A note on the group-theoretic isomorphism theorems

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Dedicated to the memory of Tibor Szele

1. Introduction

In [1, p. 94 and p. 277] one finds the following general isomorphism theorem for groups:

Theorem. *Let M, N be normal subgroups of a group G with $M \subseteq N$ and let H be an arbitrary subgroup of G. Then*

$$(*) \qquad (\{H, M\}/M)/(\{H \cap N, M\}/M) \cong \{H, N\}/N.$$

In [1] the proof of the isomorphism $(*)$ is carried through by using certain homomorphic mappings and the homomorphism theorem for groups. For the special case of $M = \{e\}$ (e being the neutral element of G) we get from this theorem the

I. Group-theoretic isomorphism theorem. Let G be a group, H a subgroup and N a normal subgroup of G. Then

$$H/H \cap N \cong \{H, N\}/N.$$

For the special case of $H = G$, we get the

II. Group-theoretic isomorphism theorem. Let G be a group and M, N normal subgroups of G with $M \subseteq N$. Then

$$(G/M)/(N/M) \cong G/N.$$

Using the theory of standard ideals in lattices (cf. [4, p. 146—151]) a similar isomorphism as $(*)$ was proved for lattices in [5]. The referee of [5] suggested to carry through this proof by using the First and Second Isomorphism Theorems for standard ideals in lattices (cf. [4, Exercises 18 and 19, p. 151]).

From this observation we got the hint that it should be possible to give a proof of $(*)$ by using the above-mentioned group theoretic isomorphism theorems. Such a proof is presented in Section 2.

Some concluding remarks are given in Section 3.

2. *A group-theoretic isomorphism theorem*

In this section we prove by other means than in [1] the following

Theorem. *Let M, N be normal subgroups of a group G with $M \subseteqq N$ and let H be an arbitrary subgroup of G. Then*

$$(*) \qquad (\{H, M\}/M)/(\{H \cap N, M\}/M) \cong \{H, N\}/N.$$

PROOF. First we show that

$$(1) \qquad N \cap \{H, M\} = \{H \cap N, M\}.$$

Because of $M \subseteqq N$ and $M \subseteqq \{H, M\}$ we have

$$(2) \qquad M \subseteqq N \cap \{H, M\}.$$

On the other hand, we have $H \cap N \subseteqq N$ and $H \cap N \subseteqq H \subseteqq \{H, M\}$ from which we get

$$(3) \qquad H \cap N \subseteqq N \cap \{H, M\}.$$

Now (2) and (3) imply

$$(4) \qquad \{H \cap N, M\} \subseteqq N \cap \{H, M\}.$$

In order to show the converse inclusion, we observe that

$$N \cap \{H, M\} = N \cap H \cdot M$$

since M is a normal subgroup. Assume now

$$(5) \qquad x \in N \cap H \cdot M.$$

From (5) it follows that

$$x \in H \cdot M$$

and therefore there exist elements $h \in H$ and $m \in M$ such that

$$x = h \cdot m.$$

This implies

$$h = x \cdot m^{-1} \in N \cdot M \subseteqq N$$

and hence

$$h \in H \cap N$$

from which we get

$$(6) \qquad x = h \cdot m \in (H \cap N) \in M = \{H \cap N, M\}.$$

Now (5) and (6) together yield

$$(7) \qquad N \cap H \cdot M = N \cap \{H, M\} \subseteqq \{H \cap N, M\}.$$

The inclusions (4) and (7) mean that equality (1) holds true. On the other hand, equality (1) implies

$$(8) \qquad \{H \cap N, M\} = N \cap \{H, M\} = N \cap HM \lhd H \cdot M.$$

After these preparations we are able to prove the theorem. By the II. Group-theoretic isomorphism theorem and using (8) we obtain

$$(9) \qquad (\{H, M\}/M)/(\{H \cap N, M\}/M) \cong \{H, M\}/\{H \cap N, M\}.$$

Moreover, we have

$$(10) \qquad \{\{H, M\}, N\} = \{H, N\}$$

since $M \subseteq N$. Using (10) and the I. Group-theoretic isomorphism theorem, we obtain further that

$$(11) \qquad \{H, M\}/(N \cap \{H, M\}) \cong \{H, N\}/N.$$

The isomorphisms (9) and (11) together mean that $(*)$ holds, which proves the theorem.

3. Concluding remarks

The original proof of $(*)$ is due to E. FRIED (Budapest) and was first published in [1]. As E. Fried wrote to the second author of the present paper, the isomorphism $(*)$ was already known to L. RÉDEI.

Moreover let us remark that a similar isomorphism as $(*)$ was used in [2] to prove a general isomorphism theorem for universal algebras.

The proof of $(*)$ given here for groups suggests that the same can be done to prove the result of [2] by using the First and the Second Isomorphism Theorems for universal algebras as formulated e.g. in [3, p. 58 and p. 62].

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