Some fixed point theorems for mappings in pseudocompact Tichonov space

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Abstract. Recently, FISHER [2] and PACHPATTE [3] have obtained some interesting results on fixed points in compact metric space. In the present paper, we have generalized some of their results over pseudocompact Tichonov space.

Introduction

A topological space X is said to be pseudocompact iff every real valued continuous function on X is bounded. It may be noted that every compact space is pseudocompact but the converse is not true. (ENGLE-KING [1], Example 5, page 150.) However, in a metric space the notions: "Compact" and "Pseudocompact" coincide. By Tichonov space we mean a completely regular Hausdorff space. It is observed that the product of two Tichonov spaces is again a Tichonov space, whereas the product of two pseudocompact spaces need not be pseudocompact.

Main results

Theorem 1. *Let P be a pseudocompact Tichonov space and μ be a non-negative real valued continuous function over $P\times P$ ($P\times P$ is Tichonov, but need not be pseudocompact) satisfying*

$$() \qquad \begin{cases} \mu(x, x) = 0 & \text{for all } x\in P \text{ and} \\ \mu(x, y) \leqq \mu(x, z)+\mu(y, z) & \text{for all } x, y, z\in P. \end{cases}$$

If T: $P\to P$ is a continuous map satisfying

$$\text{(ii)} \qquad [\mu(Tx, Ty)]^2 < \mu(x, Tx)\mu(y, Ty)+\alpha\mu(x, Ty)\mu(y, Tx)$$

for all distinct $x, y\in P$, where $\alpha\geqq 0$, Then T has a fixed point in P, which is unique whenever $\alpha\leqq 1$.

PROOF. We define φ: $P\to R$ by $\varphi(p)=\mu(Tp, p)$ for all $p\in P$, where R is the set of real numbers. Clearly φ is continuous being the composite of two continuous functions T and μ. Since P is pseudocompact Tichonov space, every real valued continuous function over P is bounded and attains its bounds. Thus there exists a point $v\in P$ such that $\varphi(v)=\inf\{\varphi(p)|p\in P\}$ where "inf" denotes the infimum or the greatest lower bound in R (note $\varphi(p)\in R$). We now affirm that v is a fixed point for T. If not, let us suppose that $Tv \neq v$. Then using (ii), we have

$$[\varphi(Tv)]^2 = [\mu(T^2v, Tv)]^2 < \\ < \mu(Tv, T^2v)\mu(v, Tv) + \alpha\mu(Tv, Tv)\mu(v, T^2v) = \\ = \mu(Tv, T^2v)\mu(v, Tv)$$

which implies

$$\varphi(Tv) < \varphi(v)$$

leading to a contradiction and hence $Tv = v$, i.e. $v \in P$ is a fixed point for T.

To prove the uniqueness of v, if possible, let $w \in P$ be another fixed point for T, i.e. $Tw = w$ and $w \neq v$. Then using (ii), we have

$$[\mu(v, w)]^2 = [\mu(Tv, Tw)]^2 < \\ < \mu(v, Tv)\mu(w, Tw) + \alpha\mu(v, Tw)\mu(w, Tv) = \\ = \alpha[\mu(v, w)]^2 \leqq [\mu(v, w)]^2 \quad (\because \alpha \leqq 1)$$

again leading to a contradiction which proves that $v \in P$ is unique. This completes the proof of the theorem.

Every metric space is a Hausdorff space. Hence an easy consequence of this theorem yields the following result due to Fisher [2].

Theorem A. *Let T be a continuous self-map of a compact metric space (X, d) satisfying*

$$[d(Tx, Ty)]^2 < d(x, Tx)d(y, Ty) + \alpha d(x, Ty)d(y, Tx)$$

for all distinct $x, y \in X$, where $\alpha \geqq 0$. Then T has a fixed point. If $\alpha \leqq 1$, then the fixed point is unique.

We next prove the following:

Theorem 2. *Let P and μ be the same as defined in Theorem 1. If $T: P \to P$ is a continuous map satisfying*

(iii) $$[\mu(Tx, Ty)]^2 < \alpha[\mu(x, Tx)\mu(y, Ty) + \mu(x, Ty)\mu(y, Tx)] + \\ + \beta[\mu(x, Tx)\mu(y, Tx) + \mu(x, Ty)\mu(y, Ty)]$$

for all distinct $x, y \in P$, where $\alpha, \beta \geqq 0$ and $0 < \alpha + 2\beta \leqq 1, \alpha < 1, \beta < 1$. Then T has a unique fixed point in P.

Proof. Let φ and v be as in the proof of Theorem 1. If $v \in P$ is not a fixed point of T, then applying (iii) we have

$$[\varphi(Tv)]^2 = [\mu(T^2v, Tv)]^2 < \\ < \alpha[\mu(Tv, T^2v)\mu(v, Tv) + \mu(Tv, Tv)\mu(v, T^2v)] + \\ + \beta[\mu(Tv, T^2v)\mu(v, T^2v) + \mu(Tv, Tv)\mu(v, Tv)] = \\ = \alpha\mu(Tv, T^2v)\mu(v, Tv) + \beta\mu(Tv, T^2v)\mu(v, T^2v) \leqq \\ \leqq \alpha\mu(Tv, T^2v)\mu(v, Tv) + \\ + \beta\mu(Tv, T^2v)[\mu(v, Tv) + \mu(Tv, T^2v)]$$

which implies

$$\varphi(Tv) < \frac{\alpha+\beta}{1-\beta}\varphi(v)$$

or,

$$\varphi(Tv) < \varphi(v) \quad (\because\ 0 < \alpha+2\beta \leqq 1)$$

leading to a contradiction and hence $Tv=v$, i.e. $v\in P$ is a fixed point for T.

To prove the uniqueness of v, if possible, let $w\in P$ be another fixed point for T, i.e. $Tw=w$ and $w\neq v$. Then using (iii), we obtain

$$\begin{aligned}[\mu(v,w)]^2 = [\mu(Tv,Tw)]^2 &< \\ &< \alpha[\mu(v,Tv)\mu(w,Tw)+\mu(v,Tw)\mu(w,Tv)]+ \\ &\quad +\beta[\mu(v,Tv)\mu(w,Tv)+\mu(v,Tw)\mu(w,Tw)] = \\ &= \alpha[\mu(v,w)]^2\end{aligned}$$

giving a contradiction since $\alpha\leqq 1$. This shows that $v\in P$ is unique, completing the proof of the theorem.

As a particular case of Theorem 2, we have the following result on a compact metric space due to PACHPATTE [3].

Theorem B. *If T is a continuous self-map of a compact metric space (X,d) satisfying*

$$\begin{aligned}[d(Tx,Ty)]^2 < \alpha[d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)]+ \\ +\beta[d(x,Tx)d(y,Tx)+d(x,Ty)d(y,Ty)]\end{aligned}$$

for all distinct $x, y\in X$, where $\alpha,\beta\geqq 0$ and $\alpha+2\beta=1$, $\alpha\leqq 1$, $\beta\leqq 1$, then T has a unique fixed point.

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