On an approximation of generalized inverses

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Summary. *Let A be a linear bounded operator from the Hilbert space X into the Hilbert space Y, and A^+ denote its generalized inverse. In this paper we find an estimate for the difference A^+y-x_ε when y belongs to a suitable subspace of Y: exactly we show that*

$$\|A^+y-x_\varepsilon\| = O(\varepsilon) \quad (0 < \varepsilon \to 0)$$

where $x_\varepsilon=(AA^*+\varepsilon I)^{-1}A^*y$

Introduction

Let A be a linear bounded operator from the Hilbert space X into the Hilbert space Y, and A^+ be the generalized inverse of it, [1]. Consider now the element $x_\varepsilon \in X$ defined as follows:

$$x_\varepsilon = (AA^*+\varepsilon I)^{-1} A^* y \tag{1}$$

for every $\varepsilon>0$, where A^* is the adjoint of A and I the identity in X. If A is a compact operator then, as it is well known, [2] (theorem of Tikhonov),

$$A^+ y = \lim_{0<\varepsilon\to 0} x_\varepsilon \tag{2}$$

holds iff $y \in R(A)\oplus R(A)^\perp$, ($R(A)$ is the range of A).

Recently it was shown [3], that (2) holds even if A is not compact, but only linear and bounded.

Now the following problem arises: how fast is the convergence in (2)? A partial answer was given to this question in [4], where it was proved that, if A has a special property, then a subspace of $R(A)\oplus R(A)^\perp$ can be constructed such that for every element y of this subspace $\|A^+y-x_\varepsilon\|^2=O(\varepsilon)$ holds.

In this paper we will give a similar estimate for $\|A^+y-x_\varepsilon\|$, without making any special supposition on A of the type as above.

Some preliminaries

Suppose X and Y are separable Hilbert spaces of the same dimension and A: $X\to Y$ a linear bounded operator. We will now construct for A two systems of numbers and two systems of orthonormal elements $\{x_1, x_2, \dots\}$, $\{y_1, y_2, \dots\}$ in X and Y respectively, which we call generalized eigenvalues and eigenelements of A according to [5], [6, p. 150].

Let us start with an element $y_1 \in Y$, with $\|y_1\|=1$ and $y_1 \in (\ker A^*)^\perp$, otherwise arbitrary. The elements x_j and y_j, as well as the numbers k_j and m_j, will be defined by recurrence:

$$(3_j) \qquad Ax_{j-1} - k_{j-1} y_{j-1} = m_{j-1} y_j \quad \|y_j\| = 1$$

$$(4_j) \qquad A^* y_j - \bar{m}_{j-1} x_{j-1} = \bar{k}_j x_j \quad \|x_j\| = 1$$

$(m_0 = k_0 = 0,\ x_0 = 0,\ y_0 = 0,\ j = 1, 2, 3, \ldots)$.

It can be proved that:

(i) $k_j \neq 0$ $(j=1, 2, \ldots)$,

(ii) the systems of elements $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ are orthonormal,

(iii) all x_j are orthogonal to $\ker A$ and all y_j are orthogonal to $\ker A^*$.

If for some j it turns out that $Ax_{j-1} - k_{j-1} y_{j-1} = 0$, then let us put $m_{j-1}=0$ and choose $y_j \in Y$ such that $\|y_j\|=1$ and y_j is orthogonal to $\ker A^*$, $(y_j, y_k)=0$ for $k=1, 2, \ldots, j-1$. In the case that A is of finite dimension it can be shown that the systems above are finite.

Interesting is the case if A is of infinite dimension. It is shown in [6 p. 150] that is not a loss of generality if we suppose that the system $\{y_j\}$ is complete in $(\ker A^*)^\perp$, and then automatically also $\{x_j\}$ appears as complete in $(\ker A)^\perp$. We can also suppose, without the restriction of generality, that $k_j > 0$ for all j [6, p.158]. In the same way it is possible to see that the construction of the systems of generalized eigenelements and eigenvalues can be chosen so that also $m_j \geqq 0$ holds.

By the systems of generalized eigenvalues and eigenelements an arbitrary linear bounded operator A has the following expansion, [5], [6 p. 153]:

$$(4) \qquad Ax := \sum_{(j)} k_j (x, x_j) y_j + \sum_{(j)} m_{j-1} (x, x_{j-1}) y_j \quad (x \in X).$$

where these series are convergent in X.

If we consider another operator $A(\varepsilon)$ depending on a positive parameter ε, defined as follows

$$(5) \qquad A(\varepsilon) x = \sum_{(j)} (k_j + \varepsilon)(x, x_j) y_j + \sum_{(j)} m_{j-1} (x, x_{j-1}) y_j \quad (x \in X),$$

then we see at once, that

$$\big(A(\varepsilon) - A\big) x = \varepsilon \sum_{(j)} (x, x_j) y_j$$

and therefore by the completeness of the system $\{x_j\}$ in $(\ker A)^\perp$ we have

$$\left\|\big(A(\varepsilon) - A\big) x\right\|^2 \leqq \varepsilon^2 \sum_{(j)} |(x, x_j)|^2 = \varepsilon^2 \|x\|^2,$$

hence

$$(6) \qquad \|A(\varepsilon) - A\|^2 = \varepsilon^2.$$

Obviously the same is valid for the adjoint too.

From this it follows, first of all, that if $\varepsilon > 0$ then $A(\varepsilon)$ tends strongly to A. This implies obviously, that $A(\varepsilon)$ is bounded in the neighbourhood of zero:

$$(7) \qquad \|A(\varepsilon)\| \leqq \|A(\varepsilon) - A\| + \|A\| \leqq \varepsilon + \|A\|$$

and the same holds for its adjoint.

Let $\varepsilon>0$ be a fixed number and $\delta>0$, then let us consider with respect to $A(\varepsilon)$ the expression corresponding to (1):

$$x_\delta(\varepsilon) = \big(A(\varepsilon)A^*(\varepsilon)+\delta I\big)^{-1}A^*(\varepsilon)y \tag{8}$$

and let us consider an $y\in[R(A)\oplus R(A)^\perp]\cap\big[R\big(A(\varepsilon)\big)\oplus R\big(A(\varepsilon)\big)^\perp\big]$. Then by the quoted generalized Tikhonov theorem $x_j(\varepsilon)\to x_0(\varepsilon)=A(\varepsilon)^+y$.

We will use further on the following lemma:

Lemma. *$A(\varepsilon)^+y\to A^+y$ in X for every $y\in[R(A)\oplus R(A)^\perp]\cap\big[R\big(A(\varepsilon)\big)\oplus R\big(A(\varepsilon)\big)^\perp\big]$.*

Proof. (1) implies

$$Nx_\varepsilon := (AA^*+\varepsilon I)x_\varepsilon = A^*y,$$

$$Mx_\delta(\varepsilon) = \big(A(\varepsilon)A^*(\varepsilon)+\delta I\big)x_\delta(\varepsilon) = A^*(\varepsilon)y.$$

The difference of the operators on the left side can be estimated as follows:

$$\|N-M\| = \|AA^*-A(\varepsilon)A^*(\varepsilon)\|+|\varepsilon-\delta| \leqq \|A\|\varepsilon+\|A^*(\varepsilon)\|+|\varepsilon-\delta| =$$

$$\leqq c\varepsilon+|\varepsilon-\delta|$$

where c is a constant independent of ε.

We will now consider the restriction of $(AA^*+\varepsilon I)^{-1}$ to the subspace $A^*\big(R(A)\oplus R(A)^\perp\big)$ and denote it, for brevity, by the same symbol. This operator is bounded with respect to ε, because for all elements $\eta\in A^*\big(R(A)\oplus R(A)^\perp\big)$, $(AA^*+\varepsilon I)^{-1}\eta\to A^+y$, where $A^*y=\eta$. Let us now choose ε and δ little enough in order to have

$$\|(AA^*+\varepsilon I)^{-1}\|(c\varepsilon+|\varepsilon-\delta|) < 1.$$

Apply now the Proposition 1 in [8 formula (9)]; we get by (6):

$$\|x_\varepsilon-x_\delta(\varepsilon)\| \leqq \frac{\|(AA^*+\varepsilon I)^{-1}\|(c\varepsilon+|\varepsilon-\delta|)\|A^*y\|}{1-\|(AA^*+\varepsilon I)^{-1}\|(c\varepsilon+|\varepsilon-\delta|)}+\|(AA^*+\varepsilon I)^{-1}\|\,\varepsilon\,\|y\|.$$

Let now $0<\delta\to 0$, then by the choice of y, $x_\delta(\varepsilon)\to A^+(\varepsilon)y$ and we can write

$$\|x_\varepsilon-A^+(\varepsilon)y\|=c'\varepsilon\frac{\|(AA^*+\varepsilon I)^{-1}\|\,\|A^*y\|}{1-c'\varepsilon\|(AA^*+\varepsilon I)^{-1}\|}+\varepsilon\|(AA^*+\varepsilon I)^{-1}\|\,\|y\|.$$

The right hand side can be arbitrarily small by choosing $\varepsilon>0$ little enough. But as $x_\varepsilon\to A^+y$ the estimate above shows that $A^+(\varepsilon)y\to A^+y$ for all $y\in[R(A)\oplus R(A)^\perp]\cap$ $\cap\big[R\big(A(\varepsilon)\big)\oplus R\big(A(\varepsilon)\big)^\perp\big]$. This completes the proof.

We conclude this section by indicating the following notations:

$$\alpha_j = (x, x_j), \tag{9}$$

$$\beta_j = (y, y_j) \quad (j = 1, 2, \dots), \tag{10}$$

$$\alpha_j(\varepsilon) = (x_\varepsilon, x_j). \tag{11}$$

The theorem and its proof

Theorem. *Let A be a linear bounded and closed operator, $A: X \to Y$, where X and Y are real separable Hilbert spaces of the same dimension. Consider the systems of generalized eigenvalues $\{k_j\}$, $\{m_j\}$ and the corresponding generalized eigen-elements $\{x_j\}$, $\{y_j\}$ in order that $\{y_j\}$ is complete in $(\ker A^*)^\perp$, $k_j>0$ and $m_j \geqq 0$ $(j=1, 2, \ldots)$. If for any $y \in \operatorname{int}[R(A)\oplus R(A)^\perp]$*

$$\sum_{j=1}^{\infty} k_j^{-4} \left| \beta_j - \frac{m_{j-1}}{(k_{j-1})^2}(k_{j-1}+k_j)\beta_{j-1} + \ldots + (-1)^{j-1} \times \right. \tag{12}$$

$$\left. \times \frac{m_1 \ldots m_{j-1}}{(k_1 \ldots k_{j-1})^2} \sum_{s=1}^{j} k_1 \ldots k_{s-1} k_{s+1} \ldots k_j \beta_1 \right|^2 < \infty$$

holds (where now $k_0=1$, if $s=j$ then $k_{s+1}=1$), then

$$\|A^+ y - x_\varepsilon\| = O(\varepsilon) \quad (0 > \varepsilon \to 0). \tag{13}$$

Remark. We can equivalently formulate our statement as follows: *Let S be the subset of* $\operatorname{int}[R(A)\oplus R(A)^\perp]$ *of all elements for which* (12) *holds; then* (13) *is valid for every element of S.*

Proof. Let us choose $\varepsilon>0$ in order that if $y \in \operatorname{int}[R(A)\oplus R(A)^\perp]$, then $y \in R(A(\varepsilon)) \oplus R(A(\varepsilon))^\perp$ should be valid too, where $A(\varepsilon)$ is the operator defined by (5). This is possible by definition of $A(\varepsilon)$:

$$A(\varepsilon)x = \sum_{(j)} k_j(x, x_j)y_j + \sum_{(j)} m_{j-1}(x, x_{j-1})y_j + \varepsilon \sum_{(j)} (x, x_j) y_j.$$

If $y \in \operatorname{int} R(A)$, and denoting for a moment by x the element corresponding to y, i.e. $Ax=y$, then $\|A(\varepsilon)x-y\| \leqq \varepsilon\|x\|$. This means that $A(\varepsilon)x$ is in a $\varepsilon\|x\|$ neighbourhood of y.

In the following let x be A^+y $(A^+y=x)$ (it is known that $\operatorname{dom} A^+ = R(A)\oplus R(A)^\perp$, see [7]), and x_ε the element defined in (1). Because of the completeness of the orthonormal systems $\{x_j\}$ and $\{y_j\}$, the relation

$$\|A^+ y - x_\varepsilon\|^2 = \sum_{(j)} |\alpha_j - \alpha_j(\varepsilon)|^2 \tag{14}$$

holds. In [3] (formulae (2.3) and (3.3)) it is proved that the Fourier-coefficients α_j fulfil the system of linear equations:

$$m_{j-1}\alpha_{j-1} + k_j\alpha_j = \beta_j \quad (j = 1, 2, \ldots). \tag{15}$$

This system is uniquely solvable and the solution is:

$$\alpha_j = \frac{1}{k_j}\left\{\beta_j - \frac{m_{j-1}}{k_{j-1}}\beta_j + \ldots + (-1)^{j-1}\frac{m_1 m_2 \ldots m_{j-1}}{k_1 k_2 \ldots k_{j-1}}\beta_1\right\} \quad (j = 1, 2, \ldots). \tag{16}$$

In [7] it is also proved that for every $y \in R(A)\oplus R(A)^\perp$ $\sum |\alpha_j|^2 < \infty$ holds. On the other hand, it can be seen [3], that the Fourier-coefficients of x_ε form the unique solution of the following system of equations:

$$(17) \qquad m_{j-1}k_j\alpha_{j-1}(\varepsilon)+(k_j^2+|m_j|^2+\varepsilon)\alpha_j(\varepsilon)+\overline{m}_j k_{j+1}\alpha_{j+1}(\varepsilon) = k_j\beta_j+\overline{m}_j\beta_{j+1}$$
$$(m_0 = 0,\ j = 1, 2, \ldots).$$

Now in order to find an estimate for (14) we need to consider the following linear system of equations:

$$(18) \qquad \begin{aligned} m_{j-1}k_j\hat{\alpha}_{j-1}(\varepsilon)+(k_j^2+|m_j|^2+k_j\varepsilon)\hat{\alpha}_j(\varepsilon)+\overline{m}_j(k_{j+1}+\varepsilon)\hat{\alpha}_{j+1}(\varepsilon) = \\ = k_j\beta_j+\overline{m}_j\beta_{j+1} \quad (j = 1, 2, \ldots). \end{aligned}$$

We see at once that the system (18) can be rewritten in the following form:

$$(19) \qquad \begin{aligned} k_j\{m_{j-1}\hat{\alpha}_{j-1}(\varepsilon)+(k_j+\varepsilon)\hat{\alpha}_j(\varepsilon)-\beta_j\}+\overline{m}_j\{m_j\hat{\alpha}_j(\varepsilon)+ \\ +(k_{j+1}+\varepsilon)\hat{\alpha}_{j+1}(\varepsilon)-\beta_{j+1}\} = 0. \end{aligned}$$

The system $\{\hat{\alpha}_j(\varepsilon)\}$ fulfils (19), obviously, if

$$(20) \qquad m_{j-1}\hat{\alpha}_{j-1}(\varepsilon)+(k_j+\varepsilon)\hat{\alpha}_j(\varepsilon) = \beta_j$$

$$(20') \qquad m_j\hat{\alpha}_j(\varepsilon)+(k_{j+1}+\varepsilon)\hat{\alpha}_{j+1}(\varepsilon) = \beta_{j+1}$$

are satisfied. We note that (20) and (20′) are members of the same system of equations and (20′) is the equation (20) with the index increased by one. But system (20) has the same form as (15), only k_j has to be replaced by $k_j+\varepsilon$. So we see that the solution of (20), i.e. that of (19), is the following:

$$(21) \quad \hat{\alpha}_j(\varepsilon) = \frac{1}{k_j+\varepsilon}\left\{\beta_j-\frac{m_{j-1}}{k_{j-1}+\varepsilon}\beta_{j-1}+\ldots+(-1)^{j-1}\frac{m_1\ldots m_{j-1}}{(k_1+\varepsilon)\ldots(k_{j-1}+\varepsilon)}\beta_1\right\}.$$

We recognize at once, that $\hat{\alpha}_j(\varepsilon)$ are exactly the Fourier-coefficients of $A^+(\varepsilon)y$ by means of the system $\{x_j\}$. But if so, by the lemma

$$\begin{aligned} |\hat{\alpha}_j(\varepsilon)-\alpha_j| = |(A^+(\varepsilon)y, x_j)-(A^+y, x_j)| = |(A^+(\varepsilon)y-A^+y, x_j)| \leqq \\ \leqq \|A^+(\varepsilon)y-A^+y\| \to 0, \end{aligned}$$

i.e. $\hat{\alpha}_j(\varepsilon)\to\alpha_j$ uniformly with respect to j. Hence

$$\sum_{(j)} |\alpha_j(\varepsilon)|^2 = A^+(\varepsilon)\, y$$

is bounded with respect to ε. Obviously, as $x_\varepsilon\to A^+y$ [3], we see that

$$\|x_\varepsilon\|^2 = \sum_{(j)} |\alpha_j(\varepsilon)|^2 \to \|A^+y\|^2 \quad (\varepsilon\to 0)$$

which implies the existence of a constant Γ_1 such that

$$\sum_{(j)} |\alpha_j(\varepsilon)|^2 < \Gamma_1.$$

Let us now return to (14) and write

$$\alpha_j-\alpha_j(\varepsilon)=[\alpha_j-\hat{\alpha}_j(\varepsilon)]+[\hat{\alpha}_j(\varepsilon)-\alpha_j(\varepsilon)]. \tag{22}$$

Find first an estimate for $\alpha_j(\varepsilon)-\hat{\alpha}_j(\varepsilon)$.

Introducing the notation

$$\gamma_j(\varepsilon)=\hat{\alpha}_j(\varepsilon)-\alpha_j(\varepsilon) \tag{23}$$

we get from equations (17) and (18):

$$\begin{aligned} m_{j-1}k_j\gamma_{j-1}(\varepsilon)+(k_j^2+|m_j|^2)\gamma_j(\varepsilon)+\overline{m}_{j+1}k_{j+1}\gamma_{j+1}(\varepsilon) &= \\ =\varepsilon[\alpha_j(\varepsilon)-k_j\hat{\alpha}_j(\varepsilon)-\overline{m}_j\hat{\alpha}_{j+1}(\varepsilon)]. \end{aligned} \tag{24}$$

Let now be

$$\zeta_j(\varepsilon)=\alpha_j(\varepsilon)-k_j\hat{\alpha}_j(\varepsilon)-\overline{m}_j\hat{\alpha}_{j+1}(\varepsilon) \tag{25}$$

and so

$$|\zeta_j(\varepsilon)|^2\leqq(1+k_j^2+|m_j|^2\big(|\alpha_j(\varepsilon)|^2+|\hat{\alpha}_j(\varepsilon)|^2+|\hat{\alpha}_{j+1}(\varepsilon)|^2\big). \tag{26}$$

Let us consider now the fact that

$$0<k_j\leqq\|A\|,\quad |m_j|\leqq\|A\|,$$

[6 p. 161], then we obtain

$$\sum_{(j)}|\zeta_j(\varepsilon)|^2\leqq C_1\sum_{(j)}|\alpha_j(\varepsilon)|^2+C_2\sum_{(j)}|\hat{\alpha}_j(\varepsilon)|^2 \tag{27}$$

where C_1 and C_2 are positive numbers. Estimate (27) implies the existence of a unique element $\zeta(\varepsilon)\in X$ such that:

$$\big(\zeta(\varepsilon),x_j\big)=\alpha_j(\varepsilon)-k_j\hat{\alpha}_j(\varepsilon)-\overline{m}_j\hat{\alpha}_{j+1}(\varepsilon)\quad(j=1,2,\ldots). \tag{28}$$

Definition (23) yields

$$\sum_{(j)}|\gamma_j(\varepsilon)|^2\leqq 2\sum_{(j)}|\alpha_j(\varepsilon)|^2+2\sum_{(j)}|\hat{\alpha}_j(\varepsilon)|^2<\infty,$$

which shows the existence of a unique element $z(\varepsilon)\in X$ such that

$$\gamma_j(\varepsilon)=\big(z(\varepsilon),x_j\big)\quad(j=1,2,\ldots).$$

It is not difficult to see that the relation between $z(\varepsilon)$ and $\zeta(\varepsilon)$ is as follows:

$$(A^*A+\varepsilon I)z(\varepsilon)=\varepsilon\zeta(\varepsilon). \tag{29}$$

We have only to consider the scalar product of both sides of (29) with x_j ($j=1,2,\ldots$) and we get by (3_j) exactly the system (24). For $\varepsilon>0$ the operator $A^*A+\varepsilon I$ is obviously not singular, so we get by (29)

$$z(\varepsilon)=\varepsilon(A^*A+\varepsilon I)^{-1}\zeta(\varepsilon)$$

and then

$$\gamma_j(\varepsilon)=\big(z(\varepsilon),x_j\big)=\varepsilon\big((A^*A+\varepsilon I)^{-1}\zeta(\varepsilon),x_j\big)\quad(j=1,2,\ldots). \tag{30}$$

We will now prove that $\eta(\varepsilon)=(A^*A+\varepsilon I)^{-1}\zeta(\varepsilon)$ is bounded as ε varies. On the other hand, as we have seen,

$$\sum_{(J)} |\hat{\alpha}_j(\varepsilon)|^2 < \Gamma_2$$

independently from ε. Therefore by (28)

$$\sum_{(J)} \left|(\zeta(\varepsilon), x_j)\right|^2 = \sum_{(J)} |\alpha_j(\varepsilon)|^2 + C\sum_{(J)} |\hat{\alpha}_j(\varepsilon)|^2 \leqq \Gamma$$

for any $\varepsilon>0$ small enough. This means that $\|\zeta(\varepsilon)\|$ is bounded, which proves the boundedness of $\eta(\varepsilon)$. By this and (30) we have our partial result

$$\text{(31)} \qquad \sum_{(J)} |\hat{\alpha}_j(\varepsilon)-\alpha_j(\varepsilon)|^2 = \sum_{(J)} |\gamma_j(\varepsilon)|^2 = \varepsilon^2 \sum_{(J)} \left|(\eta(\varepsilon), x_j)\right|^2 = \varepsilon^2 \|\eta(\varepsilon)\|^2 \leqq C\varepsilon^2,$$

where C is a positive constant.

We have now estimate $\alpha_j-\hat{\alpha}_j(\varepsilon)$. By (16) and (21)

$$\text{(32)} \qquad \alpha_j-\hat{\alpha}_j(\varepsilon) = \frac{\varepsilon}{k_j+\varepsilon}\frac{1}{k_j}\Bigg\{\beta_j - \frac{k_j+k_{j-1}+\varepsilon}{(k_{j-1}+\varepsilon)k_{j-1}} m_{j-1}\beta_{j-1} + m_{j-1}m_{j-2}\times$$
$$\times\beta_{j-2}\frac{k_j-2^k_{j-1}+k_j-2^k_j+k_{j-1}k_j+\varepsilon(k_{j-2}+k_{j-1}+k_j)+\varepsilon^2}{(k_{j-2}+\varepsilon)k_{j-2}(k_{j-1}+\varepsilon)k_{j-1}} + \dots$$
$$\dots + (-1)^{j-1} m_1 m_2 \dots m_{j-1}\beta_1 \left[\frac{\sum_{s=1}^{j} k_1\dots k_{s-1}k_{s+1}\dots k_j}{(k_1+\varepsilon)k_1\dots(k_{j-1}+\varepsilon)k_{j-1}} + \right.$$
$$\left. + \frac{\varepsilon\sum_{s=1}^{j-1} k_1\dots k_{s-1}k_{s+1}\dots k_{j-1} + \dots + \varepsilon^j}{(k_1+\varepsilon)k_1\dots(k_{j-1}+\varepsilon)k_{j-1}}\right]\Bigg\}$$

holds. Let now be

$$\text{(33)} \qquad \varphi_j(\varepsilon) = \frac{1}{\varepsilon}(\alpha_j-\hat{\alpha}_j(\varepsilon))^2 \quad (\varepsilon \neq 0)$$

and

$$\text{(34)} \qquad \psi_j(\varepsilon)^2 = \frac{1}{\varepsilon}\varphi_j(\varepsilon) \quad (\varepsilon \neq 0).$$

Therefore we have:

$$\text{(35)} \qquad \psi_j(\varepsilon) = \frac{1}{(k_j+\varepsilon)k_j}\Bigg\{\beta_j - \frac{k_j+k_{j-1}+\varepsilon}{(k_{j-1}+\varepsilon)k_{j-1}} m_{j-1}\beta_{j-1} + \dots$$
$$\dots + (-1)^{j-1} m_1 \dots m_{j-1}\beta_1 \left[\frac{\sum_{s=1}^{j} k_1\dots k_{s-1}k_{s+1}\dots k_j}{(k_1+\varepsilon)k_1\dots(k_{j-1}+\varepsilon)k_{j-1}} + \right.$$
$$\left. + \frac{\varepsilon\sum_{s=1}^{j-1} k_1\dots k_{s-1}k_{s+1}\dots k_{j-1} + \dots + \varepsilon^j}{(k_1+\varepsilon)k_1\dots(k_{j-1}+\varepsilon)k_{j-1}}\right]\Bigg\}.$$

Now we observe that the following statements hold:

$$\varphi_j(0) = 0, \tag{36}$$

$$\varphi_j'(\varepsilon) = \psi_j(\varepsilon)^2 + 2\varepsilon\psi_j(\varepsilon)\psi_j'(\varepsilon), \quad \text{hence} \tag{37}$$

$$\varphi_j'(0) = \psi_j(0)^2 = k_j^{-4}\Big\{\beta_j - \frac{k_j+k_{j-1}}{(k_{j-1})^2} m_{j-1}\beta_{j-1} + \dots$$

$$\dots + (-1)^{j-1}\frac{m_1\dots m_{j-1}}{k_1^2\dots k_{j-1}^2}\sum_{s=1}^{j} k_1\dots k_{s-1}k_{s+1}\dots k_j\beta_1\Big\}^2$$

$$(\text{obviously } \varphi_j'(0) \geqq 0, \quad j = 1, 2, \dots).$$

If the following two conditions are satisfied

$$\sum_{(j)} \psi_j(0)^2 < \infty, \tag{38}$$

$$\varphi_j''(0) < 0 \quad (j = 1, 2, \dots), \tag{39}$$

then for ε small enough we obtain

$$\varphi_j(\varepsilon) \leqq \varepsilon\varphi_j'(0) \quad (j = 1, 2, \dots). \tag{40}$$

We emphatize that condition (38) is the hypothesis (8) of our theorem, hence let us only calculate φ_j'' (0).

$$\varphi_j''(0) = 4\psi_j(0)\psi_j'(0) = \frac{4}{k_j^2}\Big\{\beta_j - \frac{k_j+k_{j-1}}{(k_{j-1})^2} m_{j-1}\beta_{j-1} + \dots \tag{41}$$

$$\dots + (-1)^{j-1}\frac{m_1\dots m_{j-1}}{k_1^2\dots k_{j-1}^2}\sum_{s=1}^{j} k_1\dots k_{s-1}k_{s+1}\dots k_j\beta_1\Big\}\times$$

$$\times\Big\{-\frac{1}{k_j^3}\Big[\beta_j - \frac{k_j+k_{j-1}}{(k_{j-1})^2} m_{j-1}\beta_{j-1} + \dots + (-1)^{j-1}\frac{m_1\dots m_{j-1}}{k_1^2\dots k_{j-1}^2}\beta_1\times$$

$$\times\sum_{s=1}^{j-1} k_1\dots k_{s-1}k_{s+1}\dots k_{j-1}\Big] + \frac{1}{k_j^2}[\dots]\Big\} \overset{\text{def}}{=} \frac{4}{k_j}\{-\psi_j(0)^2 + b_j\}.$$

Now we must distinguish between the following two cases:

$$\text{I}^\circ \quad k_j \geqq c > 0 \quad j = 1, 2, \dots \qquad \text{II}^\circ \quad k_j \to 0 \quad \text{for} \quad j \to \infty.$$

In the first, (31) and (32) yield directly the thesis, regardless of hypothesis (8). In the second case, as $k_j \to 0$, for j large enough $|b_j| < \psi_j(0)^2$. As eigenelements and eigenvalues have been chosen such that $k_j > 0$, we have from (41) $\varphi_j''(0) < 0$, that is (39) holds. Thus we have shown that

$$\sum_{(j)} |\alpha_j - \hat{\alpha}_j(\varepsilon)|^2 \leqq c_2\varepsilon^2. \tag{42}$$

Finally our thesis descends from (31) and (42). □

We recall that in the particular case of $m_j = 0$ $(j = 1, 2, \dots)$ an analogous result was shown in [4] for complex Hilbert spaces and for $y \in R(A) \oplus R(A)^\perp$.

References

[1] C. W. GROETSCH, Generalized Inverses of Linear Operators: Representation and Applications, *New York* 1977.

[2] A. N. TIKHONOV—V. Y. ARSENIN, Solutions of Ill-posed Problems, *Wiley Pubbl.* 1977.

[3] I. FENYŐ, Su un'estensione del teorema di Tikhonov, *Stochastica* **8** (1984), 219—228.

[4] I. FENYŐ—E. PAPARONI, Su una proprietà dell'inversa generalizzata, *Quaderno S 5/1984 del Dipartimento di Matematica dell'Università di Milano.*

[5] I. FENYŐ, Über die Darstellung linearen Operatoren im Hilbertraum, *Publ. Math. (Debrecen)* **25** (1978), 123—137.

[7] I. FENYŐ—H. W. STOLLE, Theorie und Praxis der linearen Integralgleichungen. Vol. I, *Basel* 1982.

[7] I. FENYŐ, Eine Darstellung der verallgemeinerten Inversen eines linearen Operatoren, *Ber. Math. Stat. Sektion Forschungs-Zentrum Graz. Ber. Nr. 117* (1979), 1—15.

[8] I. FENYŐ, An error estimate for approximate solutions of equations, In: *General Inequalities Vol. 3 Basel* 1983, 483—489.

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