A note on rings in which every finitely generated left ideal is quasi-projective

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ABSTRACT

We present an alternative proof of the following theorem by taking the ring to be one-sided perfect: If R is a local left or right perfect ring, then R is a left (qp)-ring if and only if either the square of its Jacobson radical is zero or R is a principal left ideal ring with d.c.c. We then derive that a local one-sided perfect ring with each of its finitely generated left ideals quasi-projective is a left (qp)-ring. A commutative local ring R with no non-zero nilpotent elements in which all finitely generated left ideals are quasi-projective has been shown to be a valuation domain.

1. Introduction. All rings considered in this paper are associative and have unity $1 \neq 0$. Further, each module is a unital left module. JAIN and SINGH [4] have called a ring R a perfect ring if it is both left and right perfect. As defined by them in [4], a ring R is said to be a left (qp)-ring if each of its left ideals is quasi-projective; they studied perfect left (qp)-rings and proved some results under the assumption that all rings considered in [4] are right as well as left perfect. Here an alternative proof of [4, Theorem 5] has been given by taking the ring to be only one-sided perfect. Let R be a local ring and let $J(R)$ be its Jacobson radical. The following result is proved concerning R: Let R be a right or left perfect ring. Then R is a left (qp)-ring if and only if either $J(R)^2=0$ or R is a principal left ideal ring with d.c.c. (Theorem 1). Singh and Mohammad [6] studied local rings and semi-perfect rings in which all finitely generated left ideals are quasi-projective. If R is a left (right) (qp)-ring, then clearly every finitely generated left (right) ideal of R is quasi-projective. It is proved in this note that if R is a local one-sided perfect ring with all of its finitely generated left ideals quasi-projective then R is a left (qp)-ring (Theorem 2). After this, commutative local rings in which every finitely generated ideal is quasi-projective are considered and their properties are discussed. In this connection, the following result is significant: A commutative local ring R with no non-zero nilpotent elements in which every finitely generated ideal is quasi-projective is a valuation domain. Finally, at the end of this paper, an example of a ring R which has every finitely generated left ideal quasi-projective but which is not semihereditary is given. For any ring R, $J(R)$ and $B(R)$ will always denote the Jacobson radical and prime radical respectively. For every subset X of a ring R, $l(X)$ $(r(X))$ will denote its left (right) annihilator in R.

2. Local rings. The following lemma is due to SINGH and MOHAMMAD [6, Lemma 6].

Lemma A. (i) *In a left or right perfect ring R,* $J(R)^2 \neq J(R)$ *whenever* $J(R) \neq 0$. (ii) *Any left valuation ring R with* $J(R)$ *nil, is a principal left ideal ring with d.c.c. if and only if* $J(R)^2 \neq J(R)$ *whenever* $J(R) \neq 0$.

Singh and Mohammad proved the following [6, Theorem 3].

Theorem A. *Let R be a local ring with* $J(R)$ *nil. Then every finitely generated left ideal of R is quasi-projective if and only if* (i) $J(R)^2=0$, *or* (ii) *R is a left valuation ring.*

As defined by Jain and Singh [4] a ring R is said to be a left (qp)-ring if every left ideal of R is quasi-projective as a left R-module. They proved the following [4, Theorem 5].

Theorem B. *Let R be a local perfect ring. Then R is a left* (*qp*)*-ring if and only if* (i) $J(R)^2=0$, *or* (ii) *R is a principal left ideal ring with d.c.c.*

We give an alternative proof of this theorem. We generalize its statement slightly.

Theorem 1. *Let R be a local left or right perfect ring. Then R is a left* (*qp*)*-ring if and only if* (i) $J(R)^2=0$, *or* (ii) *R is a principal left ideal ring with d.c.c.*

Proof. *Necessity*. To avoid the trivial case, we take $J(R) \neq 0$. By Lemma A(i), $J(R)^2 \neq J(R)$. Let R be a left (qp)-ring. Trivially then every finitely generated left ideal of R is quasi-projective. As R is a left or right perfect ring, it follows from Bass [2, Theorem P] that $J(R)$ is left T-nilpotent and thus $J(R)$ is a nil ideal. Hence by Theorem A, either $J(R)^2=0$, or R is a left valuation ring. In the latter case, according to Lemma A(ii), R is a principal left ideal ring with d.c.c. This completes the necessity.

Sufficiency. If R satisfies (i), $J(R)$ is a left vector space over the division ring $R/J(R)$. It follows that $J(R)$ is a completely reducible left $R/J(R)$-module, that is, $J(R)$ is a completely reducible left R-module. Let A be any left ideal of R. Since $J(R)$ is the unique maximal left ideal of R, $A \subset J(R)$ and A is a submodule of $J(R)$. Thus A is a completely reducible left R-module. By Miyashita [5, Remark, page 92], A is a quasi-projective left R-module. Consequently R is a left (qp)-ring. Now, let R satisfy (ii). Then R is also a left Noetherian ring. Since R satisfies a.c.c. on principal left ideals, we can find a principal left ideal Ra which is maximal among all principal left ideals. So we have $J(R)=Ra$ and $a \in J(R)$. We find that $a^n=0$ for some integer $n \geqq 1$, as $J(R)$ is nil. It is easy to show that every proper left ideal of R is of the form Ra^m where m is an integer such that $1 \leqq m \leqq n-1$. If A and B are any two left ideals of R, then $A=Ra^s$ and $B=Ra$ for some integers s and t^t such that $1 \leqq s \leqq n-1$, $1 \leqq t \leqq n-1$. Hence either $A \subset B$ or $B \subset A$, in that case R is a left valuation ring. By Theorem A, every finitely generated left ideal of R is quasi-projective. However, every left ideal of R is finitely generated. Therefore, every left ideal of R is quasi-projective, and R is then a left (qp)-ring. This completes the proof of the theorem.

From Theorem 1 we derive the following theorem.

Theorem 2. *Let R be a local left or right perfect ring, and further, let R have all its finitely generated left ideals quasi-projective. Then R is a left* (*qp*)*-ring.*

PROOF. To avoid the trivial case $J(R)=0$, let us take $J(R)\neq 0$. By Lemma A(i), $J(R)^2\neq J(R)$. Since R is a left or right perfect ring, $J(R)$ is a nil ideal. By Theorem A, either $J(R)^2=0$ or R is a left valuation ring. In the former case, by Theorem 1, R is a left (qp)-ring. In the latter case, by Lemma A(ii), R is a principal left ideal ring with d.c.c. Theorem 1 yields that R is a left (qp)-ring.

The following two lemmas have been stated in [6] for semi-perfect rings. For completeness and convenient references we give their proofs for local rings, as Lemma 2 is crucial to the proofs of theorems in the next section.

Lemma 1. *Let R be a local ring in which every finitely generated left ideal is quasi-projective. Then any indecomposable finitely generated left ideal of R is cyclic.*

PROOF. Let A be any indecomposable finitely generated left ideal of R. According to our hypothesis, A is *a quasi-projective left R-module*. Since R is a local ring, it follows that R is a semi-perfect ring and 1 is the only indecomposable idempotent of R. By WU and JANS [7, Theorem 3.1], $A\cong Re/Ie$ for some primitive idempotent e of R and for some two-sided ideal I of R. But $e=1$, we find that $A\cong R/I$ and thus A is a cyclic left R-module.

Lemma 2. *Let R be a local ring in which every finitely generated left ideal is quasi-projective. Then, given any two indecomposable finitely generated left ideals A and B of R, either A and B are comparable or $A\cap B=0$.*

PROOF. Let us suppose that A and B are non-comparable. Then $A\not\subset B$ and $B\not\subset A$. We prove that $A\cap B=0$. By Lemma 1, A and B are cyclic. We can write $A=Ra$ and $B=Rb$ for some elements a and b in R, so that $A+B$ is a finitely generated left ideal of R. Since R is a semi-perfect ring, any finitely generated left R-module has a projective cover. It then follows from MIYASHITA [5, Theorem 3.3] that $A+B$ is a perfect left R-module. There exists a left subideal B_0 of B that is minimal with respect to the property that $A+B_0=A+B$, that is, B_0 is a d-complement of A in $A+B$. By the definition of the perfect left R-module in the sense of MIYASHITA [5], there exists a left subideal A_0 of A that is minimal with respect to the property that $A_0+B_0=A+B$, that is, A_0 is a d-complement of B_0 in $A+B$. Notice that if $A_0=0$ then $A\subset B$. Hence $A_0\neq 0$. Similarly $B_0\neq 0$. Now A_0 and B_0 are d-complements of each other in $A+B$. By our hypothesis, $A+B$ is a quasi-projective left R-module. By MIYASHITA [5, Theorem 2.3], $A+B=A_0\oplus B_0$. Then $A\subset A_0\oplus B_0$ and $A=A\cap(A_0\oplus B_0)=A_0\oplus(A\cap B_0)$. So A_0 is a direct summand of A. But A is indecomposable, we get $A=A_0$ as $A_0\neq 0$. Similarly $B=B_0$. Hence $A\cap B=0$. This completes the proof of Lemma 2.

3. Commutative local rings. In this section we study commutative local rings in which every finitely generated ideal is quasi-projective, and discuss their properties. In this direction, we first prove a lemma.

Lemma 3. *Let R be a commutative local ring in which every finitely generated ideal is quasi-projective. If for some non-zero elements a, b in R, $Ra\cap Rb=0$, then $a\in B(R)$ and $b\in B(R)$; further $a^2=0$ and $b^2=0$.*

PROOF. It follows from SINGH and MOHAMMAD [6, Lemma 3] that $l(a)=l(b)$. We have $ab=ba\in Ra\cap Rb$. Hence $a\in l(a)$ and $b\in l(b)$; further a and b are nilpotent elements of R. Consequently $a\in B(R)$ and $b\in B(R)$, as R is commutative.

Theorem 3. *Let R be a commutative local ring in which every finitely generated ideal is quasi-projective. Then* (i) *R is a valuation ring, or* (ii) *for all non-zero elements x in $B(R)$, Rx is a uniform R-module.*

PROOF. Suppose R is not a valuation ring. By SINGH and MOHAMMAD [6, Theorem 2], we get $B(R)^2=0$. Consider any nonzero element x in $B(R)$. If Rx is not uniform, we can find two non-zero cyclic submodules Ra and Rb of Rx such that $Ra\cap Rb=0$. Since $\sigma\colon R\to Ra$, $\eta\colon R\to Rb$ defined by $\sigma(r)=ra$, $\eta(r)=rb$, $r\in R$, are projective covers of Ra and Rb respectively and R is indecomposable, we get Ra and Rb are indecomposable. By Lemma 2, Ra and Rb are not comparable. Clearly $a\in Rx$ and $b\in Rx$. There exist non-zero elements a_1 and b_1 in R such that $a=a_1x$ and $b=b_1x$. We note that Ra_1 and Rb_1 are also not comparable. Again, by using Lemma 2, we have $Ra_1\cap Rb_1=0$. It follows from Lemma 3 that $a_1\in B(R)$ and $b_1\in B(R)$. Then $a_1x\in B(R)^2$ and $b_1x\in B(R)^2$. We then have $a=0$ and $b=0$. This is a contradiction. Consequently, either R is a valuation ring or Rx is a uniform R-module for each non-zero element x in $B(R)$. This completes the proof of the theorem.

Theorem 4. *Let R be a commutative local ring in which every finitely generated ideal is quasi-projective. Then:*

(i) *Any ideal A of R, that is not contained in $B(R)$, contains $B(R)$;*

(ii) *The family of all those ideals of R that do not contain $B(R)$ is totally ordered under inclusion;*

(iii) *$B(R)$ is a prime ideal of R; and*

(iv) *For any non-zero element x in $J(R)-B(R)$, $B(R)=B(R)x$.*

PROOF. First of all we show that for given any $x, y\in R-B(R)$, the principal ideals Rx and Ry are comparable and $B(R)\subset Rx$. If Rx and Ry are not comparable, by Lemma 2, $Rx\cap Ry=0$. By Lemma 3, $x\in B(R)$ and $y\in B(R)$, which is a contradiction. This proves that Rx and Ry are comparable. Now, let us take any non-zero element z in $B(R)$. Clearly $Rx\not\subset Rz$, as $x\notin B(R)$. If $Rx\cap Rz=0$ then, again using Lemma 3, we get $x\in B(R)$. Hence $Rx\cap Rz\neq 0$, and so Rx and Rz are comparable. Then $Rz\subset Rx$ and $z\in Rx$. This proves that $B(R)\subset Rx$. This all shows that any ideal of R, that is not contained in $B(R)$, contains $B(R)$; and all such ideals are totally ordered under inclusion. In particular, the family of all prime ideals of R is totally ordered under inclusion. Hence $B(R)$, being the intersection of this totally ordered family of prime ideals, is a prime ideal. Thus (i), (ii), and (iii) are proved. For (iv), let us consider any non-zero element x in $J(R)-B(R)$. Then $x\in J(R)$ and $x\notin B(R)$. As proved above, $B(R)\subset Rx$. To show that $B(R)\subset B(R)x$, let z be any non-zero element in $B(R)$. Then $z\in Rx$ and $z=z_1x$ for some $z_1\in R$. This gives us that $z_1x\in B(R)$. Hence, as $B(R)$ is a prime ideal and $x\notin B(R)$, we get $z_1\in B(R)$. Then $z\in B(R)x$. This yields that $B(R)\subset B(R)x$. But $B(R)x\subset B(R)$. Hence $B(R)=B(R)x$.

Corollary. *Let R be a commutative local ring in which every finitely generated ideal is quasi-projective. Then the quotient ring $R/B(R)$ is a valuation domain. In particular, if $B(R)=0$ then R is a valuation domain.*

PROOF. It follows from Theorem 4(ii) that $R/B(R)$ is a valuation ring. It remains to prove that $R/B(R)$ is an integral domain. Let $a+B(R)$ be any zerodivisor in $R/B(R)$. There exists a non-zero element $b+B(R)$ in $R/B(R)$ such that $(a+B(R))(b+B(R))=\bar{0}$. Then $ab\in B(R)$ with $b\notin B(R)$. By Theorem 4(iii), $B(R)$ is a prime ideal of R. This yields that $a\in B(R)$ and $a+B(R)=\bar{0}$. Consequently $R/B(R)$ is a valuation domain.

4. Example. Clearly all left semi-hereditary rings have their finitely generated left ideals quasi-projective. We give an example of a ring R which has each of its finitely generated ideals quasi-projective but R need not be a semi-hereditary ring.

Let R be any commutative valuation ring which is not an integral domain. If A is any finitely generated ideal of R then $A=Rx$ for some element x in R. Clearly $A\cong R/l(x)$. Since any R-endomorphism of left R-module R^R is given by right multiplication by elements of R and $l(x)R\subset l(x)$, so $l(x)$ is an $(R, \mathrm{End}_R(R))$-module. By Wu and Jans [7, Proposition (2.1)], A is quasi-projective as an R-module. Now there exist non-zero elements a and b in R such that $ab=0$. If $B=Ra$ is projective, the exact sequence $0\to I\to R\xrightarrow{\sigma} B\to 0$ splits where $I=l(a)$ and σ is a natural homomorphism. Clearly $I\neq 0$ and $I\neq R$. So $R=I\oplus J$ for some non-zero ideal J of R. This is not possible, as R is a local ring. Hence B is not projective and R is not a semi-hereditary ring.

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