $|z|\to 1$ where $\beta=\frac{\alpha}{1-\alpha}$. We have the inclusions

$$H^1\subsetneqq N\subsetneqq\mathscr{H}(\alpha)\subsetneqq\mathscr{H},\quad 1/2\leqq\alpha\leqq 1.$$

Two of the most important problems in the theory of the boundary values of analytic functions in the unit disk are the problems of existence and uniqueness of the boundary values. As for the existence problem, Fatou's theorem asserts that if $f\in H^1$, then the radial limit $\lim_{r\to 1^-} f(re^{i\theta})$ exists almost everywhere on $\partial\mathscr{D}$. This result was extended to the class N by Nevanlinna. Since there is a function $f\in\mathscr{H}$ such that $\lim_{r\to 1^-} f(re^{i\theta})$ exists almost *nowhere* on $\partial\mathscr{D}$, there is no hope of extending Fatou's theorem to the class $\mathscr{H}$. However, Köthe showed that if the radial limit is replaced by the limit in the sense of hyperfunctions, then a Fatou-type theorem exists for the class $\mathscr{H}$. As for the uniqueness problem, the uniqueness theorem of F. and M. Riesz asserts that if $f\in H^1$ and $\lim_{r\to 1^-} f(re^{i\theta})=0$ on a set of positive measure on $\partial\mathscr{D}$, then f is identically zero. Hence, if f and g are in H^1 and $\lim_{r\to 1^-} f(re^{i\theta})=\lim_{r\to 1^-} g(re^{i\theta})$ on a set of positive measure on $\partial\mathscr{D}$, then $f\equiv g$. This result was also extended to the class N by Nevanlinna. Again, since there exists an analytic function $f(z)\neq 0$ such that $\lim_{r\to 1^-} f(re^{i\theta})=0$ on a set of positive measure on $\partial\mathscr{D}$, there is no hope of extending the Riesz's uniqueness theorem to the class $\mathscr{H}$. The purpose of this talk is to show that a uniqueness-type theorem exists for the class $\mathscr{H}(\alpha)$, if the radial limit is replaced by the limit in the sense of Beurling distributions.

Zharinov, V. V.: **Holomorphic functions with Fourier hyperfunction boundary values**

Holomorphic functions in tubular domains over open cones are considered. It is supposed that these functions satisfy some estimates and have Fourier hyperfunction boundary values on the real part of the boundary. It is proved that such functions are in fact holomorphic in the convex hull of the original domain and satisfy there more exact estimates.

Zsidó, L.: **The invertibility of ultradifferential operators with constant coefficients**

Let ω be an entire function of the form

$$\omega(z)=\prod_{k=1}^{\infty}\left(1+\frac{iz}{t_k}\right),$$

where $t_1, t_2, \ldots\in(0, +\infty]$, $t_1<+\infty$ are such that $\sum_{k=1}^{\infty}\frac{1}{t_k}<+\infty$. Then one can consider the space $\mathscr{D}_\omega$ of all ω-ultradifferentiable functions on $\mathbf{R}$, which is inductive limit of Fréchet spaces ([3]). The elements of the dual space $\mathscr{D}'_\omega$ are the ω-ultradistributions on $\mathbf{R}$.

An ω-ultradifferential operator is a linear operator $T\colon \mathscr{D}_\omega\to\mathscr{D}_\omega$ with

$$\operatorname{supp}(T\varphi)\subset\operatorname{supp}\varphi,\quad \varphi\in\mathscr{D}_\omega.$$

T is called with constant coefficients if it commutes with each translation operator. Every ω-ultradifferential operator with constant coefficients is of the form

$$f(D)$$

where f is an entire function of exponential type o such that

$$|f(it)|\leqq c_0\,|\omega(t)|^{n_0},\quad t\in\mathbf{R}$$

for some integer $n_0\geqq 1$ and some $c_0>0$ ([3]). Here $f(D)$ is understood as pseudo-differential operator:

$$\left(\widehat{f(D)\varphi}\right)(t)=f(it)\,\hat{\varphi}(t),\quad \varphi\in\mathscr{D}_\omega,\quad t\in\mathbf{R},$$

where $\hat{\varphi}$ denotes the Fourier transform of φ. If

$$f(z)=\sum_{k=0}^{\infty}c_k z^k$$

is the Taylor expansion of f then the series

$$f(D)=\sum_{k=0}^{\infty}c_k D^k$$

converges if and only if

$$|f(z)|\leqq c\,|\omega(|z|)|^n,\quad z\in\mathbf{C}$$

for some integer $n\geqq 1$ and some $c>0$.

Let $f(D)$ be an ω-ultradifferential operator with constant coefficients. $f(D)$ can be extended to a continuous linear operator $\mathscr{D}'_\omega\to\mathscr{D}'_\omega$ and we have $f(D)\mathscr{D}'_\omega=\mathscr{D}'_\omega$ if and only if f is ω-slowly decreasing on the imaginary axis:

$$\sup_{\substack{s\in\mathbf{R}\\ |t-s|\leqq m_0\ln|\omega(t)|}}|f(is)|\geqq\varepsilon_0\,|\omega(t)|^{-m_0},\quad t\in\mathbf{R}$$

for appropriate $m_0\geqq 1$ and $\varepsilon_0>0$ ([2]). In this case we say that $f(D)$ is invertible in $\mathscr{D}'_\omega$.

Let us consider the "union" $\bigcup_{\omega} \mathcal{D}'_{\omega}$ of all ω-ultradistributions. We remark that $\bigcup_{\omega} \mathcal{D}'_{\omega}$ is equal with the "union" of all usual Beurling or Roumieu ultradistribution spaces as these are defined in [6]. If f is an entire function with

$$\int_1^{+\infty} \frac{1}{r^2} \ln^+ \big(\sup_{|z|=r} |f(x)|\big)\, dr < +\infty$$

then there is an ω_0 and a constant $c_0 > 0$ with

$$|f(z)| \leqq c_0 \big|\omega_0(|z|)\big|, \quad z \in \mathbf{C},$$

so $f(D)$ is a convergent ω_0-ultradifferential operator with constant coefficients. Of course, $f(D)$ is a linear operator on $\bigcup_{\omega} \mathcal{D}'_{\omega}$ which maps $\mathcal{D}'_{\omega_0}$ in $\mathcal{D}'_{\omega_0}$. In [1], Chap. II, § 2, No. 1 it is asked: is $f(D)$ automatically invertible at least in some $\mathcal{D}'_{\omega_1} \supset \mathcal{D}'_{\omega_0}$? Equivalently: is the linear operator $f(D)$ on $\bigcup_{\omega} \mathcal{D}'_{\omega}$ always surjective? Equivalently: is f ω_1-slowly decreasing on the imaginary axis for an appropriate ω_1?

In [4] it is shown that $f(D)$ is invertible in the union of all ultradistribution spaces whenever f satisfies the additional condition

$$(*) \qquad \int_1^{+\infty} \frac{1}{r^2} \ln^+ \big(\sup_{|z|\leqq r} |f(x)|\big) \ln \frac{r}{\ln^+ \big(\sup_{|z|\leqq r} |f(z)|\big)}\, dr < +\infty.$$

Now, in [5] it is proved that if $\alpha\colon [0, +\infty) \to [0, +\infty)$ is an increasing function with

$$\int_1^{+\infty} \frac{1}{r^2} \alpha(r)\, dr < +\infty \quad \text{but} \quad \int_1^{+\infty} \frac{1}{r^2} \alpha(r) \ln \frac{r}{\alpha(r)}\, dr = +\infty,$$

then there exists an entire function f with

$$|f(z)| \leqq e^{\alpha(|z|)}, \quad z \in \mathbf{C},$$

such that f is not ω-slowly decreasing on the imaginary axis for any ω.

Therefore:

1) The answer to Chou's question is negative.
2) Moreover, $(*)$ is the optimal majorization condition which guarantees the global invertibility of an ultradifferential operator with constant coefficients.

References

[1] Ch.-Ch. Chou, La transformation de Fourier complexe et l'équation de convolution, *Lecture Notes in Math. Vol. 235, Springer-Verlag,* 1973.

[2] I. Cioránescu, Convolution equations in ω-ultradistribution spaces, *Rev. Roum. Math. Pures et Appl.* **25** (1980), 719—737.

[3] I. Cioránescu and L. Zsidó, ω-Ultradistributions and their application to operator theory, *Banach Center Publications Vol. 8, Warsaw* 1982, 77—220.

[4] I. CIORĂNESCU and L. ZSIDÓ, A minimum modulus theorem and applications to ultradifferential operators, *Arkiv för mat.* **17** (1979), 153—166.
[5] W. K. HAYMAN and L. ZSIDÓ, *Paper in preparation.*
[6] H. KOMATSU, Ultradistributions I, *J. Fac. Sci. Univ. Tokyo, Sec. I A* **20** (1973), 25—105.

Problems

Boehmians and their extensions

By a delta sequence we mean a sequence of smooth functions δ_n satisfying the conditions

$$1^\circ \quad \delta_n(x) = 0 \quad \text{if} \quad |x| > \alpha_n \quad \text{with} \quad \alpha_n \to 0; \qquad 2^\circ \quad \int \delta_n = 1; \qquad 3^\circ \quad \int |\delta_n| < M.$$

If f_n are continuous functions for $n \in N$ and $\{\delta_n\}$ is a delta sequence, then the sequence of pairs (f_n, δ_n) is said to be fundamental if $f_n * \delta_m = f_m * \delta_n$ for $m, n \in N$. Two fundamental sequences (f_n, δ_n) and $(g_n, \tilde{\delta}_n)$ are equivalent if $f_m * \tilde{\delta}_n = g_n * \delta_m$ for $m, n \in N$. Boehmians are classes $[(f_n, \delta_n)]$ of equivalent fundamental sequences. Operations on Boehmians are performed, as on pairs.

A Boehmians x is zero on an open interval (A, B) if for each closed interval $[A_\nu, B_\nu] \subset (A, B)$ there exists a sequence (f_n, δ_n) such that $x = [f_n, \delta_n]$ and $f_n(x) = 0$ on $[A_1, B_1]$ for sufficiently large n.

Problem. Assume that x is a Boehmian and ε is a positive member. Is there a Boehmian y such that $y = 0$ on $(-\infty, -\varepsilon)$ and $x - y = 0$ on $(0, \infty)$?

The problem was posed by J. BURZYK and P. MIKUSIŃSKI.

Dimovski, I.:

Is each function f of the DUHAMEL convolution algebra $C[0, 1]$ or $C[0, \infty)$ with $f(0) = 0$ factorizable in this algebra, i.e., does there exist functions g and h of $C[0, 1]$ or $C[0, \infty)$ such that

$$f(x) = \int_0^x g(x-t)\,h(t)\,dt?$$

(Compiled by G. Fazekas, E. Gesztelyi and Á. Száz)