Asymptotic first entrance distributions in birth and death processes

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SUMMARY

For birth and death processes with the state space $\mathbf{N}^0$, reflecting barrier at 0, birth rates $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ and death rates $\mu_j=\mu(j+\xi)$ $(j\in\mathbf{N})$, where $\lambda,\mu\geqq 0$, $\xi>0$, limit theorems for the distribution of the first entrance time into state n for $n\to\infty$ are presented. In addition, the limit distributions are stated for transition rates $\lambda_j=\lambda(j+1+\xi)$ $(j\in\mathbf{N}^0)$ and $\mu_j=\mu j$ $(j\in\mathbf{N})$.

1. Introduction

We consider time homogeneous birth and death processes with state space $\mathbf{N}^0=\{0, 1, \ldots\}$ and reflecting barrier at 0 which start from state 0 at time 0 with probability 1. For given transition rates, conditions for the existence and uniqueness of a corresponding transition matrix $\mathbf{P}=(P_{ij})$ are studied in great detail e.g. in FELLER (1940), LEDERMANN—REUTER (1954), KARLIN—MCGREGOR (1957a, b) and REUTER (1957). Particularly, for transition rates under consideration below, the existence and uniqueness are ensured. In addition, methods have been developed for finding the explicit form of the transition matrix. In this context see e.g. LEDERMANN—REUTER (1954), KARLIN—MCGREGOR (1958) and SAATY (1961a, chpt. 4). (In the last referred book there is given a comprehensive bibliography of theoretical and applied papers on birth and death processes.) For special cases see also HEATHCOTE—MOYAL (1959), KENDALL (1949) and SAATY (1961b).

For birth and death processes as described, for a given level, i.e. for some state $n\in\mathbf{N}$, the distribution of the random variable $T_n=\inf\{t>0|X_t=n\}$ (first entrance time into state n, first crossing of level n) is of special interest. Considering an auxiliary birth and death process with the finite state space $\{0, 1, \ldots, n\}$ and with the same transition rates up to state $n-1$, but with n being an absorbing barrier, let $\mathbf{P}^{(n)}=(P_{ij}^{(n)})$ denote its transition matrix. Then, the distribution of T_n for the original birth and death process is given by $P_{0n}^{(n)}$ which can be determined more or less explicitly by known methods (cf. the above referred papers). However, the usual representations for $P_{0n}^{(n)}$ are not informative concerning the dependence on the transition rates and for large n not satisfactorily effective for the numerical evaluation. Thus, in a natural way the question arises for the limit distribution of the distributions under consideration as n tends to infinity. We cannot expect an explicit general solution. However, in SOLOVYEV (1964) there is given a necessary and sufficient condition for this distribution to be an exponential one, see also GNEDENKO—BELYAYEV—SOLOVYEV (1969). The determination of this limit distribution for linear transition rates $\lambda_j=\lambda(j+1)$, $\mu_j=\mu(j+\xi)$ and $\lambda_j=\lambda(j+1+\xi)$, $\mu_j=\mu j$, respectively, is the objective of this paper. Note that the common special case $\xi=0$ was treated in EBERL (1974).

For $n\in\mathbf{N}$ let P_n denote the distribution of T_n and Φ_n its Laplace transform. Then it is well known that the reciprocals $Q_n=1/\Phi_n$ of Φ_n satisfy the following recursion formula

$$\lambda_n Q_{n+1}(s) = (\lambda_n+\mu_n+s)\, Q_n(s) - \mu_n Q_{n-1}(s) \quad (n\in\mathbf{N}) \tag{1.1}$$

with the initial conditions $Q_0\equiv 1$, $Q_1(s)=1+s/\lambda_0$. From this relation it is directly seen that Q_n is a polynomial of degree n. Furthermore, considering polynomials satisfying such a recurrence relation, FAVARD (1935) proved that these polynomials constitute an orthogonal system, see also KARLIN—MCGREGOR (1957a) and LEDERMANN—REUTER (1954). From the orthogonality and the recurrence relation (1.1) it follows that the polynomials Q_n possess exactly n simple negative (real) zeroes such that the zeroes of Q_n and of Q_{n+1} separate one another. These zeroes are relevant for determining P_n itself. (But, it is tedious to compute again and again the zeroes of Q_n for different n.)

The above recurrence relation will be used persistently. Moreover, we introduce the following notations: E_n denotes the expectation of T_n, σ_n^2 its variance. For these moments the following general formulas are valid (cf. GNEDENKO—BELYAYEV—SOLOVYEV, 1969, or EBERL, 1972):

$$E_0 = 0, \quad E_n = Q_n'(0) \quad (n\in\mathbf{N}) \tag{1.2}$$

$$\sigma_n^2 = E_n^2 - Q_n''(0) \quad (n\in\mathbf{N}) \tag{1.3}$$

$$E_n = \sum_{j=1}^{n} D_j \quad \text{with} \quad D_j = \sum_{m=0}^{j-1} \frac{1}{\lambda_m}\alpha_{m+1}\cdot\ldots\cdot\alpha_{j-1} \tag{1.4}$$

where $\alpha_k=\lambda_0\cdot\ldots\cdot\lambda_{k-1}/(\mu_1\cdot\ldots\cdot\mu_k)$ $(n\geqq 2)$

$$\sigma_n^2 = \sum_{j=1}^{n} D_j^2 + 2\sum_{m=1}^{n-1} D_m^2 \sum_{k=1}^{n-m} \alpha_m\cdot\ldots\cdot\alpha_{m+k-1} \quad (n\geqq 2). \tag{1.5}$$

The generating function Q of the polynomials Q_n, defined by

$$Q(z,s) = \sum_{n=0}^{\infty} Q_n(s) z^n, \tag{1.6}$$

will be an essential tool for deriving the limit distributions. For the transition rates under consideration below, it is readily seen that for any $s_0>0$ there exists some $r_0=r(s_0)>0$ such that the right hand side of (1.6) converges for all complex z and s with $|z|<r_0$ and $|s|<s_0$ (cf. EBERL, 1972). The generating function will be found as a solution of a differential equation where we have to distinguish $\mu=\lambda$ and $\mu<\lambda$. We do not treat the case $\mu>\lambda$; by GNEDENKO—BELYAYEV—SOLOVYEV (1969), p. 351, it follows that T_n/E_n is asymptotically distributed according to an exponential law in this case.

II. Limit distributions for $\lambda_j=\lambda(j+1)$, $\mu_j=\lambda(j+\xi)$

Subsequently, we use the following notations: For $a\in\mathbf{R}$ let

$$C_{n,a}=\binom{n+a}{n}=(1+a)_n/n! \quad (n\in\mathbf{N}^0)$$

where

$$(a)_0=1, \quad (a)_n=a\cdot\ldots\cdot(a+n-1) \quad (n\in\mathbf{N}).$$

Further, for $a\in\mathbf{R}$ and $n\in\mathbf{N}$

$$L_n^a(s)=\sum_{m=0}^{n}\binom{n+a}{n-m}(-s)^m/m!$$

is the generalized Laguerre polynomial of order a and degree n.

First we determine the polynomials Q_n. From their explicit form it becomes clear that λ represents a factor of scale, only.

Theorem 2.1. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\lambda(j+\xi)$ $(j\in\mathbf{N})$ *where* $\lambda\in(0,\infty)$ *and* $\xi\in[0,\infty)$. *Then the polynomials* Q_n *are given by*

$$(2.1) \qquad Q_n(s)=L_n^{\xi}\left(-\frac{s}{\lambda}\right)-\xi\sum_{k=1}^{n}\frac{1}{k}L_{k-1}^{-\xi}\left(\frac{s}{\lambda}\right)L_{n-k}^{\xi}\left(-\frac{s}{\lambda}\right)=$$

$$=\sum_{m=0}^{n}\frac{(s/\lambda)^m}{m!}C_{n-m,m+\xi}-\xi\sum_{m=0}^{n-1}\left(\frac{s}{\lambda}\right)^m\sum_{j=0}^{m}\frac{(-1)^j}{j!(m-j)!}\sum_{k=1}^{n}\frac{1}{k}C_{k-1,j-\xi}C_{n-k,m-j+\xi}.$$

Proof. From the recurrence relation (1.1) we get for the transition rates under consideration

$$\text{(i)} \qquad \lambda(n+1)Q_{n+1}(s)=[\lambda(2n+\xi+1)+s]Q_n(s)-\lambda(n+\xi)Q_{n-1}(s) \quad (n\in\mathbf{N})$$

with the initial conditions $Q_0\equiv 1$ and $Q_1(s)=1+s/\lambda$. Multiplying both sides of (i) with z^n and summing over all $n\in\mathbf{N}$, we obtain for the generating function Q the differential equation

$$\frac{\partial}{\partial z}Q(z,s)=\left[\frac{1+\xi}{1-z}+\frac{s}{\lambda(1-z)^2}\right]\cdot Q(z,s)-\xi/(1-z)^2$$

with the initial condition $Q(0,s)=Q_0(s)\equiv 1$.

The solution of this differential equation can be found by standard methods; it is given by

$$Q(z,s)=(1-z)^{-(1+\xi)}\exp\left\{\frac{sz}{(1-z)}\right\}\cdot\left[1-\xi\int_0^z(1-t)^{\xi-1}\exp\left\{\frac{st}{\lambda(t-1)}\right\}dt\right].$$

By means of Buchholz (1969), p. 138 (11a), this implies

$$Q(z,s)=\sum_{n=0}^{\infty}L_n^{\xi}\left(-\frac{s}{\lambda}\right)z^n-\xi\sum_{k=0}^{\infty}L_k^{\xi}\left(-\frac{s}{\lambda}\right)z^k\cdot\sum_{j=}^{\infty}L_{j-1}^{-\xi}\left(\frac{s}{\lambda}\right)\frac{z^j}{j}.$$

Therefore we have by the definition of Q

$$Q_n(s) = L_n^{\xi}\left(-\frac{s}{\lambda}\right) - \xi \sum_{k=1}^{n} \frac{1}{k} L_{k-1}^{-\xi}\left(\frac{s}{\lambda}\right) L_{n-k}^{\xi}\left(-\frac{s}{\lambda}\right)$$

which proves the first equality of (2.1). Using BUCHHOLZ (1969), p. 142 (18), and having in mind that $L_{k-1}^{-\xi} L_{n-k}^{\xi}$ is a polynomial of degree $n-1$, we get

$$L_{k-1}^{-\xi}\left(\frac{s}{\lambda}\right) L_{n-k}^{\xi}\left(-\frac{s}{\lambda}\right) = \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} L_{k-1}^{j-\xi}(0) \cdot \sum_{j=0}^{\infty} \frac{s^j}{j!} L_{n-k}^{j+\xi}(0) =$$

$$= \sum_{m=0}^{n-1} \left(\frac{s}{\lambda}\right)^m \sum_{j=0}^{m} \frac{(-1)^j}{j!\,(m-j)!} L_{k-1}^{j-\xi}(0) L_{n-k}^{m-j+\xi}(0)$$

which due to $L_n^a(0) = C_{n,a}$ implies the second equality of (2.1). □

By means of (1.2) and (1.3) we can derive from Theorem 2.1 the expectations and variances in terms of n, ξ and λ. First we provide a combinatorial lemma.

Lemma 2.2. *For* $n, j, m \in \mathbf{N}^0$ *and* $\xi \in \mathbf{R}$ *it holds*

$$(j-\xi) \sum_{k=1}^{n} \frac{1}{k} C_{k-1,\,j-\xi} C_{n-k,\,m-j+\xi} = C_{n,m} - C_{n,\,m-j+\xi}.$$

PROOF. The statement follows from the identities

$$\frac{j-\xi}{k} C_{k-1,\,j-\xi} = C_{k,\,j-\xi-1}$$

$$\sum_{k=1}^{n} C_{k,\,j-\xi-1} C_{n-k,\,m-j+\xi} = C_{n,m} - C_{n,\,m-j+\xi}$$

where the second is a consequence of GOULD (1972), p. 22 (3.2). □

Theorem 2.3. *Under the assumptions of Theorem* 2.1 *it holds for* $n \geq 2$:

$$\lambda E_n = \begin{cases} (n+1-C_{n,\xi})/(1-\xi) & \text{for } \xi \neq 1 \\ (n+1) \sum_{k=1}^{n} \frac{1}{k} - n & \text{for } \xi = 1 \end{cases} \tag{2.2}$$

(2.3)

$$\lambda^2 \sigma_n^2 = \begin{cases} \frac{C_{n,\xi}^2}{(1-\xi)^2} - \frac{4n\xi C_{n,\xi}}{(1+\xi)(1-\xi)^2} - \frac{2C_{n,\xi}}{(1-\xi)^2(2-\xi)} + \frac{(n+1)(n+\xi)}{(1-\xi)^2(2-\xi)} & \text{for } 1 \neq \xi \neq 2 \\ (n+1)^2 \left(\sum_{k=1}^{n} \frac{1}{k}\right)^2 - (n+1)(3n+2) \sum_{k=1}^{n} \frac{1}{k} + \frac{7n(n+1)}{2} & \text{for } \xi = 1 \\ \frac{n}{12}(3n^3 + 2n^2 - 27n - 38) + (n+1)(n+2) \sum_{k=1}^{n} \frac{1}{k} & \text{for } \xi = 2. \end{cases}$$

PROOF. Fix $n \geqq 2$. Using BUCHHOLZ (1969), p. 136 (8), we derive from Theorem 2.1 the relation

(i)

$$\lambda E_n = \lambda Q_n'(0) = L_{n-1}^{1+\xi}(0) - \xi \sum_{k=1}^{n} \frac{1}{k} L_{k-1}^{-\xi}(0)\, L_{n-k}^{1+\xi}(0) + \xi \sum_{k=1}^{n} \frac{1}{k} L_{k-1}^{1-\xi}(0)\, L_{n-k}^{\xi}(0) =$$

$$= C_{n-1,1+\xi} - \xi \sum_{k=1}^{n} \frac{1}{k} C_{k-1,-\xi} C_{n-k,1+\xi} + \xi \sum_{k=1}^{n} \frac{1}{k} C_{k-1,1-\xi} C_{n-k,\xi}.$$

Now we consider first the case $\xi \neq 1$; then Lemma 2.2 implies

$$\lambda E_n = C_{n-1,1+\xi} + C_{n,1} - C_{n,1+\xi} + \frac{\xi}{1-\xi}[C_{n,1} - C_{n,\xi}] =$$

$$= C_{n-1,1+\xi} - C_{n,1+\xi} + (C_{n,1} - \xi C_{n,\xi})/(1-\xi).$$

Due to $C_{n,a} - C_{n-1,a} = C_{n,a-1}$ this leads to (2.2) for $\xi \neq 1$. For $\xi = 1$ we get from (i)

$$\lambda E_n = C_{n-1,2} - \sum_{k=1}^{n} \frac{1}{k} C_{k-1,-1} C_{n-k,2} + \sum_{k=1}^{n} \frac{1}{k} C_{n-k,1} =$$

$$= \sum_{k=1}^{n} \frac{1}{k} \frac{(2)_{n-k}}{(n-k)!} = \sum_{k=1}^{n} \frac{n-k+1}{k} = (n+1) \sum_{k=1}^{n} \frac{1}{k} - n,$$

which verifies (2.2) for $\xi = 1$. To prove (2.3), we first choose $1 \neq \xi \neq 2$. From Theorem 2.1 it follows with the aid of Lemma 2.2 and BUCHHOLZ (1969), p. 136 (8), that

(ii)

$$\lambda^2 Q_n''(0) = L_{n-2}^{2+\xi}(0) - 2\xi \sum_{j=0}^{2} \frac{(-1)^j}{j!(2-j)!} \sum_{k=1}^{n} \frac{1}{k} L_{k-1}^{j-\xi}(0)\, L_{n-k}^{2-j+\xi}(0) =$$

$$= C_{n-2,2+\xi} + C_{n,2} - C_{n,2+\xi} + 2\xi(C_{n,2} - C_{n,1+\xi})/(1-\xi) - \xi(C_{n,2} - C_{n,\xi})/(2-\xi).$$

After elementary transformations this leads to

$$\lambda^2 Q_n''(0) = C_{n-2,2+\xi} - C_{n,2+\xi} + 2C_{n,2}/[(1-\xi)(2-\xi)] + \xi(C_{n,\xi} - 2C_{n,1+\xi})/(1-\xi).$$

Thus, by (1.3) and (2.2) it follows that

(iii)

$$\lambda^2 \sigma_n^2 = \lambda^2 E_n^2 - \lambda^2 Q_n''(0) = C_{n,2+\xi} - C_{n-2,2+\xi} + 2\xi C_{n,1+\xi}/(1-\xi) +$$

$$+ [(n+1)^2 + C_{n,\xi}^2 - 2(n+1) C_{n,\xi}]/(1-\xi)^2 - C_{n,\xi}/(2-\xi) - 2C_{n,2}/[(1-\xi)(2-\xi)].$$

Now, from (iii) by elementary, but tedious combinatorial transformations which are omitted the validity of (2.3) for $1 \neq \xi \neq 2$ is established. In the cases $\xi = 1$ and $\xi = 2$, respectively, the only difference is that the terms

$$\sum_{k=1}^{n} \frac{1}{k} L_{k-1}^{j-\xi}(0)\, L_{n-k}^{2-j+\xi}(0) = \sum_{k=1}^{n} \frac{1}{k} C_{k-1,j-\xi} C_{n-k,2-j+\xi}$$

in (ii) for $j=\xi=1$ and $j=\xi=2$, respectively, cannot be evaluated according to Lemma 2.2; instead one verifies that

$$\sum_{k=1}^{n} \frac{1}{k} C_{k-1,0} C_{n-k,2} = \frac{1}{2}(n+1)(n+2) \sum_{k=1}^{n} \frac{1}{k} - \frac{n}{4}(3n+5).$$

Then, the rest of the proof goes through quite analogously to the former case. □

Of course, subsequently our interest will not focus on the explicit expressions given in Theorem 2.3. However, the behaviour of E_n and σ_n^2 for $n\to\infty$ will be of essential importance.

Corollary 2.4. *Under the assumptions of Theorem* 2.1 *it holds:*

$$\lambda E_n \sim^{1)} \begin{cases} n/(1-\xi) & \text{for } 0 \leqq \xi < 1 \\ n \ln n & \text{for } \xi = 1 \\ n^{\xi}/[(\xi-1)\Gamma(1+\xi)] & \text{for } 1 < \xi < \infty \end{cases} \tag{2.4}$$

$$\lambda^2 \sigma_n^2 \sim \begin{cases} n^2/[(1-\xi)^2(2-\xi)] & \text{for } 0 \leqq \xi < 1 \\ n^2 [\ln n]^2 & \text{for } \xi = 1 \\ n^{2\xi}/[(1-\xi)\Gamma(1+\xi)]^2 & \text{for } 1 < \xi < \infty. \end{cases} \tag{2.5}$$

The proof follows from Theorem 2.3 by the well known relations

$$C_{n,\xi} \sim \frac{n^{\xi}}{\Gamma(1+\xi)}, \quad \sum_{j=1}^{n} \frac{1}{j} \sim \ln n.$$

Determining the limit distribution now, let us consider first the case $\xi \geqq 1$. Then, due to Corollary 2.4 it holds that $\sigma_n^2 \sim E_n^2$. Therefore we obtain the following theorem.

Theorem 2.5. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\lambda(j+\xi)$ $(j\in\mathbf{N})$ *where* $\lambda\in(0,\infty)$ *and* $\xi\in[1,\infty)$. *Then the normalized first entrance times* $\tilde{T}_n=T_n/E_n$ *converge in distribution to the exponential distribution on* $(0,\infty)$ *with the parameter* 1.

Since $\sigma_n^2 \sim E_n^2$ is equivalent to the condition (6.4.27) on p. 350 of GNEDENKO—BELYAYEV—SOLOVYEV (1969) which is necessary and sufficient for the convergence of $\tilde{T}_n$ in distribution to an exponential law, Theorem 2.5 follows directly from Corollary 2.4.

From the just mentioned necessary and sufficient condition for convergence to an exponential law and Corollary 2.4 it gets clear that the limit distribution cannot be an exponential one for $0\leqq\xi<1$. From the proposition on p. 351 of GNEDENKO—BELYAYEV—SOLOVYEV (1969) it follows that the birth and death process does not become stationary in that case. For the birth and death processes of Theorem 2.5 it is readily verified that there exists a stationary distribution for $\xi>1$, but not for $\xi=1$.

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[1]) *We write* $f_n \sim g_n$ *iff* $\lim_{n\to\infty} f_n/g_n = 1$.

Now, we turn over to the case $0 \leqq \xi < 1$. In view of Corollary 2.4 we introduce the normalized first entrance times $\tilde{T}_n = T_n/n$ $(n \in \mathbf{N})$. For the corresponding polynomials $\tilde{Q}_n$, the reciprocals of the Laplace transforms $\tilde{\Phi}_n$ of $\tilde{T}_n$, which are given by

$$\tilde{Q}_0 = Q_0 \equiv 1, \quad \tilde{Q}_n(s) = Q_n(s/n) \quad (n \in \mathbf{N}),$$

the following lemma holds.

Lemma 2.6. *Under the assumptions of Theorem* 2.1 *it holds for* $0 \leqq \xi < 1$:

$$\tilde{Q}(s) = \lim_{n \to \infty} \tilde{Q}_n(s) = \sum_{m=0}^{\infty} \frac{(s/\lambda)^m}{m!(1-\xi)_m}.$$

Proof. By Theorem 2.1 and Lemma 2.2 we have for $n \in \mathbf{N}$

$$\tilde{Q}_n(s) = \sum_{m=0}^{n-1} \left(\frac{s}{\lambda n}\right)^m \left(\frac{C_{n-m,m+\xi}}{m!} - \xi \sum_{j=0}^{m} \frac{(-1)^j (C_{n,m} - C_{n,m-j+\xi})}{j!(m-j)!(j-\xi)}\right) + \frac{1}{n!}\left(\frac{s}{\lambda n}\right)^n.$$

Now, we split up the above sum into three sums and then determine their limits, separately:

$$\text{(i)} \qquad \tilde{Q}_n(s) = -\sum_{m=1}^{n-1} \left(\frac{s}{\lambda n}\right)^m \frac{1}{m!} (C_{n,m+\xi} - C_{n-m,m+\xi}) -$$

$$-\xi \sum_{m=0}^{n-1} \left(\frac{s}{\lambda n}\right)^m C_{n,m} \sum_{j=0}^{m} \frac{(-1)^j}{j!(m-j)!\,(j-\xi)} +$$

$$+\xi \sum_{m=1}^{n-1} \left(\frac{s}{\lambda n}\right)^m \sum_{j=1}^{m} \frac{(-1)^j}{j!(m-j)!(j-\xi)} C_{n,m-j+\xi} + \frac{1}{n!}\left(\frac{s}{\lambda n}\right)^n.$$

Let us start with the first sum. With the aid of Gould (1972), p. 7 (1.49), we obtain

$$\text{(ii)} \qquad 0 \leqq C_{n,m+\xi} - C_{n-m,m+\xi} = \sum_{k=n-m+1}^{n} C_{k,m-1+\xi} \leqq m C_{n,m-1+\xi}.$$

Further we have

$$C_{n,m-1+\xi} \leqq C_{n,m} = \frac{n^m}{m!} \prod_{k=1}^{m} \left(1 + \frac{k}{n}\right) \leqq \frac{(2n)^m}{m!}$$

and therefore

$$\text{(iii)} \qquad 0 \leqq \frac{C_{n,m+\xi} - C_{n-m,m+\xi}}{m!(\lambda n)^m} \leqq \frac{2^m}{\lambda^m (m-1)!\,m!}$$

with

$$\text{(iv)} \qquad \sum_{m=1}^{\infty} \frac{2^m}{\lambda^m (m-1)!\,m!} < \infty.$$

Since it holds for $m \in \mathbf{N}$

$$\lim_{n \to \infty} n^{-(m-1+\xi)} C_{n,m-1+\xi} = \lim_{n \to \infty} n^{-(m-1+\xi)} \frac{\Gamma(n+m+\xi)}{\Gamma(m+\xi)\,n!} = \frac{1}{\Gamma(m+\xi)},$$

it follows in virtue of $0 \leqq \xi < 1$

(v) $$\lim_{n\to\infty} n^{-m} C_{n,m-1+\xi} = 0 \quad (m\in\mathbf{N}).$$

As a consequence of Tannery's Theorem for series (cf. BROMWICH, 1931, p. 136), (ii), (iii), (iv) and (v) imply

(vi) $$\lim_{n\to\infty} \sum_{m=1}^{n-1} \left(\frac{s}{\lambda n}\right)^m \frac{1}{m!} (C_{n,m+\xi} - C_{n-m,m+\xi}) = 0.$$

For the second sum of (i) we use the identity

$$\sum_{j=0}^{m} \frac{(-1)^j}{j!(m-j)!(j-\xi)} = -\frac{1}{\xi C_{m,-\xi} m!} = -\frac{1}{\xi(1-\xi)_m}$$

where the first equality follows from GOULD (1972), p. 6 (1.47). Then, by similar arguments as for the first sum we get

(vii) $$\lim_{n\to\infty} -\xi \sum_{m=0}^{n-1} \left(\frac{s}{\lambda n}\right)^m C_{n,m} \sum_{j=0}^{m} \frac{(-1)^j}{j!(m-j)!(j-\xi)} =$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n-1} \left(\frac{s}{\lambda n}\right)^m \frac{C_{n,m}}{(1-\xi)_m} = \sum_{m=0}^{\infty} \frac{(s/\lambda)^m}{m!(1-\xi)_m}.$$

Finally, in virtue of

$$C_{n,m-j+\xi} \leqq C_{n,m-1+\xi} \quad (1 \leqq j \leqq m)$$

and of

$$\left| \sum_{j=1}^{m} \frac{(-1)^j}{j!(m-j)!(j-\xi)} \right| \leqq \frac{1}{(1-\xi)m!} \sum_{j=0}^{m} \binom{m}{j} = \frac{2^m}{(1-\xi)m!},$$

the third sum in (i) can be treated analogously like the first one which yields

(viii) $$\lim_{n\to\infty} \xi \sum_{m=1}^{n-1} \left(\frac{s}{\lambda n}\right)^m \sum_{j=1}^{m} \frac{(-1)^j}{j!(m-j)!(j-\xi)} C_{n,m-j+\xi} = 0.$$

Now, (vi), (vii) and (viii) verify the claim. □

Now we come to the actual limit theorem for the transition rates under consideration. This theorem represents an extension of Theorem 2.1.3 in EBERL (1974) which corresponds to the case $\xi=0$. At the same time, the proof given here is simpler and more illustrative than the former one for $\xi=0$.

As usual, let J_ν denote the Bessel function (of the first kind) of order ν. For $\nu\in(-1, \infty)$ all the zeroes of J_ν are simple, real and symmetric w.r.t. $z=0$. Let $j_{\nu,m}$ $(m\in\mathbf{N})$ designate the positive zeroes of J_ν, arranged in ascending order: $0<j_{\nu,1}<$ $<j_{\nu,2}<\ldots<j_{\nu,m}<\ldots$. With these notations the following theorem holds.

Theorem 2.7. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\lambda(j+\xi)$ $(j\in\mathbf{N})$ *where* $\lambda\in(0, \infty)$ *and* $\xi\in[0, 1)$. *Then the normalized first entrance times* $\tilde{T}_n=T_n/n$ *converge in distribution to the dis-*

tribution concentrated on $(0, \infty)$ *whose distribution function* $\tilde{F}$ *and density* $\tilde{f}$ *are given by*

(2.6) $$\tilde{F}(t) = 1 - \frac{2^{1+\xi}}{\Gamma(1-\xi)} \sum_{m=1}^{\infty} \frac{\exp\{-\lambda j^2_{-\xi,m} t/4\}}{j^{1+\xi}_{-\xi,m} J_{1-\xi}(j_{-\xi,m})} \quad (t>0)$$

(2.7) $$\tilde{f}(t) = \frac{\lambda}{2^{1-\xi}\Gamma(1-\xi)} \sum_{m=1}^{\infty} \frac{j^{1-\xi}_{-\xi,m} \exp\{-\lambda j^2_{-\xi,m} t/4\}}{J_{1-\xi}(j_{-\xi,m})} \quad (t>0).$$

Remark. As will get clear from the subsequent proof, $\tilde{F}$ and $\tilde{f}$ are the distribution function and the density, respectively, of the infinite convolution of the exponential distributions with the parameters $\lambda j^2_{-\xi,1}/4, \lambda j^2_{-\xi,2}/4, \dots$. Thus, starting from our birth and death model with the assumed transition rates, the limit distribution of the normalized first entrance times is found to be the (weak) limit of a sequence of general gamma distributions (with the above mentioned parameters). These distributions are of great importance for application fields like queuing theory, reliability theory or psychology and were treated for special parameters e.g. in LIKEŠ (1967, 1968) and McGILL—GIBBON (1965).

PROOF. We consider without loss of generality the case $\lambda=1$. If Λ_ν is the function defined by

(i) $$\Lambda_\nu(z) = \Gamma(1+\nu)(z/2)^{-\nu} J_\nu(z),$$

then we have

(ii) $$\Lambda_\nu(z) = \prod_{k=1}^{\infty} \left(1-(z/j_{\nu,k})^2\right)$$

(cf. ERDÉLYI II (1953), p. 61 (1)). With the help of the fundamental series representation of J_ν, the limit function $\tilde{Q}$ of Lemma 2.6 may be represented by

$$\tilde{Q}(s) = \Lambda_{-\xi}(2\sqrt{-s}) = \prod_{k=1}^{\infty} (1+4s/j^2_{-\xi,k}).$$

Therefore the Laplace transforms $\tilde{\Phi}_n$ of the distributions of $\tilde{T}_n$ converge to the function $\tilde{\Phi}$ given by

$$\tilde{\Phi}(s) = \prod_{k=1}^{\infty} \frac{j^2_{-\xi,k}/4}{s+j^2_{-\xi,k}/4}$$

which in the complex half-plane $\{s \mid \operatorname{Re} s > -j^2_{-\xi,1}/4\}$ is the Laplace transform of the infinite convolution of the exponential distributions with the parameters $j^2_{-\xi,1}/4, j^2_{-\xi,2}/4, \dots$. Thus, the random variables $\tilde{T}_n$ are converging in distribution to this infinite convolution. We determine the distribution function $\tilde{F}$ as the limit of the distribution functions of the corresponding finite convolutions. To this end, consider the distribution functions $\hat{F}_n$ of the (finite) convolution of the exponential distributions with the parameters $j^2_{-\xi,1}/4, \dots, j^2_{-\xi,n}/4$, which can be readily seen to be given by

(iii) $$\hat{F}_n(t) = 1 - \sum_{m=1}^{n} \Big[\prod_{\substack{1\le k\le n \\ k\ne m}} (1-j^2_{-\xi,m}/j^2_{-\xi,k})^{-1} \Big] \exp\{-j^2_{-\xi,m} t/4\} \quad (t>0)$$

(see e.g. JOHNSON—KOTZ (1970), p. 222). From (i) and (ii) we get

$$\Lambda'_{-\xi}(j_{-\xi,m}) = -\Gamma(1-\xi)(j_{-\xi,m}/2)^{\xi} J_{1-\xi}(j_{-\xi,m}) \quad (m\in\mathbf{N})$$

and

$$\Lambda'_{-\xi}(j_{-\xi,m}) = \frac{-2}{j_{-\xi,m}} \prod_{\substack{1\leq k<\infty \\ k\neq m}} (1-j^2_{-\xi,m}/j^2_{-\xi,k})$$

with $\Lambda'_{-\xi}(j_{-\xi,m})\neq 0$ (since the zeroes of $J_{-\xi}$ and $J_{1-\xi}$ separate one another). Thus, we have for $m\in\mathbf{N}$

$$\text{(iv)} \qquad \lim_{n\to\infty} \prod_{\substack{1\leq k\leq n \\ k\neq m}} (1-j^2_{-\xi,m}/j^2_{-\xi,k}) = \Gamma(1-\xi)\, j^{1+\xi}_{-\xi,m} J_{1-\xi}(j_{-\xi,m})/2^{1+\xi}$$

where due to the ascending order of the $j_{-\xi,k}$ it holds for $1\leq m\leq n<\infty$

$$\Big|\prod_{\substack{1\leq k\leq n \\ k\neq m}} (1-j^2_{-\xi,m}/j^2_{-\xi,k})\Big| \geq \Gamma(1-\xi)\, j^{1+\xi}_{-\xi,m} |J_{1-\xi}(j_{-\xi,m})|/2^{1+\xi}.$$

In virtue of $|J_{1-\xi}(j_{-\xi,m})| \sim \left|\cos\left(j_{-\xi,m}+\frac{\xi\pi}{2}-\frac{3\pi}{4}\right)\right| \sqrt{2/(\pi j_{-\xi,m})}$ and $j_{-\xi,m}\sim \sim\pi\left(m-\frac{\xi}{2}-\frac{1}{4}\right)$ (cf. WATSON (1966), p. 199 and p. 506) the Dirichlet series

$$\sum_{m=1}^{\infty} (j^{1+\xi}_{-\xi,m} J_{1-\xi}(j_{-\xi,m}))^{-1} \exp\{-j^2_{-\xi,m} t/4\}$$

converges absolutely for all $t\in(0,\infty)$. (In fact, it does not converge for $t=0$.) Therefore we may apply Tannery's Theorem for series (cf. BROMWICH (1931), p. 136) to get the limit in (iii) for $n\to\infty$, which together with (iv) yields the validity of (2.6). From the absolute convergence of the Dirichlet series in (2.6) for $t\in(0,\infty)$ follows its uniform convergence in any intervall $[T,\infty)$ with $T>0$. Hence, $\hat{F}$ can be differentiated term by term, which leads to the density stated in (2.7). □

If $\tilde{E}_n$ and $\tilde{E}$ denote the expectations of $\tilde{T}_n$ and of their limit distribution and $\tilde{\sigma}^2_n$, $\tilde{\sigma}^2$ the corresponding variances, then we obtain from Corollary 2.4 and Theorem 2.7 the following corollary.

Corollary 2.8. *Under the assumptions of Theorem* 2.7 *it holds*

$$\tilde{E} = \lim_{n\to\infty} \tilde{E}_n = \frac{1}{\lambda(1-\xi)}; \quad \tilde{\sigma}^2 = \lim_{n\to\infty} \tilde{\sigma}^2_n = \frac{1}{\lambda^2(1-\xi)^2(2-\xi)}.$$

Remark. By means of the representation of the limit distribution as the infinite convolution of the exponential distributions with the parameters $j^2_{-\xi,1}/4, j^2_{-\xi,2}/4, \ldots$ we obtain

$$\sum_{m=1}^{\infty} j^{-2}_{-\xi,m} = \frac{1}{4(1-\xi)}, \quad \sum_{m=1}^{\infty} j^{-4}_{-\xi,m} = \frac{1}{16(1-\xi)^2(2-\xi)}.$$

These identities for the zeroes of $J_{-\xi}$ have been proved by Rayleigh (1874) (cf. also Watson (1966), p. 502) who used them for computing zeroes of Bessel functions. Thus, we found a probabilistic verification of these known identities.

III. Limit distributions for $\lambda_j=\lambda(j+1)$, $\mu_j=\mu(j+\xi)$ with $\mu<\lambda$

The conclusions of this section represent generalizations of the corresponding ones in section II.2 of EBERL (1974) which cover the case $\xi=0$.

First we provide a lemma on the behaviour of E_n and σ_n^2 for $n\to\infty$.

Lemma 3.1. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\mu(j+\xi)$ $(n\in\mathbf{N})$ *where* $\lambda,\mu\in(0,\infty)$, $\xi\in[0,\infty)$ *and* $0<\alpha=\mu/\lambda<1$. *Then it holds:*

$$E_n\sim\frac{1}{\beta}\ln[(1-\alpha)n]\quad with\quad \beta=\lambda-\mu; \tag{3.1}$$

$$\sigma^2=\lim_{n\to\infty}\sigma_n^2<\infty. \tag{3.2}$$

PROOF. Using (1.4) we obtain for $j\in\mathbf{N}$

$$D_j=\frac{1}{\lambda}\sum_{k=0}^{j-1}\frac{(1+\xi)_{j-1}}{(1+\xi)_{j-1-k}}\frac{(j-1-k)!}{j!}\alpha^k. \tag{i}$$

From (1.4), (1.5) and (i) it is clear that D_j, E_n and σ_n^2 are nondecreasing w.r.t. ξ. Particularly, putting $\xi=0$ it follows that

$$D_j\geqq\frac{1}{\lambda j}\sum_{k=0}^{j-1}\alpha^k=(1-\alpha^j)/(\beta j)\quad(j\in\mathbf{N})$$

and

$$E_n\geqq\frac{1}{\beta}\sum_{j=1}^{n}\frac{1-\alpha^j}{j}\sim\frac{1}{\beta}\ln[(1-\alpha)n]. \tag{ii}$$

Next we will show that for fixed $\xi\in\mathbf{N}$ with $\xi\geqq2$ there exists a constant $K_{\alpha,\xi}>0$ (not depending on j) such that it holds

$$D_j\leqq\frac{1-\alpha^j}{\beta j}+\frac{K_{\alpha,\xi}}{(j-1)j}\quad(j\geqq2). \tag{iii}$$

Fixing any $\xi\in\mathbf{N}$ with $\xi\geqq2$ we have for $j\geqq2$

$$\frac{(1+\xi)_{j-1}}{(1+\xi)_{j-1-k}}\frac{(j-1-k)!}{j!}=\frac{(j-k+\xi)_k}{(j-k)_{k+1}}\quad(1\leqq k\leqq\min(\xi,j-1)). \tag{iv}$$

Now, $(j-k+\xi)_k$ can be represented in terms of powers of $j-k$, say

$$(j-k+\xi)_k=(j-k)^k+G_k(j-k)$$

with $G_k(j-k)=\sum_{\nu=0}^{k-1}g_{k\nu}(j-k)^\nu$ where the (positive) coefficients $g_{k\nu}$ depend on ξ, but not on j. Therefore it follows for $j \geqq 2$

$$\text{(v)} \qquad \begin{aligned} \frac{(j-k+\xi)_k}{(j-k)_{k+1}} &= \frac{(j-k)^k}{(j-k)_{k+1}} + \sum_{\nu=0}^{k-1} g_{k\nu} \frac{(j-k)^\nu}{(j-k)_{k+1}} \leqq \\ &\leqq \frac{1}{j} + \frac{1}{(j-1)j} \sum_{\nu=0}^{k-1} g_{k\nu} \quad (1 \leqq k \leqq \min(\xi, j-1)). \end{aligned}$$

Writing $g = \sum_{k=1}^{\min(\xi, j-1)} \sum_{\nu=0}^{k-1} g_{k\nu}$, (iv) and (v) imply for $j \geqq 2$

$$\text{(vi)} \quad \sum_{k=0}^{\min(\xi, j-1)} \frac{(1+\xi)_{j-1}}{(1+\xi)_{j-1-k}} \frac{(j-1-k)!}{j!} \alpha^k \leqq \left(\frac{1}{j} + \frac{g}{(j-1)j}\right) \sum_{k=0}^{\min(\xi, j-1)} \alpha^k.$$

Now, (i) and (vi) yield (iii) for $2 \leqq j \leqq \xi+1$. Considering the case $j \geqq \xi+2$, we obtain for $k \in \mathbf{N}$ with $\xi < k \leqq j-1$

$$\text{(vii)} \qquad \frac{(1+\xi)_{j-1}}{(1+\xi)_{j-1-k}} \frac{(j-1-k)!}{j!} = \frac{1}{j} \frac{(j)_\xi}{(j-k)_\xi}.$$

Further, it holds for $\xi < k \leqq j-1$

$$\text{(viii)} \qquad \begin{aligned} \frac{(j)_\xi}{(j-k)_\xi} &= 1 + \sum_{\nu=1}^{\xi} k^\nu \sum^{(\nu)} [(k+i_1) \cdot \ldots \cdot (k+i_\nu)]^{-1} \leqq \\ &\leqq 1 + \sum_{\nu=1}^{\xi} \frac{k^\nu}{(j)_\nu} \binom{\xi}{\nu} \leqq 1 + \frac{(1+k)^\xi}{j} \end{aligned}$$

where the sum $\sum^{(\nu)}$ extends over all integers $i_1, \ldots, i_\nu$ with $j-k \leqq i_1 < \ldots < i_\nu \leqq$ $\leqq j-k+\xi-1$. From (vii) and (viii) it follows that

$$\text{(ix)} \qquad \begin{aligned} &\sum_{k=\xi+1}^{j-1} \frac{(1+\xi)_{j-1}}{(1+\xi)_{j-1-k}} \frac{(j-1-k)!}{j!} \alpha^k \leqq \\ &\leqq \frac{1}{j} \sum_{k=\xi+1}^{j-1} \alpha^k + \frac{1}{j^2} \sum_{k=\xi+1}^{\infty} (1+k)^\xi \alpha^k. \end{aligned}$$

Inequality (iii) for $j \geqq \xi+2$ follows from (vi) and (ix). Now, (3.1) is an immediate consequence of (ii), (iii), (1.4) and the above stated monotony of E_n w.r.t. ξ.

To prove (3.2) we consider first the second sum of (1.5) for arbitrary $\xi \geqq 1$; for $n \geqq 2$ it holds

$$\sum_{m=1}^{n-1} D_m^2 \sum_{k=1}^{n-m} \alpha_m \cdot \ldots \cdot \alpha_{m+k-1} = \sum_{m=1}^{n-1} D_m^2 \sum_{k=1}^{n-m} \frac{(m+\xi)_k}{(m+1)_k} \alpha^k \leqq A \cdot \sum_{m=1}^{n-1} D_m^2$$

with $A = \sum_{k=1}^{\infty} \alpha^k (\xi)_k / k! < \infty$. Therefore (1.5) yields for $n \geqq 2$

$$\text{(x)} \qquad \sigma_n^2 \leqq [1+2A] \sum_{m=1}^{n} D_m^2.$$

Now, (iii) implies for any $\xi\in\mathbf{N}$ with $\xi\geqq 2$ that $\sum_{m=1}^{\infty} D_m^2<\infty$. Thus, (x) together with the above stated monotony of σ_n^2 w.r.t. ξ verifies (3.2). □

In view of Lemma 3.1 we introduce the normalized first entrance times $\tilde{T}_n=T_n-\frac{1}{\beta}\ln[(1-\alpha)n]$ $(n\in\mathbf{N})$. The reciprocals $\tilde{Q}_n$ of the Laplace transforms $\tilde{\Phi}_n$ of $\tilde{T}_n$ are given by

$$\tilde{Q}_n(s)=[(1-\alpha)n]^{-s/\beta}Q_n(s).$$

In the next lemma we state the limit $\tilde{\Phi}$ of the Laplace transforms $\tilde{\Phi}_n$.

Lemma 3.2. *Under the assumptions of Lemma 3.1 the Laplace transforms $\tilde{\Phi}_n$ of the normalized first entrance times $\tilde{T}_n=T_n-\frac{1}{\beta}\ln[(1-\alpha)n]$ are converging for all s with $\operatorname{Re} s>-\beta$ to the function $\tilde{\Phi}$ given by*

$$\tilde{\Phi}(s)=\frac{\Gamma(1+s/\beta)}{F(s/\beta,\xi;\ 1+s/\beta;\ \alpha)}.\text{*)} \tag{3.3}$$

PROOF. From the recurrence relation (1.1) we obtain for the generating function Q the differential equation

$$\frac{\partial}{\partial z}Q(z,s)-\left(\frac{1+s/\beta}{1-z}+\frac{\alpha(\xi-s/\beta)}{1-\alpha z}\right)Q(z,s)=\frac{-\alpha\xi}{(1-z)(1-\alpha z)}$$

with the initial condition $Q(0,s)\equiv 1$ whose solution turns out to be given by

$$Q(z,s)=I(z,s)(1-z)^{-(1+s/\beta)}(1-\alpha z)^{-\xi+s/\beta} \tag{i}$$

with

$$I(z,s)=1-\alpha\xi\int_0^z(1-t)^{s/\beta}(1-\alpha t)^{\xi-1-s/\beta}\,dt \tag{ii}$$

where the integral on the right-hand side of (ii) is taken along the straight line connecting 0 and z. (Note that the solution (i) of the above differential equation is obtained at first stage for all reals s, z with $s>-\beta$ and $|z|<1$. Then, with the help of analytic continuation it follows that (i) holds true for all complex s with $\operatorname{Re} s>-\beta$ and all complex z within the domain $\arg(1-z)<\pi$.) By Cauchy's integral formula it holds for $n\in\mathbf{N}$

$$\tilde{Q}_n(s)=\frac{[(1-\alpha)n]^{-s/\beta}}{2\pi i}\int_{\mathscr{C}}Q(z,s)z^{-(n+1)}\,dz \tag{iii}$$

where $\mathscr{C}$ is a circle around $z=0$ such that the singularities $z=1$ and $z=1/\alpha$ of $Q(z,s)$ are outside of $\mathscr{C}$. By means of the substitution $z=n/(w+n)$, (i), (ii) and

*) *As usual, $F(a,b;c;z)$ denotes the hypergeometric series $\sum_{k=0}^{\infty}(a)_k(b)_k z^k/[(c)_k k!]$.*

(iii) yield after some elementary transformations for $n \in \mathbf{N}$ the validity of

$$\text{(iv)} \qquad \tilde{Q}_n(s) = \frac{1}{2\pi i} \int_{\mathscr{C}_n} h_n(w, s) I(n/(w+n), s)\, dw$$

with

$$\text{(v)} \qquad h_n(w, s) = \frac{1}{w}\left(\frac{1}{w} + \frac{1}{n(1-\alpha)}\right)^{s/\beta} \left(1 + \frac{w}{n}\right)^n \left(1 - \frac{\alpha n}{w+n}\right)^{-\xi}$$

where $\mathscr{C}_n$ is a simple closed positively oriented contour enclosing the singularities $w=0$, $w=-(1-\alpha)n$ and $w=-n$ of the integrand. The next step consists in showing that it holds

$$\text{(vi)} \qquad \lim_{n\to\infty} \tilde{Q}_n(s) = \tilde{Q}(s) = \frac{(1-\alpha)^{-\xi} I(1, s)}{2\pi i} \int_{\mathscr{C}^*} w^{-(1+s/\beta)} e^w\, dw$$

where the contour $\mathscr{C}^*$ starts at $-\infty$ on the real axis, encircles once the origin in the positive direction and returns to its starting point. Note that the (improper) integral in (vi) exists and that it holds by Hankel's representation of $1/\Gamma$ (see ERDÉLYI I. (1953), p. 13)

$$\text{(vii)} \qquad \frac{1}{2\pi i} \int_{\mathscr{C}^*} w^{-(1+s/\beta)} e^w\, dw = \frac{1}{\Gamma(1+s/\beta)}.$$

Without going into elementary, but tedious technical details we sketch the conclusions leading to (vi), only. To this end, fix some $\varrho \in (0, 1/2)$. For any $w_0 > 1$ let $\mathscr{C}_0^*$ consist of the straight line connecting $-w_0 - i\varrho$ and $-i\varrho$, the semi-circle $|w| = \varrho$ with $\operatorname{Re} w > 0$ and the straight line connecting $i\varrho$ and $-w_0 + i\varrho$. Further, for $n \in \mathbf{N}$ the path of integration $\mathscr{C}_n$ in (iv) is specified properly by choosing $\mathscr{C}_n$ in the same way as $\mathscr{C}_0^*$ with $w_0 = n + \varrho$ and connecting the end points $-(n+\varrho) + i\varrho$ and $-(n+\varrho) - i\varrho$ by the straight line between them. Then, for $\tilde{Q}(s)$ being given by the right-hand side of (vi) it holds for any $w_0 \in (1, \infty)$ and any $n \in \mathbf{N}$ with $n > w_0$

$$2\pi|\tilde{Q}(s) - \tilde{Q}_n(s)| \leq (1-\alpha)^{-\xi} |I(1, s)| \left| \int_{\mathscr{C}^*} w^{-(1+s/\beta)} e^w\, dw - \int_{\mathscr{C}_0^*} w^{-(1+s/\beta)} e^w\, dw \right| +$$

$$\text{(viii)} \quad + \left| \int_{\mathscr{C}_0^*} \left((1-\alpha)^{-\xi} I(1, s) w^{-(1+s/\beta)} e^w - h_n(w, s) I\left(\frac{n}{w+n}, s\right) \right) dw \right| +$$

$$+ \left| \int_{\mathscr{C}_n - \mathscr{C}_0^*} h_n(w, s) I\left(\frac{n}{w+n}, s\right) dw \right|.$$

Now, by the existence and finiteness of the (improper) integral over $\mathscr{C}^*$, the first term on the right-hand side of (viii) can be made arbitrarily small, if w_0 is chosen large enough. The second term for any fixed w_0 tends to 0 for $n \to \infty$, since it holds

$$\lim_{n\to\infty} h_n(w, s) I(n/(w+n), s) = (1-\alpha)^{-\xi} I(1, s) w^{-(1+s/\beta)} e^w$$

uniformly (w.r.t. w) on the compact path of integration $\mathscr{C}_0^*$. Note that for $\operatorname{Re} s > -\beta$ and $z=1$ the (improper) integral occurring in $I(1, s)$ is given by

$$\text{(ix)}\quad \int_0^1 (1-t)^{s/\beta}(1-\alpha t)^{\xi-1-s/\beta}\,dt = \frac{1}{1+s/\beta}F(1, 1+s/\beta-\xi; 2+s/\beta; \alpha)$$

(cf. Erdélyi I (1953), p. 59 (10)). For the remaining term on the right-hand side of (viii) one gets an appropriate upper bound for the absolute value of the integrand which ensures that the integral can be made arbitrarily small, if w_0 is chosen large enough. Now we get from (ii), (vi), (vii) and (ix) that it holds for $\operatorname{Re} s > -\beta$

(x)

$$\lim_{n\to\infty} \tilde{Q}_n(s) = \tilde{Q}(s) = \frac{(1-\alpha)^{-\xi}}{\Gamma(1+s/\beta)}\left(1-\frac{\alpha\xi}{1+s/\beta}F(1, 1+s/\beta-\xi; 2+s/\beta; \alpha)\right).$$

Expanding $(1-\alpha)^{-\xi}$ in a binomial series and with the help of Erdélyi I (1953), p. 64 (23), we obtain

$$(1-\alpha)^{-\xi}\left(1-\frac{\alpha\xi}{1+s/\beta}F(1, 1+s/\beta-\xi; 2+s/\beta; \alpha)\right) =$$

$$= \sum_{k=0}^{\infty}\frac{(\xi)_k}{k!}\alpha^k - \frac{\alpha\xi}{1+s/\beta}F(1+s/\beta, 1+\xi; 2+s/\beta; \alpha) =$$

(xi)

$$= 1+\sum_{k=1}^{\infty}\frac{(\xi)_k}{k!}\alpha^k\left(1-\frac{(1+s/\beta)_{k-1}}{(1+s/\beta)_k}\right) =$$

$$= 1+\sum_{k=1}^{\infty}\frac{(\xi)_k}{k!}\frac{s/\beta}{s/\beta+k}\alpha^k = F(s/\beta, \xi; 1+s/\beta; \alpha)$$

for $\operatorname{Re} s > -\beta$. Finally, (x) and (xi) ensure the validity of the lemma. □

The limit function $\tilde{\Phi} = \lim \tilde{\Phi}_n$ of the Laplace transforms of $\tilde{T}_n$ stated in Lemma 3.2 is continuous on the imaginary axis. Therefore it follows by the celebrated continuity theorem for characteristic functions that the normalized first entrance times $\tilde{T}_n$ are converging in distribution and that $\tilde{\Phi}$ is the Laplace transform of the limit distribution. For $\xi=0$ the limit function $\tilde{\Phi}$ is given by $\tilde{\Phi}(s) = \Gamma(1+s/\beta)$ which for $\operatorname{Re} s > -\beta$ is the Laplace transform of the distribution with the Lebesgue density

$$\tilde{f}(t) = \beta e^{-\beta t}\exp\{-e^{-\beta t}\}\ (t\in\mathbf{R}).$$

(Note that this case was treated in Eberl (1974).) In the case $\xi > 0$ we apply the method provided in Eberl (1983) for the inversion of meromorphic functions of the form

$$\Phi(s) = \Gamma(s)\left[\sum_{j=0}^{\infty} a_j/(s+j)\right]^{-1}$$

with nonnegative coefficients a_j $(j\in\mathbf{N}^0)$ such that $0<\sum_{j=0}^{\infty} a_j<\infty$. To this end, we introduce the meromorphic function

$$G(s)=\sum_{j=0}^{\infty} a_j/(s+j)$$

with the coefficients $a_j=(\xi)_j\alpha^j/j!$ $(j\in\mathbf{N}^0)$. Since it holds $a_j>0$ $(j\in\mathbf{N}^0)$ and $0<\sum_{j=0}^{\infty} a_j=(1-\alpha)^{-\xi}<\infty$, Theorem 1 of EBERL (1983) implies that the only zeroes of G are located in the intervals $(-m, -m+1)$ and that in any such interval there is exactly one simple zero s_m of G $(m\in\mathbf{N})$. The zeroes of G are the only singularities of $\Phi=\Gamma/G$. Using the representation of $\tilde{\Phi}$ by

$$\tilde{\Phi}(s)=\frac{\Gamma(1+s/\beta)}{F(s/\beta,\xi;1+s/\beta;\alpha)}=\frac{\Gamma(s/\beta)}{G(s/\beta)}=\Phi(s/\beta)$$

we obtain as an application of Theorem 2 in Eberl (1983) the following theorem.

Theorem 3.3. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\mu(j+\xi)$ $(j\in\mathbf{N})$ *where* $\lambda,\mu,\xi\in(0,\infty)$ *and* $0<\alpha=\mu/\lambda<1$. *Then, the normalized first entrance times* $\tilde{T}_n=T_n-\frac{1}{\beta}\ln[(1-\alpha)n]$ *with* $\beta=\lambda-\mu$ *are converging in distribution to the limit distribution with the Lebesgue density* $\tilde{f}$ *given by*

$$\tilde{f}(t)=-\beta\sum_{m=1}^{\infty}\Gamma(s_m)e^{\beta t s_m}\left\{\sum_{j=0}^{\infty}(\xi)_j\alpha^j/[j!(j+s_m)^2]\right\}^{-1}\qquad(t\in\mathbf{R}).$$

The following remark is of relevance for the numerical evaluation of the density $\tilde{f}$ of the limit distribution.

Remark. With the aid of Theorem 1(b) of EBERL (1983) it follows that

$$0<m+s_m<c_m=\frac{(\xi)_m\alpha^m(1-\alpha)^{\xi}}{(m-1)!}F(m+\xi,1;m;\alpha)$$

holds for $m\in\mathbf{N}$ where it is readily verified that $c_m\sim(\xi)_m\alpha^m(1-\alpha)^{\xi-1}/(m-1)!$. Particularly, we obtain $s_m\sim -m$.

IV. Limit distributions for $\lambda_j=\lambda(j+1+\xi)$, $\mu_j=\mu j$

This final section is devoted to the determination of the limit distributions of the (normalized) first entrance times for transition rates $\lambda_j=\lambda(j+1+\xi)$ and $\mu_j=\mu j$. As indicated at the end of the first section, the stationary case $\mu>\lambda$ is not to be considered, since in this case T_n/E_n is asymptotically distributed according to the exponential law on $(0,\infty)$ with parameter 1.

We start with the case $\mu=\lambda$. We call back to mind that $j_{\nu,m}$ $(m\in\mathbf{N})$ denote the positive zeroes of J_ν, the Bessel function of the first kind of order $\nu\in(-1,\infty)$ (arranged in ascending order).

Theorem 4.1. *Let be given a birth and death process with transition rates* $\lambda_j=\lambda(j+1+\xi)$ $(j\in\mathbf{N}^0)$ *and* $\mu_j=\lambda j$ $(j\in\mathbf{N})$ *where* $\lambda\in(0,\infty)$ *and* $\xi\in[0,\infty)$. *Then, the normalized first entrance times* $\tilde{T}_n=T_n/n$ *converge in distribution to the distribution concentrated on* $(0,\infty)$ *whose distribution function* $\tilde{F}$ *and density* $\tilde{f}$ *are given by*

$$(4.1)\qquad \tilde{F}(t)=1-\frac{2^{1-\xi}}{\Gamma(1+\xi)}\sum_{m=1}^{\infty}\frac{\exp\{-\lambda j_{\xi,m}^2\,t/4\}}{j_{\xi,m}^{1-\xi}J_{1+\xi}(j_{\xi,m})}\qquad (t>0)$$

$$(4.2)\qquad \tilde{f}(t)=\frac{\lambda}{2^{1+\xi}\Gamma(1+\xi)}\sum_{m=1}^{\infty}\frac{j_{\xi,m}^{1+\xi}\exp\{-\lambda j_{\xi,m}^2\,t/4\}}{J_{1+\xi}(j_{\xi,m})}\qquad (t>0).$$

Proof. We introduce the polynomials

$$R_n=\binom{n+\xi}{n}Q_n=\frac{(1+\xi)_n}{n!}Q_n\qquad (n\in\mathbf{N}^0).$$

Then, the recurrence relation (1.1) for the transition rates under consideration may be rewritten as

$$\text{(i)}\qquad (n+1)R_{n+1}(s)=[2n+\xi+1+s/\lambda]R_n(s)-(n+\xi)R_{n-1}(s)$$

with the initial conditions $R_0\equiv 1$, $R_1(s)=1+\xi+s/\lambda$. Comparing (i) with the recurrence formula satisfied by the Laguerre polynomials it follows $R_n(s)=L_n^{\xi}(-s/\lambda)$ and therefore

$$\text{(ii)}\qquad Q_n(s)=L_n^{\xi}(-s/\lambda)\Big/\binom{n+\xi}{n}\qquad (n\in\mathbf{N}).$$

To determine the expectations E_n and the variances σ_n^2 of T_n we differentiate Q_n twice:

$$\text{(iii)}\qquad Q_n'(s)=\frac{1}{\lambda}L_{n-1}^{1+\xi}(-s/\lambda)\Big/\binom{n+\xi}{n}\qquad (n\geqq 1)$$

$$\text{(iv)}\qquad Q_n''(s)=\frac{1}{\lambda^2}L_{n-2}^{2+\xi}(-s/\lambda)\Big/\binom{n+\xi}{n}\qquad (n\geqq 2)$$

where we have used the differentiation formula $\frac{d}{dx}L_n^{\alpha}(x)=-L_{n-1}^{1+\alpha}(x)$ for the Laguerre polynomials. Now, (iii) implies

$$\text{(v)}\qquad E_n=Q_n'(0)=\frac{1}{\lambda}\binom{n+\xi}{n-1}\Big/\binom{n+\xi}{n}=\frac{n}{\lambda(1+\xi)}\qquad (n\geqq 1).$$

From (iv) and (v) we obtain

$$\text{(vi)}\qquad \begin{aligned}\sigma_n^2=E_n^2-Q_n''(0)&=\frac{n^2}{\lambda^2(1+\xi)^2}-\frac{1}{\lambda^2}\binom{n+\xi}{n-2}\Big/\binom{n+\xi}{n}=\\&=\frac{n^2}{\lambda^2(1+\xi)^2}-\frac{(n-1)n}{\lambda^2(1+\xi)(2+\xi)}\qquad (n\geqq 2).\end{aligned}$$

On account of (v) and (vi) we consider the normalized first entrance times $\tilde{T}_n = T_n/n$ $(n \in \mathbf{N})$. Due to (ii), the reciprocals $\tilde{Q}_n$ of their Laplace transforms $\tilde{\Phi}_n$ are given by

$$\tilde{Q}_n(s) = L_n^{\xi}(-s/(\lambda n)) \Big/ \binom{n+\xi}{n} \qquad (n \in \mathbf{N}).$$

Now, it holds

$$\lim_{n\to\infty} n^{-\xi} L_n^{\xi}(-s/(\lambda n)) = (-s/\lambda)^{-\xi/2} J_{\xi}(2\sqrt{-s/\lambda})$$

(cf. Erdélyi II (1953), p. 191 (36)). Additionally using the relation $\binom{n+\xi}{n} \sim n^{\xi}/\Gamma(1+\xi)$, we obtain

$$\tilde{Q}(s) = \lim_{n\to\infty} \tilde{Q}_n(s) = \Gamma(1+\xi)(-s/\lambda)^{-\xi/2} J_{\xi}(2\sqrt{-s/\lambda}).$$

Thus, the limit function $\tilde{Q}$ of the reciprocals $\tilde{Q}_n$ of the Laplace transforms $\tilde{\Phi}_n$ is of the same kind as in Theorem 2.7 with $-\xi \in (-1, 0]$ being replaced by $\xi \in [0, \infty)$. Therefore the remainder of the proof follows along the same lines as in the proof of Theorem 2.7. (Note that the used formulas for the zeroes of J_{ν} with $\nu \in (-1, 0]$ hold for the zeroes of J_{ν} with $\nu \in [0, \infty)$, too.) □

Remark. From the proof of Theorem 4.1 (see (v) and (vi)) it is immediately clear that the expectation $\tilde{E}$ and the variance $\tilde{\sigma}^2$ of the limit distribution are given by

$$\tilde{E} = 1/[\lambda(1+\xi)], \quad \tilde{\sigma}^2 = 1/[\lambda^2(1+\xi)^2(2+\xi)].$$

Now we switch over to the case $\mu < \lambda$. First we provide a lemma on the behaviour of the expectations and variances for $n \to \infty$.

Lemma 4.2. *Let be given a birth and death process with transition rates* $\lambda_j = \lambda(j+1+\xi)$ $(j \in \mathbf{N}^0)$ *and* $\mu_j = \mu j$ $(j \in \mathbf{N})$ *where* $\lambda, \mu \in (0, \infty)$, $\xi \in [0, \infty)$ *and* $0 < \alpha = \mu/\lambda < 1$. *Then it holds:*

$$E_n \sim \frac{1}{\beta} \ln [(1-\alpha)n] \quad \textit{with} \quad \beta = \lambda - \mu; \tag{4.3}$$

$$\sigma^2 = \lim_{n\to\infty} \sigma_n^2 < \infty. \tag{4.4}$$

The proof may be accomplished similarly like that of Lemma 3.1 and is omitted.

Theorem 4.3. *Let be given a birth and death process with transition rates* $\lambda_j = \lambda(j+1+\xi)$ $(j \in \mathbf{N}^0)$ *and* $\mu_j = \mu j$ $(j \in \mathbf{N})$ *where* $\lambda, \mu \in (0, \infty)$, $\xi \in [0, \infty)$ *and* $0 < \alpha = \mu/\lambda < 1$. *Then, the normalized first entrance times* $\tilde{T}_n = T_n - \frac{1}{\beta} \ln [(1-\alpha)n]$ *are converging in distribution to the limit distribution with the Lebesgue density* $\tilde{f}$ *given by*

$$\tilde{f}(t) = \frac{\beta}{\Gamma(1+\xi)} e^{-\beta(1+\xi)t} e^{-e^{-t}} \qquad (t \in \mathbf{R}). \tag{4.5}$$

PROOF. Introducing the polynomials

$$R_n = \alpha^{-n}\binom{n+\xi}{n}Q_n = (1+\xi)_n\alpha^{-n}Q_n/n! \qquad (n\in\mathbf{N}^0)$$

and their generating function $R(z,s)=\sum_{n=0}^{\infty}R_n(s)z^n$, the recurrence relation (1.1) for the transition rates under consideration leads to the homogeneous partial differential equation

$$[\lambda z^2-(\lambda+\mu)z+\mu]\frac{\partial R(z,s)}{\partial z}-[s+\lambda(1+\xi)(1-z)]R(z,s) = 0$$

with the initial condition $R(0,s)\equiv 1$. The solution of this differential equation with the required initial condition turns out to be

$$R(z,s) = (1-z)^{s/\beta}(1-z/\alpha)^{-(1+\xi+s/\beta)}$$

where $\beta=\lambda-\mu$. Therefore it follows by Cauchy's integral formula for $n\in\mathbf{N}$

$$Q_n(s) = \frac{n!\,\alpha^n}{(1+\xi)_n}R_n(s) =$$

(i)

$$= \frac{n!\,\alpha^n}{2\pi i(1+\xi)_n}\int_{\mathscr{S}}(1-z)^{s/\beta}(1-z/\alpha)^{-(1+\xi+s/\beta)}z^{-(n+1)}dz$$

where $\mathscr{S}$ is a closed contour encircling the origin once counter-clockwise such that the singularities $z=1$ and $z=\alpha$ of the integrand are outside of $\mathscr{S}$. Passing over to the normalized first entrance times $\tilde{T}_n=T_n-\frac{1}{\beta}\ln[(1-\alpha)n]$ $(n\in\mathbf{N})$, the reciprocals of the Laplace transforms of $\tilde{T}_n$ are given by $\tilde{Q}_n(s)=[(1-\alpha)n]^{-s/\beta}Q_n(s)$ $(n\in\mathbf{N})$. Thus, for any $n\in\mathbf{N}$ performing the transformation $w=n(\alpha-z)/z$, we obtain from (i) for all $n\in\mathbf{N}$

(ii) $$\tilde{Q}_n(s) = \frac{n!\,n^{\xi}}{2\pi i(1+\xi)_n}\int_{\mathscr{S}_n}\left[\frac{1}{w}+\frac{1}{n(1-\alpha)}\right]^n\left[\frac{1}{w}+\frac{1}{n}\right]^{\xi}\frac{dw}{w}$$

where $\mathscr{S}_n$ is a closed contour encircling the origin $w=0$ once counter-clockwise and containing the singularities $w=0$, $w=-n(1-\alpha)$ and $w=-n$ of the integrand in its interior.

Now it holds

(iii) $$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\mathscr{S}_n}\left[\frac{1}{w}+\frac{1}{n(1-\alpha)}\right]^{s/\beta}\left[1+\frac{w}{n}\right]^n\left[\frac{1}{w}+\frac{1}{n}\right]^{\xi}\frac{dw}{w} =$$

$$= \frac{1}{2\pi i}\int_{\mathscr{S}^*}e^w w^{-(1+\xi+s/\beta)}dw$$

where $\mathscr{S}^*$ is a contour with $|\arg w|\leqq\pi$ for all $w\in\mathscr{S}^*$ which starts at $-\infty$, encircles the origin $w=0$ once counter-clockwise and returns to its starting point. This can be seen completely analogously like for $\xi=0$ in the proof of Satz 2.2.3 in EBERL

(1974). With the aid of Hankel's representation of $1/\Gamma$ (see e.g. ERDÉLYI I (1953), p. 13) and by the well-known relation $(1+\xi)_n/n! \sim n^\xi/\Gamma(1+\xi)$, (ii) and (iii) yield the validity of

$$\tilde{Q}(s) = \lim_{n\to\infty} \tilde{Q}_n(s) = \Gamma(1+\xi)/\Gamma(1+\xi+s/\beta).$$

Thus, the Laplace transforms $\tilde{\Phi}_n$ of $\tilde{T}_n$ converge to the limit function $\tilde{\Phi}$ given by

$$\tilde{\Phi}(s) = \lim_{n\to\infty} \tilde{\Phi}_n(s) = \Gamma(1+\xi+s/\beta)/\Gamma(1+\xi).$$

Finally, it is readily seen that $\tilde{\Phi}$ for $\operatorname{Re} s > -\beta(1+\xi)$ is the Laplace transform of the Lebesgue density $\tilde{f}$ given by (4.5). □

Remark. The expectation $\tilde{E}$ and the variance $\tilde{\sigma}^2$ of the limit distribution of Theorem 4.3 are

$$\tilde{E} = \frac{\gamma}{\beta} - \frac{\xi}{\beta}\sum_{k=1}^{\infty}\frac{1}{k(k+\xi)} \qquad (\gamma = 0{,}57721\ldots\text{: Euler's constant})$$

$$\tilde{\sigma}^2 = \frac{1}{\beta^2}\sum_{k=1}^{\infty}\frac{1}{(k+\xi)^2}.$$

This may be seen by differentiating the Laplace transform $\tilde{\Phi}$ twice and using the corresponding representation of the function $\psi = \Gamma'/\Gamma$.

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