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Generic submanifolds of generalized complex space forms

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Abstract. In the present paper we study generic submanifolds (in the sense of Ronsse) of generalized complex space forms, where such submanifolds generalize/imply holomorphic, totally real, slant, CR-, anti-holomorphic, f-, generic (in the sense of Chen), generalized CR-, and skew CR submanifolds. Some examples along with an open problem are given. A necessary and sufficient condition for integrability of totally real distribution has been found. Ricci tensor and scalar curvature of generic submanifolds have been studied. The paper ends with some results for totally umbilical generic submanifolds.

1. Introduction

The theory of submanifolds of an almost Hermitian manifold is one of the most interesting topics in differential geometry. In an almost Hermitian manifold, its almost complex structure J transforms a vector into a vector perpendicular to it. Perhaps this was the natural motivation to study submanifolds of an almost Hermitian manifold, according to the behaviour of its tangent bundle under the action of the almost complex structure J of the ambient manifold.

There are two well-known classes of submanifolds, namely, holomorphic (invariant) submanifolds and totally real (anti-invariant) submanifolds. In the first case the tangent space of the submanifold remains invariant under the action of the almost complex structure J where as in the second case it is mapped into the normal space.

Study of differential geometry of CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost Hermitian manifold was initiated by A. Bejancu in 1978 [2] and was followed by several

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geometers (see [3], [7], [14], [33] and references cited therein). A submanifold M of an almost Hermitian manifold is called a *CR-submanifold* if the tangent bundle TM of M can be decomposed as the direct sum of a holomorphic (invariant) and a totally real (anti-invariant) distributions. *CR*-submanifolds have good interactions with other parts of mathematics and substantial applications to (pseudo-) conformal mappings and relativity (see [3], [11] and references cited therein).

If a submanifold M of an almost Hermitian manifold admits a holomorphic distribution $\mathcal{D} = TM \cap J(TM)$ then it is called a *f-submanifold* (Yano and Ishihara [31]). Later on this submanifold was defined as *generic submanifold* (Chen [6]) and this is a further generalization of the concept of *CR*-submanifold.

Later, Chen introduced *slant submanifolds* [8] as another generalization of invariant and anti-invariant submanifolds of almost Hermitian manifolds. On a submanifold M of an almost Hermitian manifold, for a vector $0 \neq X_x \in T_xM$, the angle $\theta(X_x)$ between JX_x and the tangent space T_xM is called the *Wirtinger angle* of X_x. If the Wirtinger angle is independent of $x \in M$ and $X_x \in T_xM$, then M is called a *slant submanifold* [8]. Invariant and anti-invariant submanifolds are slant submanifolds with $\theta = 0$ and $\theta = \pi/2$ respectively. Slant submanifolds of almost Hermitian manifolds are characterized by the condition $P^2 + \lambda^2 I = 0$ for some real number $\lambda \in [0,1]$, where PX is the tangential part of JX for $X \in TM$ and I is the identity transformation.

In 1990, Ronsse [22] introduced *generic* and in particular, *skew CR-submanifolds* of an almost Hermitian manifold which differs from but implies the generic submanifold given by Chen and the *generalized CR-submanifold* introduced by Mihai [17]. On a submanifold M of an almost Hermitian manifold, the tangent space $T_xM,\ x \in M$ can be decomposed as the direct sum of the mutually orthogonal P-invariant distinct eigenspaces $\mathrm{Ker}(P^2 + \lambda_i^2(x)I)_x$ of P_x^2 where $\lambda_i(x) \in [0,1],\ i = 1,\dots,q$. If dimensions of the eigenspaces $\mathrm{Ker}(P^2 + \lambda_i^2(x)I)_x$ and q are independent of $x \in M$, then M is called a generic submanifold [22]. Moreover, if λ_i's are also independent of $x \in M$ then a generic submanifold is called a skew *CR*-submanifold.

We observe that skew *CR*-submanifolds also generalize slant submanifolds. Thus generic submanifolds (in the sense of Ronsse) generalize holomorphic, totally real, *CR*- and slant submanifolds.

In this paper we study generic submanifolds of generalized complex space forms. Section 2 is devoted to some preliminaries. Section 3 contains some examples and an open problem. Integrability of totally real distribution is the subject matter of Section 4. Some results on generic submanifolds of generalized complex space forms are given in Section 5.

Ricci tensor and scalar curvature of generic submanifolds have been studied in Section 6. In the last section totally umbilical generic submanifolds have been studied.

2. Preliminaries

(a) Almost Hermitian manifolds and its different classes

Let $\overline{M}$ be an almost Hermitian manifold (AH-manifold) with an almost Hermitian structure (J, g). If J is integrable, i.e. the Nijenhuis tensor $[J, J]$ of J vanishes then the AH-manifold is called a *Hermitian manifold*. The fundamental 2-form Ω of an AH-manifold is defined by $\Omega(X, Y) \equiv g(X, JY)$ for all $X, Y \in T\overline{M}$. An AH-manifold is called an *almost Kähler manifold* if the fundamental 2-form Ω is closed. An AH-manifold becomes

a *nearly Kähler manifold* [12] if $(\overline{\nabla}_X J)X = 0$,

a *Kähler manifold* if $\overline{\nabla} J = 0$,

a *locally conformal Kähler manifold* [29] if $d\Omega = \Omega \wedge \omega$ and $[J, J] = 0$

for all $X \in T\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of the Riemannian metric g and ω is certain closed 1-form (the Lee form) on $\overline{M}$.

An AH-manifold with J-invariant Riemannian curvature tensor $\overline{R}$, i.e.

$$\overline{R}(JX, JY, JZ, JW) = \overline{R}(X, Y, Z, W), \qquad X, Y, Z, W \in T\overline{M},$$

is called an RK-manifold (Vanhecke [30]).

All nearly Kähler and para-Kähler [21] (F-space [23]) manifolds belong to the class of RK-manifolds. There are examples of flat para-Kähler manifolds (and hence of RK-manifolds) which are not Kähler [15, 24, 26].

An AH-manifold $\overline{M}$ is said to have *(pointwise) constant type* if for each $x \in \overline{M}$ and for all $X, Y, Z \in T_x\overline{M}$ such that

$$g(X, Y) = g(X, Z) = g(X, JY) = g(X, JZ) = 0,$$
$$g(Y, Y) = 1 = g(Z, Z)$$

we have

$$\overline{R}(X, Y, X, Y) - \overline{R}(X, Y, JX, JY) = \overline{R}(X, Z, X, Z) - \overline{R}(X, Z, JX, JZ).$$

The notion of constant type was first introduced by A. Gray for a nearly Kähler manifold [12].

It is known that if $\overline{M}$ is an RK-manifold then it has (pointwise) constant type iff there is a differentiable function α on $\overline{M}$ satisfying (Van-

HECKE [30])

$$\overline{R}(X,Y,X,Y) - \overline{R}(X,Y,JX,JY) = \alpha(g(X,X)g(Y,Y) - g(X,Y)^2 - g(X,JY)^2)$$

for all $X,Y,Z \in T\overline{M}$. Furthermore, $\overline{M}$ has global constant type if α is constant. The function α is called the constant type of $\overline{M}$.

An RK-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\overline{M}(c,\alpha)$. For $\overline{M}(c,\alpha)$ it is known that [30]

$$4\overline{R}(X,Y)Z = (c+3\alpha)(g(Y,Z)X - g(X,Z)Y) + (c-\alpha)(g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ)$$

for all $X,Y,Z \in T\overline{M}$. If $c = \alpha$ then $\overline{M}(c,\alpha)$ is a space of constant curvature.

A complex space form $\overline{M}(c)$ (a Kähler manifold of constant holomorphic sectional curvature c) belongs to the class of AH-manifolds $\overline{M}(c,\alpha)$ (with the constant type zero).

An AH-manifold $\overline{M}$ is called a *generalized complex space form* $\overline{M}(f_1,f_2)$ (TRICERRI and VANHECKE [25]) if its Riemannian curvature tensor $\overline{R}$ satisfies

$$\overline{R} = f_1\overline{R}_1 + f_2\overline{R}_2 \tag{2.1}$$

where f_1 and f_2 are smooth functions on $\overline{M}$ and

$$\overline{R}_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.2}$$

$$\overline{R}_2(X,Y)Z = g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ \tag{2.3}$$

for all $X,Y,Z \in T\overline{M}$. We have the inclusion relation $\overline{M}(c) \subset \overline{M}(c,\alpha) \subset \overline{M}(f_1,f_2)$.

(b) Submanifolds of a Riemannian manifold

Let M be a submanifold of a Riemannian manifold $\overline{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N$$

for all $X,Y \in TM$ and $N \in T^\perp M$, where $\overline{\nabla}$, ∇ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\overline{M}$, M and the normal bundle $T^\perp M$ of M respectively, and h is the second fundamental form related to A by $g(h(X,Y),N) = g(A_N X, Y)$. Moreover, if J is a (1,1) tensor field on $\overline{M}$, for $X, Y \in TM$ and $N \in T^{\perp}M$ we put

$$JX = PX + FX, \qquad PX \in TM,\ FX \in T^{\perp}M, \tag{2.4}$$

$$JN = tN + fN, \qquad tN \in TM,\ fN \in T^{\perp}M. \tag{2.5}$$

In this case we have

$$(\overline{\nabla}_X J)Y = ((\nabla_X P)Y - A_{FX}Y - th(X,Y)) + ((\nabla_X F)Y + h(X,PY) - fh(X,Y)), \tag{2.6}$$

where

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y \qquad \text{and} \qquad (\nabla_X F)Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y.$$

Let $\overline{R}$ (resp. R) be the curvature tensor of $\overline{M}$ (resp. M). Then the equations of Gauss and Codazzi are given by

$$g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)), \tag{2.7}$$

$$(\overline{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.8}$$

respectively, where $(\overline{R}(X,Y)Z)^{\perp}$ is the normal component of $\overline{R}(X,Y)Z$, and

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.9}$$

The submanifold M is defined [5] to be
totally geodesic in $\overline{M}$ if $h = 0$,
minimal if $\mathbf{H} \equiv \mathrm{Trace}(h)/\mathrm{Dim}(M) = 0$, and
totally umbilical if $h(X,Y) = g(X,Y)\mathbf{H}$.

(c) Generic and other classes of submanifolds of AH-manifolds

Let M be a submanifold of an AH-manifold $\overline{M}$. Then the operator P^2_x is symmetric $((g(P^2X,Y) = g(X,P^2Y))$ on T_xM and therefore its eigenvalues are real and it is diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0. For each $x \in M$ we may set

$$\mathcal{D}^{\lambda}_x = \mathrm{Ker}(P^2 + \lambda^2(x)I)_x$$

where I is the identity transformation and $\lambda(x)$ belongs to the closed real interval $[0,1]$ such that $-\lambda^2(x)$ is an eigenvalue of P^2_x. Since P^2_x is symmetric and diagonalizable, there is some integer q such that $-\lambda^2_1(x), \dots, -\lambda^2_q(x)$

are distinct eigenvalues of P_x^2 and T_xM can be decomposed as the direct sum of the mutually orthogonal P-invariant eigenspaces, i.e.

$$T_xM = \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_q}.$$

If $\lambda_i(x) > 0$ then $\mathcal{D}_x^{\lambda_i}$ is even-dimensional. Note that $\mathcal{D}_x^1 = \operatorname{Ker}(F_x)$ and $\mathcal{D}_x^0 = \operatorname{Ker}(P_x)$. Here $\mathcal{D}_x^1$ is the maximal J-invariant while $\mathcal{D}_x^0$ is the maximal anti-J-invariant subspace of T_xM. For more details we refer to [22, 28].

Now, we recall the definitions of generic and skew CR-submanifolds of an AH-manifold defined by Ronsse [22].

Definition. A submanifold M of an AH-manifold $\overline{M}$ is called a *generic submanifold* of $\overline{M}$ if there are k functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that the following two conditions hold:

(i) $-\lambda_1^2(x), \ldots, -\lambda_k^2(x)$ are distinct eigenvalues of P^2 at $x \in M$ with

$$T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k},$$

where $\mathcal{D}_x^1 = \operatorname{Ker}(F_x)$, $\mathcal{D}_x^0 = \operatorname{Ker}(P_x)$ and $\mathcal{D}_x^{\lambda_i} = \operatorname{Ker}(P^2 + \lambda_i^2(x)I)_x$, $i \in \{1, \ldots, k\}$,

(ii) the dimensions of $\mathcal{D}_x^1, \mathcal{D}_x^0, \mathcal{D}_x^{\lambda_1}, \ldots, \mathcal{D}_x^{\lambda_k}$ are independent of $x \in M$.

If in addition, each λ_i is constant, then M is called a *skew CR-submanifold*. If $k = 0$ (i.e. in (i) $T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0$) then (i) implies (ii) (Remark 2 [27]).

Condition (ii) in the above definition enables one to define P-invariant mutually orthogonal distributions

$$\mathcal{D}^\lambda = \bigcup_{x \in M} \mathcal{D}_x^\lambda, \qquad \lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\},$$

on M such that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k}.$$

In view of the study of Nomizu [18], these distributions are differentiable.

For $X \in TM$ we write

$$X = U^1X + U^0X + U^{\lambda_1}X + \cdots + U^{\lambda_k}X, \tag{2.11}$$

where $U^1, U^0, U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$ respectively.

For a generic submanifold M of $\overline{M}$ we have

$$T^\perp M = \underline{\mathcal{D}}^1 \oplus \underline{\mathcal{D}}^0 \oplus \underline{\mathcal{D}}^{\lambda_1} \oplus \cdots \oplus \underline{\mathcal{D}}^{\lambda_k},$$

where $\underline{\mathcal{D}}^1 = \mathrm{Ker}(t)$, $\underline{\mathcal{D}}^0 = \mathrm{Ker}(f)$, $F\mathcal{D}^\lambda = \underline{\mathcal{D}}^\lambda$ and $t\underline{\mathcal{D}}^\lambda = \mathcal{D}^\lambda$, $\lambda \in \{0, \lambda_1, \dots, \lambda_k\}$.

A generic submanifold of an AH-manifold becomes

- a *CR-submanifold* [3] if $k = 0$,
- a *proper CR-submanifold* [3] if $k = 0$ and $\mathcal{D}^1 \neq \{0\} \neq \mathcal{D}^0$,
- a *holomorphic (invariant) submanifold* [34] if $k = 0$ and $\mathcal{D}^0 = \{0\}$,
- a *totally real (anti-invariant) submanifold* [32] if $k = 0$ and $\mathcal{D}^1 = \{0\}$,
- a *slant submanifold* [8] if $\mathcal{D}^1 = \{0\} = \mathcal{D}^0$, $k = 1$ and λ_1 is constant,
- an *anti-holomorphic submanifold* [3] (*generic submanifold* in the sense of Yano and Kon [33]) if $k = 0$ and $J\mathcal{D}^0 = T^\perp M$.

The generic submanifold in the sense of Ronsse also implies the generic submanifold in the sense of Chen [6, 20] (f-submanifold in the sense of Yano and Ishihara [31]) and generalized *CR*-submanifold in the sense of Mihai [17]. Throughout the paper generic submanifolds are in the sense of Ronsse unless specifically stated otherwise.

3. Some examples

First we give an example of a generic submanifold of an AH-manifold.

Example 3.1. We consider the Euclidean space $\Re^8$ and denote its points by $x = (x^i)$. Let (e_j), $j = 1, \dots, 8$ be the natural basis defined by $e_j = \partial/\partial x^j$. We define a $(1,1)$ tensor field J by

$$Je_1 = -e_2, \quad Je_2 = e_1, \quad Je_3 = -e_8, \quad Je_8 = e_3,$$
$$Je_4 = -\cos\nu(x)e_5 + \sin\nu(x)e_6, \quad Je_5 = \cos\nu(x)e_4 + \sin\nu(x)e_7,$$
$$Je_6 = -\sin\nu(x)e_4 + \cos\nu(x)e_7, \quad Je_7 = -\sin\nu(x)e_5 - \cos\nu(x)e_6,$$

where $\nu : \Re^8 \to (0, \pi/2)$ is some smooth function. Then $\Re^8$ possesses an almost Hermitian structure (J, g), where g is the canonical metric on $\Re^8$ given by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, \dots, 8$.

The submanifold $\Re^5 = \{x \in \Re^8 | x^6, x^7, x^8 = 0\}$ of $\Re^8$ is a generic submanifold with $\mathcal{D}^1 = \mathrm{Span}\{e_1, e_2\}$, $\mathcal{D}^0 = \mathrm{Span}\{e_3\}$ and $\mathcal{D}^\lambda = \mathrm{Span}\{e_4, e_5\}$, where for $x \in \Re^5$, $\lambda(x) \equiv \cos\nu(x)$.

Example 3.2. Let $1 < \min(h_1, h_2) < h_1 + h_2 < n$ such that M_1 is a complex submanifold of C^{h_1}, M_2 is a totally real submanifold of C^{h_2} and M_3 is a proper slant submanifold of $C^{n-h_1-h_2}$. Then the product $M_1 \times M_2 \times M_3$ is a skew CR submanifold of the complex manifold C^n.

It is known that a differentiable manifold M admits a *CR-structure* (in the sense of GREENFIELD [13]) iff there is a differentiable distribution $\mathcal{D}$ and a $(1,1)$ tensor field J on M such that for all vector fields X and Y in $\mathcal{D}$

$$J^2X = -X \qquad \text{and} \qquad [JX, JY] - [X, Y] = J[JX, Y] - J[X, JY] \in \mathcal{D}.$$

A manifold endowed with a *CR*-structure is called a *CR-manifold* (BEJANCU [3], pp. 128–130).

In [4], BLAIR and CHEN proved the following theorem (as justification of the name *CR*-submanifold).

Theorem. *Every CR-submanifold of a Hermitian manifold is a CR-manifold.*

Thus, for a *CR*-submanifold M of an AH-manifold $\overline{M}$ with $\mathcal{D}^1 \neq \{0\}$ to be a *CR*-manifold, it is sufficient that $\overline{M}$ is Hermitian. However, this is not necessary, and in the following example we find a *CR*-submanifold $M = \Re^3$ of an AH-manifold $\overline{M} = \Re^4$ in which M is a *CR*-manifold and the almost Hermitian structure of $\overline{M}$ is not Hermitian.

Example 3.3. Consider the Euclidean space $\Re^4$ and denote its points by $x = (x^1, x^2, x^3, x^4)$. Let (e_j), $j = 1, \dots, 4$ be the natural basis defined by $e_j = \partial/\partial x^j$, g be the canonical metric defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, \dots, 4$. For every $x \in \Re^4$, the set (E_j) defined by

$$E_1 = e_1, \qquad E_2 = \cos(x^1)e_2 + \sin(x^1)e_3,$$
$$E_3 = -\sin(x^1)e_2 + \cos(x^1)e_3, \quad E_4 = e_4,$$

forms an orthonormal basis, i.e. $g(E_i, E_j) = \delta_{ij}$. As the point x varies in $\Re^4$ the above set of equations defines four vector fields also denoted by (E_j). Now, the following identities

$$J(E_1) = E_2, \quad J(E_2) = -E_1, \quad J(E_3) = E_4, \quad J(E_4) = -E_3$$

define an almost complex structure J on $\Re^4$. Then $\Re^4$ is a non-Hermitian AH-manifold with the almost Hermitian structure (J, g) (KONDERAK [15]).

Consider the hypersurface $\Re^3 = \{x \in \Re^4 \mid x^4 = 0\}$. Since each real hypersurface of an AH-manifold is a *CR*-submanifold ([3], p. 21), $\Re^3$ is a *CR*-submanifold of $\Re^4$. Here the holomorphic distribution is $\mathcal{D}^1 = \operatorname{Span}\{E_1, E_2\}$, and the anti-invariant distribution is $\mathcal{D}^0 = \operatorname{Span}\{E_3\}$.

It is straightforward to check that $(\mathcal{D}, J)$ defines a *CR*-structure on the *CR*-hypersurface $\Re^3$. Moreover, since $[E_1, E_2] = E_3$, $\mathcal{D}^1$ is not integrable. On the other hand, $\mathcal{D}^0$ is integrable.

In view of the above example we have the following open problem.

PROBLEM 3.4. Does a non-Hermitian AH-manifold admit a CR-submanifold which is not a CR-manifold?

4. Integrability of the totally real distribution

First we prove a lemma.

Lemma 4.1. *Let M be a generic submanifold of an AH-manifold. Then the totally real distribution $\mathcal{D}^0$ is integrable iff*

$$d\Omega(X, Y, Z) = 0, \qquad Y, Z \in \mathcal{D}^0,\ X \in TM. \tag{4.1}$$

PROOF. For $X \in TM,\ Y, Z \in \mathcal{D}^0$, we have

$$\begin{aligned} 3d\Omega(X, Y, Z) &= X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) - \Omega([X, Y], Z) \\ &\quad - \Omega([Y, Z], X) - \Omega([Z, X], Y) \\ &= -g([Y, Z], JX) = g(P[Y, Z], X). \end{aligned}$$

Thus (4.1) implies and is implied by the integrability of $\mathcal{D}^0$. □

Using the above lemma we can prove the following

Theorem 4.2. *Let $\overline{M}$ be one of the almost Kähler, Kähler or locally conformal Kähler manifold. Then in order that a submanifold M of $\overline{M}$ is one of a generic, anti-holomorphic, CR- or generalized CR-submanifold it is necessary that the totally real distribution $\mathcal{D}^0$ is integrable.*

5. Generic submanifolds

Let M be a submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$. If M is anti-invariant or invariant then it is easy to verify that TM and $T^{\perp}M$ are invariant under the action of $\overline{R}(X, Y)$ for all $X, Y \in TM$, i.e. $\overline{R}(X, Y)Z \in TM$ and $\overline{R}(X, Y)N \in T^{\perp}M$ for all $X, Y, Z \in TM$ and $N \in T^{\perp}M$.

If $f_2 \neq 0$ and TM is invariant under the action of $\overline{R}(X, Y)$ then

$$\overline{R}(X, Y)X = f_1\overline{R}(X, Y)X - 3f_2 g(JX, Y)JX$$

which implies that $g(JX, Y)JX \in TM$ so that either $JX \in TM$ or $g(JX, Y) = 0$. Since J is linear, M is either invariant or anti-invariant.

Thus we are able to state

Theorem 5.1. *Let M be a submanifold of $\overline{M}(f_1, f_2)$ with $f_2 \neq 0$. Then M is invariant or anti-invariant iff $\overline{R}(X,Y)Z \in TM$ for all $X, Y, Z \in TM$. Consequently, if M is a generic submanifold of $\overline{M}(f_1, f_2)$ such that $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k}$, then $\overline{R}(X,Y)Z \in TM$ for all $X, Y, Z \in TM$ iff $f_2 = 0$.*

In particular, the above theorem provides Proposition 3.1 of [9], Proposition 2 of [16] and Theorem 3.2 of [1].

Proposition 5.2. *Let M be a generic submanifold of an AH-manifold $\overline{M}$ and $\mathcal{D}$ be a distribution on M. For $X, Y \in \mathcal{D}$ the following two statements are equivalent:*

$$\text{(a)} \quad h(X, PY) = h(PX, Y),$$
$$\text{(b)} \quad (A_N PX + PA_N X) \perp \mathcal{D}, \quad N \in T^\perp M.$$

Moreover, if $\overline{M}$ is Kähler then (a) is equivalent to each of the following equivalent statements:

$$\text{(c)} \quad (\nabla_X F)Y - (\nabla_Y F)X = 0, \quad \text{(d)} \quad F[X,Y] = \nabla^\perp_X FY - \nabla^\perp_Y FX.$$

Proof. In view of $g(h(X,Y), N) = g(A_N X, Y)$, (a) $\iff$ (b). Using (2.6) we can prove the equivalence of (a), (c) and (d). □

The above proposition is an improvement over Lemma 4.2 of [22]. In view of the above proposition we make the following

Definition 5.3. For a distribution $\mathcal{D}$ on a submanifold M of an AH-manifold $\overline{M}$ we say that P is $\mathcal{D}$-*commutative* if one of the equivalent statements of (a) and (b) of the above proposition holds.

Note that P is $\mathcal{D}$-commutative for each distribution $\mathcal{D}$ on M iff $PA_N + A_N P = 0$ for all $N \in T^\perp M$. If M is a generic submanifold, then P is $\mathcal{D}^0$-commutative. If M is a generic submanifold of a Kähler manifold, then P is $\mathcal{D}^1$-commutative iff $\mathcal{D}^1$ is integrable. If M is a generic submanifold of a nearly Kähler manifold then P is $\mathcal{D}^1$-commutative if $\mathcal{D}^1$ is integrable.

For each $\mathcal{D}^\lambda$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$ on a generic submanifold of an AH-manifold we choose a local orthonormal basis: $E_1, \ldots, E_{n(\lambda)}$, where $n(\lambda) = \mathrm{Dim}(\mathcal{D}^\lambda)$ and put

$$\mathbf{H}_\lambda = \sum_i h(E_i, E_i), \qquad i \in \{1, \ldots, n(\lambda)\}. \tag{5.1}$$

A generic submanifold of an AH-manifold with $\mathbf{H}_\lambda = 0$ will be called $\mathcal{D}^\lambda$-*minimal* and it will be minimal if $\mathbf{H}_0 + \mathbf{H}_1 + \mathbf{H}_{\lambda_1} + \cdots + \mathbf{H}_{\lambda_k} = 0$.

Proposition 5.4. *Let M be a generic submanifold of an AH-manifold. If P is $\mathcal{D}^\lambda$-commutative, $\lambda \neq 0$, then M is $\mathcal{D}^\lambda$-minimal.*

Proof. Choose a local orthonormal basis for $\mathcal{D}^\lambda$: $E_1, \ldots, E_{n(\lambda)/2}, \ldots, E_{n(\lambda)}$, where $E_{(n(\lambda)/2)+i} = PE_i/\lambda$, $\quad (1 \leq i \leq n(\lambda)/2)$. Then we have

$$\begin{aligned} h(E_i, E_i) + h(PE_i/\lambda, PE_i/\lambda) &= h(E_i, E_i) + h(P^2E_i, E_i)/\lambda^2 \\ &= h(E_i, E_i) + h(-\lambda^2 E_i, E_i)/\lambda^2 = 0. \end{aligned}$$

Consequently $\mathbf{H}_\lambda = 0$. □

As an application of the above proposition we get

Corollary 5.5. *If M is an invariant submanifold of a Kähler or nearly Kähler manifold then M is minimal.*

Let M be a submanifold of $\overline{M}(f_1, f_2)$. Then Gauss equation becomes

$$\begin{aligned} R(X, Y, Z, W) = {} & f_1(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ & + f_2(g(X, PZ)g(PY, W) - g(Y, PZ)g(PX, W) \\ & + 2g(X, PY)g(PZ, W)) + g(h(X, W), h(Y, Z)) \\ & - g(h(X, Z), h(Y, W)) \end{aligned} \tag{5.2}$$

for all $X, Y, Z, W \in TM$. In particular,

$$\begin{aligned} -R(X, Y, X, Y) = {} & f_1(g(X, X)g(Y, Y) - g(X, Y)^2) + 3f_2 g(X, PY)^2 \\ & + g(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2. \end{aligned} \tag{5.3}$$

If X and Y are orthogonal unit vectors in T_xM then the sectional curvature of a plane section determined by X and Y will be

$$\begin{aligned} K_M(X \wedge Y) = {} & f_1 + 3f_2 g(X, PY)^2 \\ & + g(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2. \end{aligned} \tag{5.4}$$

Theorem 5.6. *If M is an anti-invariant totally geodesic submanifold of $\overline{M}(f_1, f_2)$, then*

$$R(X, Y, Y, X) = f_1(g(X, X)g(Y, Y) - g(X, Y)^2).$$

Thus if f_1 is constant then M is a space of constant curvature f_1 .

Proof. Putting $h = 0$ and $P = 0$ in (5.3) we get the proof. □

The above theorem leads to the following two corollaries.

Corollary 5.7. *If M is an anti-invariant totally geodesic submanifold of $\overline{M}(c,\alpha)$, then M is a space of constant curvature $(c+3\alpha)/4$.*

Corollary 5.8 (Proposition 3.2 [9]). *If M is a totally geodesic anti-invariant submanifold of a complex space form $\overline{M}(c)$, then M is a space of constant curvature $c/4$.*

If $\nabla F = 0$ for a submanifold M of an AH-manifold $\overline{M}$, then in view of Theorem 6.3 of [28] the following three statements follow:
(a) M is a skew CR-submanifold,
(b) each of the distributions $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ is parallel and consequently M is locally product of leaves of these distributions (In fact, on $\mathcal{D}^{\lambda_i}$ one gets the structure defined by $P^2 = -\lambda_i^2 I$ [10]),
(c) each of the subbundles $\underline{\mathcal{D}}^1, \underline{\mathcal{D}}^0, \underline{\mathcal{D}}^{\lambda_1}, \dots, \underline{\mathcal{D}}^{\lambda_k}$ of $T^\perp M$ is parallel with respect to $\nabla^\perp$.

If $\overline{M}$ is a nearly Kähler manifold then for a submanifold M of $\overline{M}$, from (2.6) it follows that

$$(\nabla_X F)X + h(X,PX) - fh(X,X) = 0, \qquad X \in TM. \tag{5.5}$$

Proposition 5.9. *If $\nabla F = 0$ for a submanifold M of a nearly Kähler manifold $\overline{M}$, then for $X \in \mathcal{D}^\lambda$, $\lambda \in \{0,1,\lambda_1,\dots,\lambda_k\}$, $h(X,X)$ is zero or an eigenvector of f^2 with eigenvalue $-\lambda^2$. In both cases $h(X,X) \in \underline{\mathcal{D}}^\lambda$.*

Proof. Since $\nabla F = 0$, (5.5) $\Rightarrow$ $f^2 h(X,X) = h(X,P^2X)$
$= -\lambda^2 h(X,X), \quad X \in \mathcal{D}^\lambda$. □

A submanifold M of a Riemannian manifold is said to be
$(\mathcal{D}\text{-}\mathcal{D}')$-*mixed totally geodesic* if $h(\mathcal{D},\mathcal{D}') = 0$,
$\mathcal{D}$-*totally geodesic* if $h(\mathcal{D},\mathcal{D}) = 0$,
where $\mathcal{D}$ and $\mathcal{D}'$ are differentiable distributions on M.

Next we prove

Theorem 5.10. *Let M be a submanifold of a generalized complex space form $\overline{M}(f_1,f_2)$ such that $\overline{M}$ is nearly Kähler. If $\nabla F = 0$ then for any unit orthogonal vectors $X \in \mathcal{D}^\lambda$, $Y \in \mathcal{D}^\mu$, $\lambda \neq \mu$, we have*

$$K_M(X \wedge Y) = f_1 - \|h(X,Y)\|^2.$$

Moreover, if M is $(\mathcal{D}^\lambda\text{-}\mathcal{D}^\mu)$-mixed totally geodesic then $K_M(X \wedge Y) = f_1$.

Proof. From Proposition 5.9, $h(X,X) \in \underline{\mathcal{D}}^\lambda$ and $h(Y,Y) \in \underline{\mathcal{D}}^\mu$. Since $\mathcal{D}^\mu$ is P-invariant, $g(X,PY) = 0$. Therefore from (5.4) we get the proof. □

The Proposition 5.1 of [22] can be obtained from the above theorem.

For a unit vector $X \in \mathcal{D}^\lambda$, $\lambda \neq 0$, of a generic submanifold of an AH-manifold we define the $\mathcal{D}^\lambda$-*sectional curvature* for X by

$$\mathbf{H}_\lambda(X) = K_M(X \wedge PX/\lambda).$$

In particular, if M is a CR-submanifold then $\mathcal{D}^1$-sectional curvature becomes the holomorphic sectional curvature $\mathbf{H}(X) = K_M(X \wedge JX)$ (Barros and Urbano [1]).

From (5.4) we obtain

$$\begin{aligned}(5.6)\quad \mathbf{H}_\lambda(X) &= f_1 + 3\lambda^2 f_2 \\ &\quad + (1/\lambda^2)(g(h(X,X), h(PX,PX)) - \|h(X,PX)\|^2).\end{aligned}$$

Theorem 5.11. *If M is a generic submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ such that P is $\mathcal{D}^\lambda$-commutative then*

$$(5.7)\qquad \mathbf{H}_\lambda(X) = f_1 + 3\lambda^2 f_2 - \|h(X,X)\|^2 - (1/\lambda^2)\|h(X,PX)\|^2).$$

Consequently,

$$(5.8)\qquad \mathbf{H}_\lambda(X) \leq f_1 + 3\lambda^2 f_2$$

and equality holds if M is $\mathcal{D}^\lambda$-totally geodesic.

Proof. If P is $\mathcal{D}^\lambda$-commutative then for $X \in \mathcal{D}^\lambda$ we get $h(PX, PX) = -\lambda^2 h(X,X)$. Using above equation in (5.6) we get (5.7). Rest of the proof is straight forward. □

Remark 5.12. If $\overline{M}(f_1, f_2)$ is $\overline{M}(c, \alpha)$, then (5.8) becomes

$$4\mathbf{H}_\lambda(X) \leq c(1 + 3\lambda^2) + 3\alpha(1 - \lambda^2).$$

In case of complex space form $\overline{M}(c)$, $\alpha = 0$, and we get Proposition 5.2 of [22]. If M is a CR-submanifold of $\overline{M}(c)$ then the holomorphic sectional curvature $\mathbf{H}(X)$ of M satisfies $\mathbf{H}(X) \leq c$, which is Theorem 4.3 of [1] on page 359.

If $\overline{M}(f_1, f_2)$ is nearly Kähler then for a unit vector $X \in \mathcal{D}^1$ we get

$$h(X, PX) = fh(X,X) = F\nabla_X X.$$

Consequently,

$$h(PX, PX) = f^2 h(X,X) = fF\nabla_X X + F\nabla_{PX} X.$$

Thus $\mathcal{D}^1$-sectional curvature is

$$\begin{aligned}\mathbf{H}_1(X) &= f_1 + 3f_2 - 2\|fh(X,X)\|^2 - \|F\nabla_X X\|^2 \\ &\quad + g(h(X,X), fF\nabla_X X + F\nabla_{PX} X - 2F\nabla_X X).\end{aligned}$$

In particular, if M is an invariant submanifold of $\overline{M}(f_1, f_2)$, then

$$\mathbf{H}_1(X) = f_1 + 3f_2 - 2\|h(X,X)\|^2.$$

In a special case, if M is an invariant submanifold of $\overline{M}(c,\alpha)$, then

$$\mathbf{H}_1(X) = c - 2\|h(X,X)\|^2,$$

which is the last equation on page 358 of [1].

Let M be a generic submanifold of an AH-manifold. Let $\{E_1, \ldots, E_{n(\lambda)}\}$ and $\{F_1, \ldots, F_{n(\mu)}\}$ be local orthonormal bases for $\mathcal{D}^\lambda$ and $\mathcal{D}^\mu$ respectively. Then $(\mathcal{D}^\lambda - \mathcal{D}^\mu)$-*sectional curvature* is defined by [22]

$$\rho_{\lambda\mu} = \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} K_M(E_i \wedge F_j).$$

If $\lambda \neq \mu$ then from (5.4) we obtain

$$\rho_{\lambda\mu} = n(\lambda)n(\mu)f_1 + g(\mathbf{H}_\lambda, \mathbf{H}_\mu) - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} \|h(E_i, F_j)\|^2.$$

For $\lambda = 0$ we get

$$\rho_{00} = n(0)^2 f_1 + \|\mathbf{H}_0\|^2 - \sum_{i=1}^{n(0)} \sum_{j=1}^{n(0)} \|h(E_i, E_j)\|^2.$$

In order to calculate $\rho_{\lambda\lambda}$ for $\lambda \neq 0$ we choose a local orthonormal basis for $\mathcal{D}^\lambda$:

$$E_1, \ldots, E_{n(\lambda)/2}, \ldots, E_{n(\lambda)}, \quad \text{where}$$
$$E_{(n(\lambda)/2)+i} = PE_i/\lambda, \quad (1 \leq i \leq n(\lambda)/2).$$

Using this basis we obtain

$$\sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} g(E_i, PE_j)^2 = n(\lambda)\lambda^2.$$

Therefore for $\lambda \neq 0$ we get

$$\rho_{\lambda\lambda} = n(\lambda)^2 f_1 + 3n(\lambda)\lambda^2 f_2 + \|\mathbf{H}_\lambda\|^2 - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} \|h(E_i, E_j)\|^2.$$

In view of the above discussion we can state

Theorem 5.13. *Let M be a generic submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$. Then*

(1) *If $\mathbf{H}_\lambda$ is perpendicular to $\mathbf{H}_\mu$, $\lambda \neq \mu$, then $\rho_{\lambda\mu} \leq n(\lambda)n(\mu)f_1$ and equality holds iff M is $(\mathcal{D}^\lambda$-$\mathcal{D}^\mu)$-mixed totally geodesic.*

(2) *If M is $\mathcal{D}^\lambda$-minimal then $\rho_{\lambda\lambda} \leq n(\lambda)^2 f_1 + 3n(\lambda)\lambda^2 f_2$ and equality holds iff M is $\mathcal{D}^\lambda$-totally geodesic.*

6. Ricci tensor and scalar curvature

Let M be a submanifold of dimension m of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension $2n$. Let $\{E_1, \ldots, E_m\}$ be a local orthonormal basis of TM and $\{N_1, \ldots, N_{2n-m}\}$ be a local basis of normal sections and let $A_{N_\nu} \equiv A_\nu$. Then the Ricci tensor S of M is given by

$$\begin{aligned} S(X,Y) &= \sum_{i=1}^{m} R(X,E_i,E_i,Y) = \sum_{i=1}^{m}[f_1(g(E_i,E_i)g(X,Y) \\ &\quad - g(X,E_i)g(E_i,Y)) + 3f_2 g(PX,E_i)g(E_i,PY) \\ &\quad + g(h(X,Y),h(E_i,E_i)) - g(h(X,E_i),h(E_i,Y))] \\ &= (m-1)f_1 g(X,Y) + 3f_2 g(PX,PY) \\ &\quad + \sum_{\nu=1}^{2n-m}\{(\mathrm{Trace}\, A_\nu)g(A_\nu X,Y) - g(A_\nu X, A_\nu Y)\}. \end{aligned}$$

Thus we have

Theorem 6.1. *Let M be an m-dimensional submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension $2n$. Then*

$$\begin{aligned} S(X,Y) &= (m-1)f_1 g(X,Y) + 3f_2 g(PX,PY) \\ &\quad + \sum_{\nu=1}^{2n-m}\{(\mathrm{Trace}\, A_\nu)g(A_\nu X,Y) - g(A_\nu X, A_\nu Y)\}. \end{aligned} \tag{6.1}$$

Next we prove

Theorem 6.2. *Let M be an m-dimensional generic submanifold of a* *generalized complex space form $\overline{M}(f_1, f_2)$ of dimension $2n$. Then*

$$S(X,Y) = \sum_{\lambda}((m-1)f_1 + 3\lambda^2 f_2)g(U^\lambda X, U^\lambda Y) + \sum_{\nu=1}^{2n-m}\{(\text{Trace } A_\nu)g(A_\nu X, Y) - g(A_\nu X, A_\nu Y)\}, \quad \lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}. \tag{6.2}$$

Proof. Using (2.11) in (6.1) we get (6.2). □

The above theorem leads to the following

Corollary 6.3. *Let M be an m-dimensional generic submanifold of a generalized complex space form $\overline{M}(c, \alpha)$ of dimension $2n$. Then*

$$S(X,Y) = \sum_{\lambda}\left(\frac{m-1+3\lambda^2}{4}c + \frac{3(m-1+3\lambda^2)}{4}\alpha\right)g(U^\lambda X, U^\lambda Y) + \sum_{\nu=1}^{2n-m}\{(\text{Trace}\, A_\nu)g(A_\nu X, Y) - g(A_\nu X, A_\nu Y)\}, \quad \lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}. \tag{6.3}$$

In particular, if M is a CR-submanifold of $\overline{M}(c, \alpha)$ (resp. $\overline{M}(c)$), then (6.3) becomes (5.4) of [1] (resp. (4.7) of [2]). In particular, if M is a totally real submanifold of a complex space form $\overline{M}(c)$ then (6.3) becomes (5.4) of [9].

Let M be an m-dimensional generic submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$. Let $n(\lambda)$ be dimension of $\mathcal{D}^\lambda$, $\lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}$. Then considering a local orthonormal basis:

$$E_1^0, \dots, E_{n(0)}^0, E_1^1, \dots, E_{n(1)/2}^1, E_{(n(1)/2)+1}^1 = PE_1^1, \dots, E_{n(1)}^1 = PE_{n(1)/2}^1,$$
$$E_1^{\lambda_1}, \dots, E_{n(\lambda_1)/2}^{\lambda_1}, E_{(n(\lambda_1)/2)+1}^{\lambda_1} = PE_1^{\lambda_1}/\lambda_1, \dots, E_{n(\lambda_1)}^{\lambda_1} = PE_{n(\lambda_1)/2}^{\lambda_1}/\lambda_1, \dots,$$
$$E_1^{\lambda_k}, \dots, E_{n(\lambda_k)/2}^{\lambda_k}, E_{(n(\lambda_k)/2)+1}^{\lambda_k} = PE_1^{\lambda_k}/\lambda_k, \dots, E_{n(\lambda_k)}^{\lambda_k} = PE_{n(\lambda_k)/2}^{\lambda_k}/\lambda_k,$$

in view of (6.2) we get the following

Theorem 6.4. *Let M be an m-dimensional generic submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension $2n$. Then the* *scalar curvature* ρ *of* M *is given by*

$$\rho = \sum_{\lambda} n(\lambda)((m-1)f_1 + 3\lambda^2 f_2) + m^2\|\mathbf{H}\|^2 - \|h\|^2, \quad \lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}. \tag{6.4}$$

In particular, if $\overline{M}(f_1, f_2)$ is $\overline{M}(c, \alpha)$ then the above equation becomes

$$\rho = \frac{1}{4}\left(m^2 - m + 3\sum_{\lambda} n(\lambda)\lambda^2\right)c + \frac{3}{4}\left(m^2 - m - \sum_{\lambda} n(\lambda)\lambda^2\right)\alpha + m^2\|\mathbf{H}\|^2 - \|h\|^2, \lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}. \tag{6.5}$$

When $k = 0$, i.e. M is a *CR*-submanifold then the above equation becomes (5.5) of [1] and when M is a *CR*-submanifold of a complex space form $\overline{M}(c)$, then (6.5) becomes (4.8) of [2].

In view of (6.2) and (6.4) we have the following two theorems.

Theorem 6.5. *Let* M *be an* m*-dimensional minimal generic submanifold of a generalized complex space form* $\overline{M}(f_1, f_2)$ *of dimension* $2n$. *Then*

(a) $S - \sum_{\lambda}((m-1)f_1 + 3\lambda^2 f_2)g \circ (U^{\lambda} \times U^{\lambda})$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$ *is negative semi-definite,*

(b) $\rho \leq \sum_{\lambda} n(\lambda)((m-1)f_1 + 3\lambda^2 f_2)$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$.

Theorem 6.6. *For an* m*-dimensional minimal generic submanifold* M *of a generalized complex space form* $\overline{M}(f_1, f_2)$ *of dimension* $2n$, *the following three conditions are equivalent:*

(1) M *is totally geodesic,*

(2) $S = \sum_{\lambda}((m-1)f_1 + 3\lambda^2 f_2)g \circ (U^{\lambda} \times U^{\lambda})$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$,

(3) $\rho = \sum_{\lambda} n(\lambda)((m-1)f_1 + 3\lambda^2 f_2)$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$.

The above two theorems lead to the following two corollaries.

Corollary 6.7. *Let* M *be an* m*-dimensional minimal generic submanifold of* $\overline{M}(c, \alpha)$ *of dimension* $2n$. *Then*

(a) $S - \sum_{\lambda}(\frac{m-1+3\lambda^2}{4}c + \frac{3(m-1+3\lambda^2)}{4}\alpha)g \circ (U^{\lambda} \times U^{\lambda})$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$ *is negative semi-definite,*

(b) $\rho \leq \frac{1}{4}(m^2 - m + 3\sum_{\lambda} n(\lambda)\lambda^2)c + \frac{3}{4}(m^2 - m - \sum_{\lambda} n(\lambda)\lambda^2)\alpha$, $\lambda \in \{1, \lambda_1, \ldots, \lambda_k\}$.

In particular, we get also Proposition 3.3 of [9].

Corollary 6.8. *For an m-dimensional minimal generic submanifold M of $\overline{M}(c,\alpha)$ of dimension $2n$, the following three conditions are equivalent:*

(1) M *is totally geodesic,*

(2) $S = \sum_\lambda \left(\frac{m-1+3\lambda^2}{4}c + \frac{3(m-1+3\lambda^2)}{4}\alpha\right) g\circ(U^\lambda \times U^\lambda)$,
$\lambda \in \{0,1,\lambda_1,\dots,\lambda_k\}$,

(3) $\rho = \frac{1}{4}(m^2 - m + 3\sum_\lambda n(\lambda)\lambda^2)c + \frac{3}{4}(m^2 - m - \sum_\lambda n(\lambda)\lambda^2)\alpha$,
$\lambda \in \{1,\lambda_1,\dots,\lambda_k\}$.

7. Totally umbilical generic submanifolds

First we prove

Proposition 7.1. *If M is a totally umbilical generic submanifold of a Kähler manifold then either $\mathcal{D}^0 = \{0\}$ or 1-dimensional or the mean curvature vector $\mathbf{H}$ is perpendicular to $\underline{\mathcal{D}}^0$.*

Proof. If $\mathcal{D}^0 = \{0\}$ or $\operatorname{Dim}(\mathcal{D}^0) = 1$, then the conclusion is obvious. If $\operatorname{Dim}(\mathcal{D}^0) > 1$, let $X, W \in \mathcal{D}^0$ such that $g(X,W) = 0$ and $\|X\| = 1$. Then

$$\begin{aligned} g(\mathbf{H}, FW) &= g(h(X,X), FW) = g(A_{FW}X, X) \\ &= g(A_{FX}W, X) = g(h(X,W), FX) = 0. \end{aligned}$$ □

In particular, this proposition leads to the Corollary 1 of [4]. Next, we prove the following

Theorem 7.2. *If M is a totally umbilical generic submanifold of a generalized complex space form $\overline{M}(f_1,f_2)$ with $f_2 \neq 0$, then M is not proper.*

Proof. Let $X \in \mathcal{D}^1$, $Y \in \mathcal{D}^0$ be two non-null vectors. Then from (2.8), (2.9), $PY = 0$ and the fact that M is totally umbilical we get $(\overline{R}(X,PX)Y)^\perp = 0$. Also from (2.1) we get

$$\overline{R}(X,PX)Y = -2f_2 g(PX,PX)JY \neq 0 = (\overline{R}(X,PX)Y)^\perp.$$

Thus we have a contradiction. □

As a consequence of the above theorem we get

Corollary 7.3. *There exist no totally geodesic proper generic submanifold of a generalized complex space form $\overline{M}(f_1,f_2)$ with $f_2 \neq 0$.*

Remark 7.4. The similar results hold for generalized *CR*-submanifolds.

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