On some properties of the stable sequences of random elements

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1. Let $\mathfrak{X}_S$ be the set of all random elements (r.e.) defined on a probability space $(\Omega, \mathscr{A}, P)$ with values in a separable, complete metric space (S, ϱ), i.e.

$$\mathfrak{X}_S = \{X: \Omega \to S, X^{-1}(\mathscr{B}) \subset \mathscr{A}\},$$

where $\mathscr{B}$ stands for the σ-field, generated by the open sets of S. When $S=\mathbf{R}$, we set

$$\mathfrak{X}_{\mathbf{R}} = \{X: \Omega \to R, X^{-1}(-\infty, x)\in\mathscr{A}, x\in\mathbf{R}\},$$

for the set of all random variables (r.v.) defined on $(\Omega, \mathscr{A}, P)$.

Definition 1. A sequence $\{X_n, n\geqq 1\}$ of r.e. is called stable if for every $B\in\mathscr{A}$, $P(B)>0$, there exists a probability measure μ_B such that

$$\lim_{n\to\infty} P([X_n\in A]|B) = \mu_B(A)$$

for every $A\in\mathscr{C}_{\mu_B}=\{A\in\mathscr{B}: \mu_B(\partial A)=0\}$, where ∂A denotes the boundary of A and $P(D|B)=\dfrac{P(D\cap B)}{P(B)}$.

If $\mu_B(A)=\mu(A)$ for every $B\in\mathscr{A}$, $P(B)>0$, then the sequence $\{X_n, n\geqq 1\}$ of r.e. is called μ-mixing.

It is easy to observe that the sequence $\{X_n, n\geqq 1\}$ of r.v. is stable iff

$$\lim_{n\to\infty} P([X_n < x]|B) = F_B(x) \quad \text{for every} \quad B\in\mathscr{A}, P(B)>0,$$

and $x\in\mathscr{C}_{F_B}$, where $F_B(x)$ denotes the distribution function.

Let us denote

$$Q_A(B)= \mu_B(A)P(B), \quad B\in\mathscr{A},\ P(B)>0, \quad A\in\mathscr{B},$$

and

$$Q_x(B) = F_B(x)P(B), \quad B\in\mathscr{A},\ P(B)>0, \quad x\in\mathbf{R}.$$

The measure Q_A and Q_x are absolutely continuous with respect to the measure P and by the Radon—Nikodym theorem there exist density functions α_A and α_x such that

$$Q_A(B) = \int_B \alpha_A\, dP \quad \text{and} \quad Q_x(B) = \int_B \alpha_x\, dP, \quad B\in\mathscr{A}.$$

If α_A is a constant a.e. for every $A\in\mathscr{B}$, then the sequence $\{X_n, n\geqq 1\}$ of r.e. is mixing. It is easy to see that the sequence $\{X_n, n\geqq 1\}$ of r.v. is mixing iff α_x is a constant a.e. for every $x\in\mathbf{R}$.

The aim of this paper is to give some properties of the stable sequences of random elements taking values in a metric space. The main theorem (Theorem 4) extends results of [5] concerning the stability and the density function of a sequence $\{g(X_n, Y), n\geqq 1\}$, where $\{X_n, n\geqq 1\}$ is a stable sequence of random variables, g is a real function, and Y is a random vector to the case when X_n, $n\geqq 1$, Y are random elements taking values in some metric space and g is a continuous mapping. Moreover, we generalize some results of [1], [10], and [11].

2. First we shall give a characterization of a probability space in terms of the equivalence of some kind of convergence of a sequence $\{X_n, n\geqq 1\}$ of r.e.

In what follows we shall use the well known result [3]: $X_n \xrightarrow{P} X$, $n\to\infty$ (P — in probability) iff

$$(1)\qquad \lim_{n\to\infty} P([X_n\in A]\triangle[X\in A]) = 0 \quad \text{for every} \quad A\in\mathscr{C}_{P_X},$$

where $A\triangle B$ denotes the symmetric difference of A and B.

It is well known [7] that any sample space Ω can be represented as

$$(2)\qquad \Omega = B\cup\bigcup_{k=1}^{\infty} B_k,\quad B_m\cap B_n=\emptyset,\quad m\neq n,\quad B\cap B_m=\emptyset,\quad m=1,2,\ldots,$$

each B_k is either an atom or an empty set, and B has the property that, for any given $A\in\mathscr{A}$ such that $A\subset B$ and any ε, $0<\varepsilon<P(A)$, there exists $C\in\mathscr{A}$, $C\subset A$, such that $P(C)=\varepsilon$. Random elements are constant on atoms.

Theorem 1. *The following statements are equivalent:*

a) *every stable sequence $\{X_n, n\geqq 1\}$ of r.e. converges in probability;*

b) *for any r.e. X and any sequence $\{X_n, n\geqq 1\}$ of r.e. $X_n\xrightarrow{P} X$, $n\to\infty$. iff $X_n\xrightarrow{\text{a.s.}} X$, $n\to\infty$;*

c) *Ω is at most a countable union of disjoint atoms.*

PROOF. The part b)$\Leftrightarrow$c) has been proved in [10].

c)$\Rightarrow$a). Let $\Omega=\bigcup_{i=1}^{\infty} B_i$ and $\{X_n, n\geqq 1\}$ be a stable sequence of r.e. Then

$$P([X_n\in A]\cap B_i)\to\mu_{B_i}(A)P(B_i),\quad B_i\in\mathscr{A},\quad A\in\mathscr{B}.$$

If

$$X_n(\omega)=a_{n,i}\quad\text{for}\quad \omega\in B_i,\quad i=1,2,\ldots,\quad n=1,2,\ldots,$$

then

$$[X_n\in A]\cap B_i=\begin{cases}B_i & \text{if}\quad a_{n,i}\in A,\\ 0 & \text{if}\quad a_{n,i}\notin A.\end{cases}$$

For every $A\in\mathscr{C}_{\mu_{B_i}}$ the limit

$$\lim_{n\to\infty} P([X_n\in A]\cap B_i)$$

exists, so there exists n_0 such that $a_{n,i}\in A$ for all $n\geqq n_0$ or $a_{n,i}\notin A$ for all $n\geqq n_0$, which proves that the sequence $\{a_{n,i},\ n\geqq 1\}$ converges. Indeed, by the stability of the sequence $\{X_n,\ n\geqq 1\}$ we know that $\{X_n,\ n\geqq 1\}$ is tight, i.e. for any given $\varepsilon>0$ there exists a compact set $K\subset S$ such that

$$\sup_n P[X_n\notin K] < \varepsilon.$$

Therefore, for every $i\in N$ we can choose a compact set $K_i\subset S$ such that $\sup_n P([X_n\notin K_i]\cap B_i)<\frac{1}{4}P(B_i)$ and $\mu_{B_i}(K_i)>\frac{3}{4}$. Thus $a_{n,i}\in K_i$, $n=1,2,\ldots$. Assume that the sequence $\{a_{n,i},\ n\geqq 1\}$ does not converge as $n\to\infty$. Then there exist subsequences $\{a_{n_k,i},\ k\geqq 1\}$ and $\{a_{n'_k,i},\ k\geqq 1\}$ such that

$$a_{n_k,i}\to a_i,\quad k\to\infty,\quad a_{n'_k,i}\to a'_i,\quad k\to\infty,\quad \text{and}\quad \varrho(a_i,a'_i)>0.$$

Putting now $A=\{s\in S\colon \varrho(s,a_i)<r\}$, where $r<\varrho(a_i,a'_i)$ is such that $A\in\mathscr{C}_{\mu_{B_i}}$ we see that $\lim_{n\to\infty} P([X_n\in A]\cap B_i)$ does not exists for every i which contradicts the assumption that $\{X_n,\ n\geqq 1\}$ is a stable sequence. Let

$$a_i=\lim_{n\to\infty} a_{n,i},\quad i=1,2,\ldots,$$

and put

$$X(\omega)=a_i\quad\text{for}\quad \omega\in B_i.$$

Then

$$X_n\xrightarrow{\text{a.s.}} X,\quad n\to\infty.$$

a)$\Rightarrow$c). Assume that $\Omega=B\cup\bigcup_{i=1}^{\infty} B_i$ and $P(B)>0$. Then there exists a mixing sequence $\{D_n,\ n\geqq 1\}$ of events such that $B\supset D_n\in\mathscr{A}$, $P(D_n)=\frac{1}{2}P(B)$, and

$$\lim_{n\to\infty} P(D_n\cap B')=\frac{1}{2}P(B'),\quad B'\subset B,\quad B'\in\mathscr{A}.$$

Indeed, let $B_0\in\mathscr{A}$, $B_0\subset B$ with $P(B_0)=\frac{1}{2}P(B)$, and put $B_1=B\setminus B_0$. Now let $B_{i_1,0}\in\mathscr{A}$, $B_{i_1,0}\subset B_{i_1}$ with $P(B_{i_1,0})=\frac{1}{2}P(B_{i_1})$, $i_1=0,1$, and put $B_{i_1,1}=B_{i_1}\setminus B_{i_1,0}$. By the induction argument we see that there exist sets

$$B_{i_1,i_2,\ldots,i_k,0}\in\mathscr{A},\quad B_{i_1,i_2,\ldots,i_k,0}\subset B_{i_1,i_2,\ldots,i_k}$$

with

$$P(B_{i_1,i_2,\ldots,i_k,0})=\frac{1}{2}P(B_{i_1,i_2,\ldots,i_k}),$$

and

$$B_{i_1,i_2,\ldots,i_k,1}=B_{i_1,i_2,\ldots,i_k}\setminus B_{i_1,i_2,\ldots,i_k,0}.$$

Setting

$$D_1 = B_1, \quad D_2 = B_{0,1} \cup B_{1,1}, \ldots,$$

$$D_n = \bigcup_{\{(i_1, i_2, \ldots, i_{n-1}):\, i_k = 0 \text{ or } 1,\, k=1,2,\ldots,n-1\}} B_{i_1, i_2, \ldots, i_{n-1}, 1}$$

we see that

$$P(D_n \cap D_k) = \frac{1}{2} P(D_k), \quad k < n,$$

which proves, by the Rényi theorem 3 of [8], the existence of the mixing sequence $\{D_n, n \geqq 1\}$ given above. Let

$$X_n(\omega) = \begin{cases} a & \text{if} \quad \omega \in D_n \\ b & \text{if} \quad \omega \notin D_n, \quad a, b \in S, \quad \varrho(a, b) > 0. \end{cases}$$

It is easy to see that the sequence $\{X_n, n \geqq 1\}$ of r.e. is stable but it does not converge in probability. Indeed, if

$$X(\omega) = \begin{cases} a, & \omega \in H \\ b, & \omega \notin H, \; H \in \mathscr{A}, \; H \subset B, \end{cases}$$

then

$$\lim_{n\to\infty} P\left[\varrho(X_n, X) \geqq \frac{1}{2} \varrho(a, b)\right] = \lim_{n\to\infty} \big(P([X_n = b], [X = a]) +$$

$$+ P([X_n = a], [X = b])\big) = \frac{1}{2} P(B) > 0,$$

which proves that $X_n \overset{P}{\nrightarrow} X$, $n \to \infty$. Thus Ω is at most a countable union of disjoint atoms.

3. In what follows we shall use the following

Lemma 1. *Suppose that* $\{X_n, n \geqq 1\}$ *and* $\{Y_n, n \geqq 1\}$ *are sequences of r.e. such that*

$$\varrho(X_n, Y_n) \xrightarrow{P} 0, \quad n \to \infty. \tag{3}$$

If $\{X_n, n \geqq 1\}$ *is stable, then* $\{Y_n, n \geqq 1\}$ *is also stable with the same density function.*

Proof. Note that for every B, $P(B) > 0$, and any given $A \in \mathscr{C}_{\mu_B}$, we have for any given $\varepsilon > 0$,

$$P([Y_n \in A] \| B) = P\big([Y_n \in A] \cap [\varrho(X_n, Y_n) < \varepsilon] \| B\big) + P\big([Y_n \in A] \cap [\varrho(X_n Y_n) \geqq \varepsilon] \| B\big).$$

We see that

$$P\big([X_n \in (\overline{A})^\varepsilon] \cap [\varrho(X_n, Y_n) < \varepsilon] \| B\big) + P\big([Y_n \in A] \cap [\varrho(X_n, Y_n) \geqq \varepsilon] \| B\big) \leqq P([Y_n \in A] \| B)$$

$$\leqq P\big([X_n \in A^\varepsilon] \cap [\varrho(X_n, Y_n) < \varepsilon] \| B\big) + P\big([Y_n \in A] \cap [\varrho(X_n, Y_n) \geqq \varepsilon] \| B\big),$$

where $\bar{A}=\Omega\setminus A$, and $A^\varepsilon=\{x\colon \inf_{y\in A}\varrho(x,y)\leqq\varepsilon\}$. Hence, if A^ε, $(\bar{A})^\varepsilon\in\mathscr{C}_{\mu_B}$, then by the assumption that $\{X_n,\ n\geqq 1\}$ is stable and (3), we get

$$\mu_B(\overline{(\bar{A})^\varepsilon}) = \lim_{n\to\infty} P([X_n\in\overline{(\bar{A})^\varepsilon}]|B) \leqq \liminf_{n\to\infty} P([Y_n\in A]|B)$$

$$\leqq \limsup_{n\to\infty} P([Y_n\in A]|B) \leqq \lim_{n\to\infty} P([X_n\in A^\varepsilon]|B) = \mu_B(A^\varepsilon).$$

Letting now $\varepsilon\to 0$, we obtain

$$\mu_B(A) = \lim_{n\to\infty} P([Y_n\in A]|B),$$

what completes the proof of Lemma 1.

For the local density $\alpha_A(\omega)=\alpha(A,\omega)$, we have the following

Lemma 2. *Let $\alpha(A,\omega)$ be the local density of the stable sequence $\{X_n,\ n\geqq 1\}$ of r.e. Then there is a variant $\lambda(A,\omega)$ of $\alpha(A,\omega)$ such that with probability 1 $\lambda(\cdot,\omega)$ is a probability measure on $(S,\mathscr{B})$ i.e. $P\{\omega\colon \lambda(A,\omega)\neq\alpha(A,\omega)\}=0$ for every $A\in\mathscr{B}$.*

Proof. If $A_1\subset A_2$, $A_1, A_2\in\mathscr{B}$, then by the stability of $\{X_n,\ n\geqq 1\}$, we have

$$P\{\omega\colon \alpha(A_1,\omega)\leqq\alpha(A_2,\omega)\}=1. \tag{4}$$

Moreover, we see that for every $n\geqq 1$

$$P\{\omega\colon \alpha(\bigcup_{i=1}^n A_i,\omega)\neq\sum_{i=1}^n\alpha(A_i,\omega)\}=0,\quad A_i\cap A_j=\emptyset,\quad i\neq j,\quad A_i\in\mathscr{B}, \tag{5}$$

and also that

$$P\{\omega\colon \alpha(A,\omega)=\sum_{i=1}^\infty\alpha(A_i,\omega)\}=1,\quad \text{where}\quad A_i\cap A_j=\emptyset,\quad i\neq j,\quad A=\bigcup_{i=1}^\infty A_i. \tag{6}$$

Let $\{s_i,\ i\geqq 1\}$ be a dense subset of S and $K(s_i, r_l)$, $i\geqq 1$, $l\geqq 1$ be a family of all balls such that $r_l\in W$ — the set of rational numbers. One can see, by (4), that there exists a set $T\in\mathscr{A}$ with $P(T)=0$ such that for every $\omega\notin T$ $K(s_i,r_l)\subset\subset K(s_{i'},r_{l'})$ implies $\alpha(K(s_i,r_l),\omega)\leqq\alpha(K(s_{i'},r_{l'}),\omega)$ where i, i', l, l' are positive integers. Let $\mathscr{K}$ be the field generated by the finite sums of finite intersections of balls and their complements [4], p. 23. Using (4) and (5) one can conclude that there exists a set $T^0\in\mathscr{A}$, $T\subset T^0$ with $P(T^0)=0$ such that for $\omega\notin T^0$, $\alpha(\cdot,\omega)$ is a finitely additive set function on $\mathscr{K}$.

Let now for $\omega\notin T^0$

$$\alpha_\varepsilon(A,\omega)=\inf\{\sum_{i=1}^\infty\alpha(K_i,\omega),\ K_i\in\mathscr{K},\ d(K_i)<\varepsilon,\ A\subset\bigcup_{i=1}^\infty K_i\},$$

where $d(K_i)$ denotes the diameter of K_i. Then

$$\lambda_e(A,\omega)=\lim_{\varepsilon\to 0}\alpha_\varepsilon(A,\omega) \tag{7}$$

defines an outer metrical measure on the class of all subsets of S and its restriction λ to $\mathscr{B}$ is a probability measure on $(S,\mathscr{B})$ [6].

Put $\mathscr{B}^*=\{A\colon \alpha(A,\omega)=\lambda(A,\omega)\ \text{a.s.},\ A\in\mathscr{B}\}$.

If $A \in \mathscr{K}$, then we see that

$$\alpha_{1/n}(A, \omega) = \inf\left\{\sum_{i=1}^{\infty} \alpha(K_i, \omega): K_i \in \mathscr{K}, d(K_i) < \frac{1}{n}, A \subset \bigcup_{i=1}^{\infty} K_i\right\} =$$

$$= \inf\left\{\sum_{i=1}^{\infty} \alpha(K_i', \omega): K_i' \in \mathscr{K}, d(K_i') < \frac{1}{n}, A = \bigcup_{i=1}^{\infty} K_i', \ K_i' \cap K_j' = \emptyset, \ i \neq j\right\}.$$

Let $\{K_i^m, i \geqq 1, m \geqq 1\}$ be a family of sets of $\mathscr{K}$ such that

$$d(K_i^m) < \frac{1}{n}, \ i \geqq 1, \ n \geqq 1, \quad \bigcup_{i=1}^{\infty} K_i^m = A, \ K_i^m \cap K_j^m = \emptyset, \quad i \neq j, \quad n \geqq 1, \quad m \geqq 1$$

and

$$\alpha_{1/n}(A, \omega) + \frac{1}{m} > \sum_{i=1}^{\infty} \alpha(K_i^m, \omega), \quad \omega \notin T^0, \quad n \geqq 1, \quad m \geqq 1.$$

By (6), we have for $m \geqq 1$

$$\alpha(A, \omega) = \sum_{i=1}^{\infty} \alpha(K_i^m, \omega), \quad \omega \notin T_m, P(T_m) = 0.$$

Hence, for $n \geqq 1$

$$\alpha_{1/n}(A, \omega) = \alpha(A, \omega) \quad \text{a.s.} \quad (\text{i.e. } \omega \notin T^0 \cup \bigcup_{m=1}^{\infty} T_m),$$

and by (7),

$$\lambda(A, \cdot) = \alpha(A, \cdot) \quad \text{a.s.}$$

as $A \in \mathscr{K}$. Thus $\mathscr{K} \subset \mathscr{B}^*$.

We shall now show that $\mathscr{B}^*$ is a σ-field. Indeed, if $A \in \mathscr{B}^*$, then by (4)

$$\lambda(S \setminus A, \cdot) = \lambda(S, \cdot) - \lambda(A, \cdot) = \alpha(S, \cdot) - \alpha(A, \cdot) = \alpha(S \setminus A, \cdot) \quad \text{a.s.}$$

If now $A_n \in \mathscr{B}^*$, $n \geqq 1$, $A_n \subset A_{n+1}$, then

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n, \cdot\right) = \lim_{n\to\infty} \lambda(A_n, \cdot) = \lim_{n\to\infty} \alpha(A_n, \cdot) = \alpha\left(\bigcup_{n=1}^{\infty} A_n, \cdot\right) \quad \text{a.s.}$$

as α satisfies (6). Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{B}^*$, which completes the proof that $\mathscr{B}^*$ is the σ-field generated by $\mathscr{K}$, i.e. $\mathscr{B}^* = \mathscr{B}$ [4], p. 27. Thus we have proved that there exists a variant $\lambda(A, \omega)$ of $\alpha(A, \omega)$ such that $\lambda(\cdot, \omega)$ is a probability measure on $(S, \mathscr{B})$ with probability 1.

We now shall prove the following

Theorem 2. *Let (S, ϱ), (S, ϱ') be metric spaces, and $g: S \to S'$ be a continuous map. If $\{X_n, n \geqq 1\}$ is a stable sequence of r.e. with density α_A, then the sequence $\{g(X_n), n \geqq 1\}$ is also stable with density given by*

(8) $$\beta(A', \omega) = \alpha[g(s) \in A', \omega], \quad A' \in \mathscr{B}'.$$

PROOF. By the assumption for every $B \in \mathscr{A}$, $P(B) > 0$,

$$(P_B)_{X_n} \Rightarrow \mu_B, \quad n \to \infty, \quad (\Rightarrow\text{-weakly converges})$$

takes place, where values of $(P_B)_{X_n}$ are defined by $P([X_n\in A]|B)$, $A\in\mathscr{B}$. Using Theorem 5.1 of [2], p. 30, we have

$$(P_B)_{X_n} g^{-1} \Rightarrow \mu_B g^{-1} \quad (\mu_B g^{-1}(A') = \mu_B(A'),\ A'\in\mathscr{B}'),$$

what proves that $\{g(X_n),\ n\geqq 1\}$ is stable, and moreover,

$$\int_B \beta_{A'}\, dP = Q'_{A'}(B) = \mu'_B(A')\,P(B) = \mu_B\big(g^{-1}(A')\big)\,P(B) = = Q_{(g^{-1}(A'))}(B) = \int_B \alpha_{g^{-1}(A')}\, dP,$$

where $\beta_{A'}$ is the density of the stable sequence $\{g(X_n),\ n\geqq 1\}$. Hence

$$\beta_{A'} = \alpha_{g^{-1}(A')} \quad \text{a.e.} \quad A'\in\mathscr{B}'.$$

Remark 1. It is easy to see that Theorem 2 is true when $g\colon S\to S'$ and $\mu_\Omega(D_g)=0$, where D_g denotes the set of all discontinuity points of g.

From Theorem 2 we get the following well-known facts:

Corollary 1. *If $\{X_n,\ n\geqq 1\}$ is a mixing sequence of r.e., then $\{g(X_n),\ n\geqq 1\}$ is a mixing sequence.*

Theorem 3. *Let $\{X_n,\ n\geqq 1\}$ be a stable sequence of r.v. with the density $\alpha_x=0$ or 1 a.e. for every $x\in R$. Then the density $\alpha_A=0$ or 1 a.e. for every $A\in\mathscr{B}$.*

PROOF. Let $\mathscr{H}$ stand for the class off all sets $H\subset R$ such that

$$\lim_{n\to\infty} P([X_n\in H]|B) = \frac{1}{P(B)}\int_B \alpha_H\, dP,$$

and the density $\alpha_H=0$ or 1 a.e. (of course, $(-\infty, x)\in\mathscr{H}$). We shall prove that $\mathscr{H}$ is a σ-field. By Lemma 2 we can assume and do that the density α_A of $\{X_n,\ n\geqq 1\}$ is a probability measure with respect to A for almost all $\omega\in\Omega$. Therefore,

$$\alpha(\overline{H}, \cdot) = 1-\alpha(H, \cdot) \quad \text{a.e.,} \quad H\in\mathscr{H},$$

whence $\overline{H}\in\mathscr{H}$.

Since

$$\alpha\big((a, b\rangle, \cdot\big) = \alpha\big((-\infty, b\rangle, \cdot\big)-\alpha\big((-\infty, a\rangle, \cdot\big),$$

it follows that $(a, b\rangle\in\mathscr{H}$.

Let now $H_1, H_2\in\mathscr{H}$, $H_1\cap H_2=\emptyset$. Then

$$0\leqq \alpha(H_1\cup H_2, \cdot) = \alpha(H_1, \cdot)+\alpha(H_2, \cdot) \leqq 1 \quad \text{a.e.,}$$

what proves that $\alpha(H_1\cup H_2, \cdot)=0$ or 1 a.e., so $H_1\cup H_2\in\mathscr{H}$. Thus we have proved that $\mathscr{H}$ contains a field of the sets of the following form:

$$H = \bigcup_{i=1}^{n} (a_i, b_i\rangle, \quad (a_i, b_i\rangle\cap(a_j, b_j\rangle = \emptyset, \quad i\neq j. \tag{9}$$

Let $H_n\in\mathscr{H}$, $n\geqq 1$, $H_1\subset H_2\subset\ldots\subset H_n\subset\ldots$, and $H_0=\bigcup_{n=1}^{\infty} H_n$. Then

$$\alpha(H_0,\cdot)=\lim_{n\to\infty}\alpha(H_n,\cdot)$$

which implies that $\alpha(H_0,\cdot)=0$ or 1 a.e., so $H_0\in\mathscr{H}$. Therefore, by Theorem A of [4], p. 27, we see that $\mathscr{H}$ is a σ-field, which completes the proof of Theorem 3.

It is known that if a stable sequence $\{X_n, n\geqq 1\}$ of r.e. contains a mixing subsequence, then the sequence $\{X_n, n\geqq 1\}$ is mixing.

We see that the following facts hold true [12].

Lemma 3. *If a stable sequence $\{X_n, n\geqq 1\}$ of r.e. contains a subsequence which converges in probability to a r.e. X $(X_{n_k}\xrightarrow{P} X, k\to\infty)$, then the sequence $\{X_n, n\geqq 1\}$ converges in probability to X.*

Remark 2. Let $\{X_n, n\geqq 1\}$ be a sequence of r.e. convergent in probability to X. The sequence $\{X_n, n\geqq 1\}$ is mixing iff X is a degenerate r.e.

4. Let (S,ϱ), (S',ϱ') and (S^*,ϱ^*) be nondegenerate polish metric spaces with Borel σ-fields $\mathscr{B}$, $\mathscr{B}'$ and $\mathscr{B}^*$, respectively. By $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{X}^*$ we denote the sets all r.e. defined on $(\Omega,\mathscr{A},P)$ taking values in S, S' and S^*, respectively.

To prove the next theorem we need the following lemma.

Lemma 4. *Let $\{X_n\in\mathscr{X}, n\geqq 1\}$ be a sequence of r.e. such that $X_n\xrightarrow{D} X$, $n\to\infty$. If a sequence $\{Y_n\in\mathscr{X}', n\geqq 1\}$ of r.e. converges in probability to a r.e. Y, then for every continuous function $g\colon S\times S'\to S^*$, we have*

$$\varrho^*\big(g(X_n,Y_n),g(X_n,Y)\big)\xrightarrow{P} 0,\quad n\to\infty.$$

Proof. Since $\{X_n, n\geqq 1\}$ and $\{Y_n, n\geqq 1\}$ are tight, then for any given $\varepsilon>0$ there exist compact sets $K\subset S$ and $K'\subset S'$ such that

$$P\big([X_n\in K]\cap[Y_n\in K']\big)>1-\varepsilon,\quad n\geqq 1.$$

The function g is uniformly continuous on the set $K\times K'$. Thus there exists a $\delta>0$ such that

$$\varrho^*\big(g(x,y),g(x',y')\big)<\varepsilon\quad\text{if}\quad\sqrt{\varrho^2(x,x')+\varrho'^2(y,y')}<\delta.$$

Let $\{D_i, i=1,2,\ldots,m\}$ be a sequence of sets such that

$$\bigcup_{i=1}^{m} D_i\supset K',\ d'(D_i)<\delta,\quad\text{and}\quad D_i\in\mathscr{C}_{P_Y}$$

$(d'(D)$ denotes the diameter of $D)$. It is obvious that

$$P\big(\varrho^*(g(X_n,Y_n),g(X_n,Y))>\varepsilon'\big)\leqq$$

$$\leqq\sum_{i=1}^{m}P\big([\varrho^*(g(X_n,Y_n),g(X_n,Y))>\varepsilon'],Y\in D_i,Y_n\in D_i,X_n\in K\big)+$$

$$+\sum_{i=1}^{m}P\big([\varrho^*(g(X_n,Y_n),g(X_n,Y))>\varepsilon', Y\notin D_i,Y_n\in D_i,X_n\in K\big)+\varepsilon.$$

The first sum equals 0 and

$$\lim_{n\to\infty} P\big(\varrho^*(g(X_n, Y_n), g(X_n, Y)) > \varepsilon'\big) \leqq$$

$$\leqq \lim_{n\to\infty} \sum_{i=1}^{m} P(Y\notin D_i, Y_n\in D_i)+\varepsilon = \varepsilon,$$

which completes the proof.

Definition 2. A sequence $\{X_n, n\geqq 1\}$ of r.v. is said to be *w*-star convergent to a r.v. $X (X_n \xrightarrow{w^*} X$ as $n\to\infty)$ if for every $D\in\mathscr{A}$

$$\lim_{n\to\infty} \int_D X_n\, dP = \int_D X\, dP.$$

Theorem 4. *Let $\{X_n\in\mathscr{X}, n\geqq 1\}$ be a stable sequence of r.e. with the density function $\alpha(A, \cdot)$. Suppose that a sequence $\{Y_n\in\mathscr{X}', n\geqq 1\}$ of r.e. converges in probability to a r.e. $Y\in\mathscr{X}'$.*

If a function $g\colon S\times S'\to S^*$ is continuous, then the sequence $\{Z_n\in\mathscr{X}^*, n\geqq 1\}$ of r.e. defined by $Z_n=g(X_n, Y_n)$, $n\geqq 1$, is stable with the density $\beta(A^*, \cdot)$, given by

$$\beta(A^*, \omega) = \alpha\big([x\colon\ g(x, Y(\omega))\in A^*], \omega\big), \quad A^*\in\mathscr{B}^*. \tag{10}$$

Proof. Let $Z_n'=g(X_n Y)$, $n\geqq 1$. From Lemmas 4 and 1, we conclude that it is enough to show that the sequence $\{Z_n', n\geqq 1\}$ is stable with the density (10).

Now by results of [1] it is known that a sequence $\{X_n\in\mathscr{X}, n\geqq 1\}$ of r.e. is stable iff for every r.e. $V\in\mathscr{X}'$ the sequence $\{(X_n, V), n\geqq 1\}$ converges weakly in $(S\times S', \mathscr{B}\times\mathscr{B}')$. Hence, we see that the sequence $\{(X_n, Y), n\geqq 1\}$ is stable with the density $\alpha'(A', \cdot)$, $A'\in\mathscr{B}\times\mathscr{B}'$, and Theorem 2 establishes the stability of the sequence $\{Z_n', n\geqq 1\}$.

We see that the measures

$$Q_A^n(D) = P\big([X_n\in A], D\big) \tag{11}$$

are absolutely continuous with respect to the measure P, so by the Radon—Nikodym theorem there exist measurable functions $\alpha_n(A, \cdot)$ such that

$$Q_A^n(D) = \int_D \alpha_n(A, \cdot)\, dP. \tag{12}$$

We can assume and do that $\alpha_n(A, \omega)$ for almost all ω are measures on $(S, \mathscr{B})$. Moreover, by the stability of $\{X_n, n\geqq 1\}$, we have for every $D\in\mathscr{A}$

$$\lim_{n\to\infty} \int_D \alpha_n(A, \cdot)\, dP = \int_D \alpha(A, \cdot)\, dP = Q_A(D) = \mu_D(A)\, P(D), \quad A\in\mathscr{C}_{\mu_\Omega}. \tag{13}$$

Hence

$$\alpha_n(A, \cdot) \xrightarrow{w^*} (A, \cdot), \quad n\to\infty, \quad \text{for} \quad A\in\mathscr{C}_{\mu_\Omega}.$$

Note now that for $Y\in\mathscr{X}'$, $B\in\mathscr{B}'$ and $D\in\mathscr{A}$, $P(D)>0$, we have by (11) and (12)

$$(14)\qquad P([X_n\in A],[Y\in B],D)=\int\limits_{[Y\in B]\cap D}\alpha_n(A,\cdot)\,dP=$$

$$=\int\limits_D I_{[Y\in B]}\alpha_n(A,\cdot)\,dP=\int\limits_D \mu_{Y(\cdot)}(B)\,\alpha_n(A,\cdot)\,dP,$$

where $\mu_{Y(\cdot)}(B)=I_{[Y\in B]}(\cdot)$.

Now we see also that the measures $Q_A^n(D)$ defined on the space $(\Omega,\mathscr{A})$ by the formula

$$(15)\qquad Q_{A'}^n(D)=P([(X_n,Y)\in A'],D),\quad A'\in\mathscr{B}\times\mathscr{B}'$$

are absolutely continuous with respect to the measure P and that there exists a measurable function $\alpha_n'(A',\cdot)$ such that

$$Q_{A'}^n(D)=\int\limits_D\alpha_n'(A',\cdot)\,dP.$$

In the particular case with $A'=A\times B$, we have

$$Q_{A\times B}^n(D)=\int\limits_D\alpha_n'(A\times B,\cdot)\,dP.$$

Similarly, as above, we can assume that $\alpha_n'(A',\cdot)$ are measures on $(S\times S',\mathscr{B}\times\mathscr{B}')$, and state that

$$\alpha_n'(A\times B,\cdot)=I_{[Y\in B]}(\cdot)\alpha_n(A,\cdot)=\mu_{Y(\cdot)}(B)\alpha_n(A,\cdot)\quad\text{a.e.}$$

Moreover, we have

$$\alpha_n'(A\times B,\cdot)=\mu_{Y(\cdot)}(B)\,\alpha_n(A,\cdot)\xrightarrow{w^*}\mu_{Y(\cdot)}(B)\,\alpha(A,\cdot),$$

and

$$\alpha_n'(A\times B,\cdot)\xrightarrow{w^*}\alpha'(A\times B,\cdot),\quad\text{for}\quad A\in\mathscr{C}_{\mu_\Omega}\quad\text{and}\quad B\in\mathscr{C}_{P_Y},$$

whence

$$\alpha'(A\times B,\cdot)=\mu_{Y(\cdot)}(B)\alpha(A,\cdot)\quad\text{a.e.}\quad A\in\mathscr{C}_{\mu_\Omega},\quad B\in\mathscr{C}_{P_Y}.$$

Let now $A_x'=\{y:\,(x,y)\in A'\}$, and $A_y'=\{x:\,(x,y)\in A'\}$. Then

$$\alpha'(A',\cdot)=\int\limits_S\alpha'(A_y'\times S,\cdot)\alpha'(S\times dy,\cdot)=\int\limits_S I_{[Y\in S]}\alpha(A_y',\cdot)\mu_{Y(\cdot)}(dy)\alpha(S,\cdot)=$$

$$=\int\limits_S\alpha(A_y',\cdot)\,\mu_{Y(\cdot)}(dy)=\alpha(A_{Y(\cdot)}',\cdot),$$

what proves that the density function of the sequence $\{(X_n,Y),\,n\geqq1\}$ is given by

$$(16)\qquad \alpha'(A',\omega)=\alpha(A_{Y(\omega)},\omega)=\alpha([s:\,(s,Y(\omega))\in A'],\omega).$$

Therefore, if a function $g:\,S\times S'\to S^*$ is continuous, then, by Theorem 2, the sequence $\{Z_n,\,n\geqq1\}$, $Z_n=g(X_n,Y)$, $n\geqq1$, is stable with the density function

$$\beta(A^*,\omega)=\alpha([s:\,g(s,Y(\omega))\in A^*],\omega),$$

which gives the proof that the density function of $\{Z_n \in \mathscr{X}^*, n \geqq 1\}$ is given by (10).

The following example shows that the assumption of the stability of the sequence $\{X_n, n \geqq 1\}$ cannot be omitted.

Example. Let X be a r.v. having the normal distribution with mean 0 and variance σ^2. Put

$$X_n = \begin{cases} X, & \text{when } n \text{ is even,} \\ -X, & \text{when } n \text{ is odd.} \end{cases}$$

Setting $Y_n = X$, and $g(x, y) = |x - y|$, we get

$$g(X_n, Y_n) = \begin{cases} 0, & \text{when } n \text{ is even,} \\ 2|X|, & \text{when } n \text{ is odd.} \end{cases}$$

Hence, we conclude that the sequence $\{Z_n, n \geqq 1\}$, $Z_n = g(X_n, Y_n)$, $n \geqq 1$, does not converge in law.

Theorem 4 allows us to prove the following property of the stable sequence $\{X_n, n \geqq 1\}$ of r.e.

Theorem 5. *Let* $\{X_n, n \geqq 1\}$ *be a stable sequence of r.e. Then for every r.e.* $Y \in \mathscr{X}$ *there exists a limit*

$$a(Y) = \lim_{n\to\infty} r(X_n, Y), \tag{17}$$

where $r(X, Y) = \inf\{\varepsilon > 0\colon P[\varrho(X, Y) > \varepsilon] < \varepsilon\}$.

Proof. Define the function g by $g(x, y) = \varrho(x, y)$. It is obvious that g is continuous. Thus, by Theorem 4, the sequence $\{g(X_n, Y), n \geqq 1\}$ converges weakly to a r.v. Z. Hence, for every $\varepsilon \in \mathscr{C}_{F_Z}$ there exists

$$\lim_{n\to\infty} P[\varrho(X_n, Y) > \varepsilon] = b(Y, \varepsilon).$$

Let

$$a(Y) = \inf\{\varepsilon > 0\colon b(Y, \varepsilon) < \varepsilon\}.$$

We now prove that

$$\lim_{n\to\infty} r(X_n, Y) = \lim_{n\to\infty} \inf\{\varepsilon > 0\colon P[\varrho(X_n, Y) > \varepsilon] < \varepsilon\} = a(Y).$$

For every $\delta > 0$, $\delta \in \mathscr{C}_{F_Z}$, there exists n_0 such that for $n \geqq n_0$

$$a(Y) - \delta \leqq P[\varrho(X_n, Y) > a(Y) - \delta] \to b(Y, a(Y) - \delta),$$

and

$$a(Y) + \delta \geqq P[\varrho(X_n, Y) > a(Y) + \delta] \to b(Y, a(Y) + \delta).$$

Hence

$$a(Y) - \delta \leqq r(X_n, Y) \leqq a(Y) + \delta, \quad n \geqq n_0.$$

Letting $\delta \to 0$, we get

$$\lim_{n\to\infty} r(X_n, Y) = a(Y),$$

which completes the proof of Theorem 5.

Theorem 5'. *Let* $\{X_n,\ n\geqq 1\}$ *be a stable sequence of r.e. Then for every r.e.* $Y\in\mathscr{X}$, *there exists a limit*

$$a_1(Y)=\lim_{n\to\infty} r_1(X_n, Y),$$

where $r_1(X, Y)=E\dfrac{\varrho(X, Y)}{1+\varrho(X, Y)}$.

PROOF. We see that the sequence $\left\{\dfrac{\varrho(X_n, Y)}{1+\varrho(X_n, Y)},\ n\geqq 1\right\}$ is uniformly integrable, and by Theorem 4, it is also stable. Therefore, by Theorem 3 of [5], there exists

$$\lim_{n\to\infty} r_1(X_n, Y)=a_1(Y).$$

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