On P-Sasakian manifolds which admit certain tensor fields

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0. Introduction. G. P. POKHARIYAL and R. S. MISHRA ([5]) have introduced new tensor fields named as W_2 — and E-tensor fields in a Riemannian manifold and studied their properties. Next, G. P. POKHARIYAL has studied some properties of these tensor fields in a Sasakian manifold ([6]).

On the other hand, I. SATO introduced the notion of an almost paracontact, a P-Sasakian and an SP-Sasakian manifold and he had a lot of very interesting results about such manifolds ([2], [3]).

In this paper, we shall study P-Sasakian manifolds which admit the W_2 — or the E-tensor field that satisfy certain conditions. In § *1*, we recall the notion of a P-Sasakian and an SP-Sasakian manifold and the essential properties of these manifolds. In § *2*, we shall get some formulas of the W_2- and the E-tensor fields in a P-Sasakian manifold. In § *3*, we shall show that a W_2-symmetric P-Sasakian manifold is of constant curvature and an E-symmetric P-Sasakian manifold is an SP-Sasakian manifold of constant curvature or an η-Einstein P-Sasakian manifold with the harmonic structure vector field. In the last § *4*, we shall prove that an E-recurrent P-Sasakian manifold is necessarily of constant curvature.

In this paper, we always assume that manifolds and tensor fields are differentiable.

1. Preliminaries. An n-dimensional P-Sasakian manifold M^n is a Riemannian manifold with positive definite Riemannian metric g which admits a unit 1-form η satisfying

$$\text{(1.1)} \quad \begin{aligned} &\text{a)}\quad \nabla_\mu \eta_\lambda - \nabla_\lambda \eta_\mu = 0, \\ &\text{b)}\quad \nabla_\nu \nabla_\mu \eta_\lambda = -g_{\nu\mu}\eta_\lambda - g_{\nu\lambda}\eta_\mu + 2\eta_\nu \eta_\mu \eta_\lambda, \end{aligned}$$

where ∇_λ denotes the covariant differentiation with respect to g and the indices $\nu, \mu, \ldots, \lambda$ run over the range $1, 2, \ldots, n$. If we put

$$\text{(1.2)} \quad \xi^\lambda = g^{\lambda\varepsilon}\eta_\varepsilon, \quad \varphi_\mu{}^\lambda = \nabla_\mu \xi^\lambda,$$

then we have

$$\text{(1.3)} \quad \begin{aligned} &\eta_\varepsilon \xi^\varepsilon = 1, \quad \varphi_\varepsilon{}^\lambda \xi^\varepsilon = 0, \quad \eta_\varepsilon \varphi_\mu{}^\varepsilon = 0, \\ &\varphi_{\mu\lambda} = \varphi_{\lambda\mu} \, (\varphi_{\mu\lambda} = \varphi_\mu{}^\varepsilon g_{\varepsilon\lambda}), \quad \operatorname{rank}(\varphi_\mu{}^\lambda) = n-1, \\ &\varphi_\mu{}^\varepsilon \varphi_\varepsilon{}^\lambda = \delta_\mu{}^\lambda - \eta_\mu \xi^\lambda, \quad g_{\gamma\varepsilon}\varphi_\mu{}^\gamma \varphi_\lambda{}^\varepsilon = g_{\mu\lambda} - \eta_\mu \eta_\lambda. \end{aligned}$$

In an n-dimensional P-Sasakian manifold M^n, the following identities are established ([4]);

$$(1.4)\qquad R_{\omega\nu\mu}{}^{\varepsilon}\eta_{\varepsilon} = g_{\omega\mu}\eta_{\nu} - g_{\nu\mu}\eta_{\omega}, \quad R_{\mu\varepsilon}\xi^{\varepsilon} = -(n-1)\eta_{\mu},$$

$$(1.5)\qquad \begin{aligned} &R_{\mu}{}^{\varepsilon}\varphi_{\varepsilon\lambda} - R_{\mu\varepsilon\gamma\lambda}\varphi^{\varepsilon\gamma} = (n-2)\varphi_{\mu\lambda} - \varphi g_{\mu\lambda} + 2\varphi\eta_{\mu}\eta_{\lambda},\\ &R_{\mu}{}^{\varepsilon}\varphi_{\varepsilon\lambda} = R_{\lambda}{}^{\varepsilon}\varphi_{\varepsilon\mu},\end{aligned}$$

$$(1.6)\qquad \begin{aligned} R_{\omega\nu\varepsilon\gamma}\varphi_{\mu}{}^{\varepsilon}\varphi_{\lambda}{}^{\gamma} = {}& R_{\omega\nu\mu\lambda} + g_{\nu\lambda}g_{\omega\mu} - g_{\omega\lambda}g_{\nu\mu} + \varphi_{\omega\lambda}\varphi_{\nu\mu}\\ &- \varphi_{\nu\lambda}\varphi_{\omega\mu} + 2(g_{\nu\mu}\eta_{\omega}\eta_{\lambda} - g_{\omega\mu}\eta_{\mu}\eta_{\lambda} + g_{\omega\lambda}\eta_{\nu}\eta_{\mu} - g_{\nu\lambda}\eta_{\omega}\eta_{\mu}),\end{aligned}$$

where $R_{\omega\nu\mu\lambda}$ and $R_{\mu\lambda}$ denote the Riemannian curvature tensor and the Ricci tensor with respect to g, respectively, and $\varphi = \text{trace}\,(\varphi_{\mu}{}^{\lambda})$.

In a P-Sasakian manifold M^n, if the unit 1-form η satisfies the condition

$$(1.7)\qquad \nabla_{\mu}\eta_{\lambda} = \varepsilon(-g_{\mu\lambda} + \eta_{\mu}\eta_{\lambda}), \quad (\varepsilon = \pm 1),$$

then the manifold M^n is called an SP-Sasakian manifold ([1]). From (1.7), we can easily see that an SP-Sasakian manifold is characterized by $\varphi^2 = (n-1)^2$ ([1]).

Remark. At first, I. Sato ([2]) defined an SP-Sasakian manifold as a P-Sasakian manifold satisfying $\nabla_{\mu}\eta_{\lambda} = -g_{\mu\lambda} + \eta_{\mu}\eta_{\lambda}$.

A P-Sasakian manifold is called an η-Einstein manifold if the Ricci tensor $R_{\mu\lambda}$ is written by the form

$$(1.8)\qquad R_{\mu\lambda} = a g_{\mu\lambda} + b\eta_{\mu}\eta_{\lambda}$$

for certain scalar fields a and b. The scalar fields a and b are related by $a+b = -(n-1)$ and $na+b = R$, where R denotes the scalar curvature respect to g. The following proposition was proved in [4];

Proposition 1.1. *An n-dimensional $(n \neq 3)$ non-Einstein η-Einstein P-Sasakian manifold has a constant scalar curvature R if and only if the vector field ξ is harmonic.*

2. *W_2- and E-tensor fields in a P-Sasakian manifold.* Let M^n be an n-dimensional P-Sasakian manifold. We define tensor fields W_2 and E in M^n as follows;

$$(2.1)\qquad W_{2\,\omega\nu\mu\lambda} = R_{\omega\nu\mu\lambda} + \frac{1}{n-1}(g_{\omega\mu}R_{\nu\lambda} - g_{\nu\mu}R_{\omega\lambda}),$$

and

$$(2.2)\qquad \begin{aligned} E_{\omega\nu\mu\lambda} = \frac{1}{2}(W_{2\,\omega\nu\mu\lambda} - W_{2\,\omega\nu\lambda\mu}) = {}& R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)}(g_{\omega\mu}R_{\nu\lambda} -\\ &- g_{\omega\lambda}R_{\nu\mu} + g_{\nu\lambda}R_{\omega\mu} - g_{\nu\mu}R_{\omega\lambda}).\end{aligned}$$

In this section, we state the properties of W_2- and E-tensor fields in M^n. The proof of the following two propositions are trivial. So, we omit their proof.

Proposition 2.1. *In an n-dimensional P-Sasakian manifold, we have*

$$(2.3)\qquad \begin{cases} \text{a)}\ \ W_{2_{\omega\nu\mu\varepsilon}}\xi^{\varepsilon}=0, \quad \text{b)}\ \ W_{2_{\varepsilon\nu\mu\lambda}}\xi^{\varepsilon}=\left((g_{\nu\lambda}+\frac{1}{n-1}R_{\nu\lambda}\right)\eta_{\mu} \\ \text{c)}\ \ W_{2_{\tau\varkappa\gamma\varepsilon}}\varphi_{\omega}{}^{\tau}\varphi_{\nu}{}^{\varkappa}\varphi_{\mu}{}^{\gamma}\varphi_{\lambda}{}^{\varepsilon}=W_{2_{\omega\nu\mu\lambda}}+\left\{\left(g_{\omega\lambda}+\frac{1}{n-1}R_{\omega\lambda}\right)\eta_{\nu}-\right. \\ \qquad \left.-\left(g_{\nu\lambda}+\frac{1}{n-1}R_{\nu\lambda}\right)\eta_{\omega}\right\}\eta_{\mu}=W_{2_{\omega\nu\mu\lambda}}+(W_{2_{\varepsilon\omega\nu\lambda}}-W_{2_{\varepsilon\nu\omega\lambda}})\,\xi^{\varepsilon}\eta_{\mu}. \end{cases}$$

Proposition 2.2. *In an n-dimensional P-Sasakian manifold M^n, the E-tensor field satisfies*

$$(2.4)\qquad \begin{cases} \text{a)}\ \ E_{\omega\nu\mu\varepsilon}\xi^{\varepsilon}=\frac{1}{2}(g_{\omega\mu}\eta_{\nu}-g_{\nu\mu}\eta_{\omega})+\frac{1}{2(n-1)}(R_{\omega\mu}\eta_{\nu}-R_{\nu\mu}\eta_{\omega}), \\ \text{b)}\ \ E_{\varepsilon\nu\mu\lambda}\xi^{\varepsilon}=\frac{1}{2}(g_{\nu\lambda}\eta_{\mu}-g_{\nu\mu}\eta_{\lambda})+\frac{1}{2(n-1)}(R_{\nu\lambda}\eta_{\mu}-R_{\nu\mu}\eta_{\lambda}), \\ \text{c)}\ \ E_{\varepsilon\nu\mu\gamma}\xi^{\varepsilon}\xi^{\gamma}=-\frac{1}{2}\left(g_{\nu\mu}+\frac{1}{n-1}R_{\nu\mu}\right). \end{cases}$$

3. Symmetric P-Sasakian manifolds.

Definition 3.1. An n-dimensional P-Sasakian manifold M^n is called W_2-*symmetric* if it satisfies

$$(3.1)\qquad \nabla_{\varkappa}W_{2_{\omega\nu\mu\lambda}}=0,$$

and an n-dimensional P-Sasakian manifold is called *E-symmetric* if it satisfies

$$(3.2)\qquad \nabla_{\varkappa}E_{\omega\nu\mu\lambda}=0.$$

Now, let M^n be an n-dimensional W_2-symmetric P-Sasakian manifold. Then we have from (3.1) and the second Bianchi identity ([7])

$$(3.3)\qquad R_{\tau\varkappa\omega}{}^{\varepsilon}W_{2_{\varepsilon\nu\mu\lambda}}+R_{\tau\varkappa\nu}{}^{\varepsilon}W_{2_{\omega\varepsilon\mu\lambda}}+R_{\tau\varkappa\mu}{}^{\varepsilon}W_{2_{\omega\nu\varepsilon\lambda}}+R_{\tau\varkappa\lambda}{}^{\varepsilon}W_{2_{\omega\nu\mu\varepsilon}}=0.$$

Transvecting (3.3) with ξ^{λ} and taking account of (1.4) and (2.3), we have

$$(3.4)\qquad \eta_{\varkappa}W_{2_{\omega\nu\mu\tau}}-\eta_{\tau}W_{2_{\omega\nu\mu\varkappa}}=0,$$

from this, we obtain

$$(3.5)\qquad W_{2_{\omega\nu\mu\lambda}}=0,$$

that is, the manifold M^n is W_2-flat. (3.5) implies

$$(3.6)\qquad R_{\omega\nu\mu\lambda}=-\frac{1}{n-1}(g_{\omega\mu}R_{\nu\lambda}-g_{\nu\mu}R_{\omega\lambda}).$$

Transvecting (3.6) with $g^{\omega\lambda}$, we get

$$(3.7)\qquad R_{\nu\mu}=\frac{R}{n}g_{\nu\mu}.$$

By virtue of (1.4) and (3.7), we obtain $R=-n(n-1)$. Thus (3.6) and (3.7) imply

$$R_{\omega\nu\mu\lambda}=-(g_{\omega\lambda}g_{\nu\mu}-g_{\omega\mu}g_{\nu\lambda}). \tag{3.8}$$

Thus we have

Theorem 3.1. *An n-dimensional W_2-symmetric P-Sasakian manifold is of constant curvature* -1.

Next, let M^n be an n-dimensional E-symmetric P-Sasakian manifold. Then we have from (3.2) and the second Bianchi identity

$$R_{\tau\kappa\omega}{}^{\varepsilon}E_{\varepsilon\nu\mu\lambda}+R_{\tau\kappa\nu}{}^{\varepsilon}E_{\omega\varepsilon\mu\lambda}+R_{\tau\kappa\mu}{}^{\varepsilon}E_{\omega\nu\varepsilon\lambda}+R_{\tau\kappa\lambda}{}^{\varepsilon}E_{\omega\nu\mu\varepsilon}=0. \tag{3.9}$$

Transvecting (3.9) with $\xi^\kappa\xi^\lambda$ and taking account of (1.2), (1.3), (1.4) and (2.4), we get

$$\begin{aligned}R_{\omega\nu\mu\lambda}= &-\frac{1}{2}(g_{\omega\lambda}g_{\nu\mu}-g_{\omega\mu}g_{\nu\lambda}+g_{\omega\lambda}\eta_\nu\eta_\mu-g_{\nu\lambda}\eta_\omega\eta_\mu)+\\ &+\frac{1}{2(n-1)}(g_{\nu\mu}R_{\omega\lambda}-g_{\omega\mu}R_{\nu\lambda}-\eta_\nu\eta_\mu R_{\omega\lambda}+\eta_\omega\eta_\mu R_{\nu\lambda}).\end{aligned} \tag{3.10}$$

The equation (3.10) implies

$$R_{\mu\lambda}=\frac{R-(n-1)^2}{2n-1}g_{\mu\lambda}-\frac{R+n(n-1)}{2n-1}\eta_\mu\eta_\lambda \tag{3.11}$$

that is, the manifold M^n is of η-Einstein. Substituting (3.11) into (3.10), the curvature tensor $R_{\omega\nu\mu\lambda}$ is written by

$$\begin{aligned}R_{\omega\nu\mu\lambda}= &\frac{R-(n-1)(3n-2)}{2(n-1)(2n-1)}(g_{\omega\lambda}g_{\nu\mu}-g_{\omega\mu}g_{\nu\lambda})-\\ &-\frac{R+n(n-1)}{2(n-1)(2n-1)}(g_{\omega\lambda}\eta_\nu\eta_\mu-g_{\nu\lambda}\eta_\omega\eta_\mu+g_{\nu\mu}\eta_\omega\eta_\lambda-g_{\omega\mu}\eta_\nu\eta_\lambda).\end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) into (1.5) and by the straightforward calculation, we have

$$(R-7n^2+11n-4)\varphi(\eta_\omega\eta_\lambda-g_{\omega\lambda})+(2n-1)\{R-(n-1)(3n-4)\}\varphi_{\omega\lambda}=0. \tag{3.13}$$

Thus it is sufficient to consider the following two cases;

(i) $\varphi\neq 0$. Then, transvecting (3.13) with $g^{\omega\lambda}$, we have

$$-(n-1)(R-7n^2+11n-4)+(2n-1)\{R-(n-1)(3n-4)\}=0 \tag{3.14}$$

By virtue of (3.13) and (3.14), we get

$$(n-1)\varphi_{\omega\lambda}+\varphi(\eta_\omega\eta_\lambda-g_{\omega\lambda})=0,$$

that is,

$$\varphi_{\omega\lambda}=\varepsilon(-g_{\omega\lambda}+\eta_\omega\eta_\lambda). \tag{3.15}$$

Moreover, the equation (3.14) gives us

$$R = -n(n-1). \tag{3.16}$$

Thus we have from (3.12) and (3.16)

$$R_{\omega\nu\mu\lambda} = -(g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}),$$

that is, the manifold M^n is of constant curvature -1.

(ii) $\varphi=0$. Then we have from (3.13)

$$R = (n-1)(3n-4). \tag{3.17}$$

Substituting (3.17) into (3.12), the curvature tensor $R_{\omega\nu\mu\lambda}$ is written by

$$R_{\omega\nu\mu\lambda} = \frac{-1}{2n-1}(g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}) - \tag{3.18}$$

$$-\frac{2(n-1)}{2n-1}(g_{\omega\lambda}\eta_\nu\eta_\mu - g_{\nu\lambda}\eta_\omega\eta_\mu + g_{\nu\mu}\eta_\omega\eta_\lambda - g_{\omega\mu}\eta_\nu\eta_\lambda).$$

Thus we have

Theorem 3.2. *For an n-dimensional E-symmetric P-Sasakian manifold M^n, we have the following two cases;*

(i) *If $\varphi\neq 0$, then the manifold M^n is SP-Sasakian with constant curvature -1, and*

(ii) *If $\varphi=0$, then the vector field ξ^λ is harmonic, the manifold M^n is η-Einstein and the curvature tensor $R_{\omega\nu\mu\lambda}$ is given by* (3.18).

4. Recurrent P-Sasakian manifolds.

Definition 4.1. An n-dimensional P-Sasakian manifold M^n is called *W_2-recurrent* if it satisfies

$$\nabla_\varkappa W_{2_{\omega\nu\mu\lambda}} = \theta_\varkappa W_{2_{\omega\nu\mu\lambda}} \tag{4.1}$$

for certain non-zero vector field θ_λ, and an n-dimensional P-Sasakian manifold M^n is called *E-recurrent* if it satisfies

$$\nabla_\varkappa E_{\omega\nu\mu\lambda} = \theta_\varkappa E_{\omega\nu\mu\lambda} \tag{4.2}$$

for a certain non-zero vector field θ_k.

It is trivial that a W_2-recurrent P-Sasakian manifold is an E-recurrent one.

Let M^n be an n-dimensional E-recurrent P-Sasakian manifold. Then we have (4.2) and this equation means

$$\nabla_\varkappa R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)}(g_{\omega\mu}\nabla_\varkappa R_{\nu\lambda} - g_{\omega\lambda}\nabla_\varkappa R_{\nu\mu} + g_{\nu\lambda}\nabla_\varkappa R_{\omega\mu} - g_{\nu\mu}\nabla_\varkappa R_{\omega\lambda}) =$$

$$= \theta_\varkappa\{R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)}(g_{\omega\mu} R_{\nu\lambda} - g_{\omega\lambda} R_{\nu\mu} + g_{\nu\lambda} R_{\omega\mu} - g_{\nu\mu} R_{\omega\lambda})\}. \tag{4.3}$$

Transvecting (4.3) with $g^{\omega\lambda}$, we get

$$n\nabla_\varkappa R_{\nu\mu} = n\theta_\varkappa R_{\nu\mu} - R\theta_\varkappa g_{\nu\mu} + g_{\nu\mu}\nabla_\varkappa R. \tag{4.4}$$

Moreover, transvecting (4.4) with $g^{\nu\mu}$, we obtain

$$\nabla_\varkappa R = R\theta_\varkappa + n(n-1)\theta_\varkappa. \tag{4.5}$$

By virtue of (4.3), (4.4) and (4.5), we have

$$\nabla_\varkappa R_{\omega\nu\mu\lambda} = \theta_\varkappa (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}). \tag{4.6}$$

Covariant differentiation of (4.6) gives

$$\begin{aligned}\nabla_\tau\nabla_\varkappa R_{\omega\nu\mu\lambda} &= \nabla_\tau\theta_\varkappa (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}) + \\ &+ \theta_\tau\theta_\varkappa (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}).\end{aligned} \tag{4.7}$$

From this, we obtain

$$\begin{aligned} R_{\tau\varkappa\omega}{}^{\varepsilon} R_{\varepsilon\nu\mu\lambda} + R_{\tau\varkappa\nu}{}^{\varepsilon} R_{\omega\varepsilon\mu\lambda} + R_{\tau\varkappa\mu}{}^{\varepsilon} R_{\omega\nu\varepsilon\lambda} + R_{\tau\varkappa\lambda}{}^{\varepsilon} R_{\omega\nu\mu\varepsilon} &= \\ = (\nabla_\tau\theta_\varkappa - \nabla_\varkappa\theta_\tau)(R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}). \end{aligned}$$

Transvection of the above equation with $\xi^\varkappa \xi^\lambda$ gives that the manifold M^n is of constant curvature -1. Thus we have

Theorem 4.1. *An E-recurrent P-Sasakian manifold is of constant curvature* -1.

Corollary 4.2. *A* W_2*-recurrent P-Sasakian manifold is of constant curvature* -1.

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