On additive arithmetical functions with values in topological groups I.

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1. Throughout this paper we shall use the following standard notations: $\mathbf{N}$=natural numbers; Z=rational integers; Q_x=multiplicative group of positive rationals; $\mathbf{R}_x$=multiplicative group of positive reals; Q=additive group of rationals; $\mathbf{R}$=additive group of reals; T=one-dimensional circle group (torus); each of them in the usual topology.

Let G be an Abelian group. We shall say that a mapping $\varphi\colon \mathbf{N}\to G$ is a completely additive function, if

$$\varphi(mn) = \varphi(m)+\varphi(n) \quad \forall m, n\in\mathbf{N} \tag{1.1}$$

holds.

If we consider G as a multiplicative group, then the mapping $V\colon \mathbf{N}\to G$ satisfying the relation

$$V(mn) = V(m)V(n) \quad \forall m, n\in\mathbf{N} \tag{1.2}$$

is called a completely multiplicative function.

We can extend the domain of φ and V to Q_x by the relations

$$\varphi\left(\frac{m}{n}\right) := \varphi(m)-\varphi(n), \quad V\left(\frac{m}{n}\right) = V(m)\,V^{-1}(n), \tag{1.3}$$

uniquely. Furthermore the relations

$$\varphi(rs) = \varphi(r)+\varphi(s), \quad V(rs) = V(r)V(s) \tag{1.4}$$

hold. So the extensions of the functions φ, V define a $Q_x\to G$ homomorphism

Let now G be an Abelian topological group, $\varphi\colon Q_x\to G$ be a homomorphism. We shall say that φ is continuous at the point 1, if $r_\nu\in Q_x$, $r_\nu\to 1$ implies that

$$\varphi(r_\nu)\to 0. \tag{1.5}$$

Lemma 1. *Let G be an additively written closed Abelian topological group, $\varphi\colon Q_x\to G$ be a homomorphism that is continuous at the point 1. Then its domain can be extended by the relation*

$$\varphi(\alpha) := \lim_{\substack{r_\nu\to\alpha \\ r_\nu\in Q_x}} \varphi(r_\nu) \quad (\alpha\in\mathbf{R}_x) \tag{1.6}$$

uniquely. The so obtained $\varphi\colon \mathbf{R}_x \to G$ *is a continuous homomorphism, consequently*

$$\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta) \quad (\forall\, \alpha, \beta \in \mathbf{R}_x) \tag{1.7}$$

holds.

Proof, Let $\alpha \in \mathbf{R}_x$, $r_\nu \to \alpha$ be an arbitrary sequence of rationals. Since $r_\nu / r_\mu \to 1$ as $\nu, \mu \to \infty$, therefore

$$\varphi\left(\frac{r_\nu}{r_\mu}\right) = \varphi(r_\nu) - \varphi(r_\mu) \to 0 \quad \text{as} \quad \nu, \mu \to \infty,$$

consequently $\varphi(r_\nu)$ is a Cauchy-sequence, and so it is convergent. Hence it follows immediately that the limit is well defined. The further assertions in the lemma are obvious consequences of this. ∎

We are interested in such completely additive functions for which

$$\Delta\varphi(n) := \varphi(n+1) - \varphi(n) \to 0 \quad (n \to \infty) \tag{1.8}$$

holds.

An old theorem of P. Erdős asserts that for $G = \mathbf{R}$ (1.8) implies that φ is a constant multiple of log, in other words that φ is a continuous homomorphism $Q_x \to \mathbf{R}$ [1].

A very recent, until now unpublished result due to E. Wirsing [2] asserts that: if $G = T$ written additively, then (1.8) implies that φ is a constant multiple of $\log \pmod{2\pi}$, i.e. that φ is a continuous homomorphism $Q_x \to T$.

Now we state this theorem in multiplicative form as

Lemma 2. *Let* $T = \{z \in \mathbf{C} \,\big|\, |z| = 1\}$ *be the unit circle, and* $V\colon \mathbf{N} \to T$ *be a completely multiplicative function, such that*

$$\delta V(n) := V(n+1)V^{-1}(n) \to 1 \quad (\in T) \quad (n \to \infty). \tag{1.9}$$

Then $V(n) = n^{i\tau}$, τ *is a real number.*

This is the crucial point of the proof of our

Theorem 1. *Let G be an additively written, metrically compact Abelian topological group. Let* $\varphi\colon \mathbf{N} \to G$ *be a completely additive function satisfying the condition* (1.8). *Then its extension* $\varphi\colon Q_x \to G$ *defined by* (1.3) *is continuous at* 1, *consequently its extension* $\varphi\colon \mathbf{R}_x \to G$ *defined by* (1.6) *is a continuous homomorphism.*

2. Proof of Theorem 1. Let us assume that (1.1), (1.8) hold. Let $\chi\colon G \to T$ be any continuous character,

$$V(n) := \chi(\varphi(n)).$$

Then

$$\delta V(n) = V(n+1)V(n)^{-1} = \chi(\Delta\varphi(n)) \to \chi(0) = 1 \quad (n \to \infty).$$

Furthermore V is completely multiplicative, and so by Lemma 2, $V(n) = e^{i\tau \log n}$ $(\tau \in \mathbf{R})$.

Now we prove that $\varphi\colon Q_x\to G$ is continuous in 1. Let $N_j/M_j\to 1$, N_j, $M_j\in\mathbf{N}$ $(j\to\infty)$. We consider $A_j=\varphi(N_j)-\varphi(M_j)$. Since G is metrically therefore it is sequentially compact. Then there exists a convergent subsequence, $A_{j_l}\to B(\in G)$. Then $\chi(A_{j_l})\to\chi(B)$. By Lemma 2 we get

$$\chi(A_{j_l}) = \exp\left(i\tau\log\frac{N_{j_l}}{M_{j_l}}\right)\to 1.$$

So $\chi(B)=1$ for each continuous character χ, consequently $B=O(\in G)$, and so $\varphi\colon Q_x\to G$ is continuous in 1. Lemma 1 completes the proof of the theorem. ∎

3. *Some remarks*

1. It is known that every locally compact, compactly generated Abelian group G is topologically isomorphic with $R^a\times Z^b\times F$ for some nonnegative integers a and b and some compact Abelian group F. For the proof see [4] Theorem 9.8 p. 90. Theorem 1 and theorem of Erdős imply immediately

Theorem 2. *Let G be an additively written, metrizable, locally compact, compactly generated Abelian topological group. Let us assume that the conditions* (1.1), (1.8) *for $\varphi\colon\mathbf{N}\to G$ hold. Then the assertions stated in Theorem* 1 *remain true.*

Proof. Since $G=\mathbf{R}^a\times Z^b\times F$, it is enough to prove the assertion for $G=\mathbf{R}, Z, F$. This was proved for $G=\mathbf{R}, F$ earlier. Furthermore for $G=Z$, $\Delta\varphi\to 0$ implies that $\varphi(n)\equiv 0$. ∎

This completes the proof.

2. Let G be an Abelian topological group. G is said to be solenoidal, if there exists a continuous homomorphism of $\mathbf{R}$ into G, such that the image $\tau(\mathbf{R})$ is dense in G. The mapping $v\to e^v$ is a topological isomorphism of $\mathbf{R}$ onto $\mathbf{R}_x$, so we may change $\mathbf{R}$ by $\mathbf{R}_x$ in the definition of solenoidal groups. It is known that a compact Abelian group is solenoidal if and only if it is connected and its cardinality is not greater than the continuum. For the proof see [4] Theorem 25.18.

Let us assume that the conditions of Theorem 1 or 2 are satisfied. Let $H:=\varphi(\mathbf{R}_x)$ be the image of the continuous homomorphism. Then the mapping $\varphi\colon\mathbf{R}_x\to H$ is a topological isomorphism, or H^- (H^- denotes the closure of H) is a compact subgroup in G. (See [4], Theorem 9.1.)

These theorems allow us to give a quite complete characterization of the sets that can occur as images for a suitable continuous homomorphism $\varphi\colon\mathbf{R}_x\to G$.

For a mapping $\varphi\colon\mathbf{N}\to G$ under the conditions (1.1), (1.8) let $K_\varphi=$closure of $\varphi(\mathbf{R}_x)$, where the extension of the domain of φ to $\mathbf{R}_x$ is defined by (1.6).

Theorem 3. 1. *Let G be a metrically compact Abelian group, S be a subgroup in G. Then there exists a completely additive function $\varphi\colon\mathbf{N}\to G$ satisfying* (1.8) *and $K_\varphi=S$ if and only if S is compact, connected and the cardinality of it is not greater than the continuum.*

2. *Let G be a metrizable, locally compact, compactly generated Abelian group, S be a subgroup in G. Then there exists a completely additive function $\varphi\colon \mathbf{N}\to G$ satisfying* (1.8) *and $K_\varphi=S$ if and only if*

a) *S is topologically isomorphic to $\mathbf{R}$, or*

b) *S is compact, connected and the cardinality of it is not greater than the continuum.*

3. *Let G be an arbitrary Abelian topological group, $\varphi,\psi\colon \mathbf{N}\to G$ be completely additive functions, such that*

$$\psi(n+1)-\varphi(n)\to 0 \quad (n\to\infty). \tag{3.1}$$

Then $\psi(n)=\varphi(n)\ \forall n\in\mathbf{N}$.

This assertion is almost obvious. Let $H(n):=\psi(n)-\varphi(n)$.

Starting from the relation

$$\psi(n+1)-\varphi(n)=-H(2)+\psi(2n+2)-\varphi(2n)=$$
$$=-H(2)+\psi(2n+2)-\varphi(2n+1)-H(2n+1)+\psi(2n+1)-\varphi(2n),$$

from (3.1) we deduce that

$$H(2n+1)+H(2)\to 0 \quad (n\to\infty). \tag{3.2}$$

Let m be an arbitrary odd number. From (3.2) we get that $H((2n+1)m)\to -H(2)$ $(n\to\infty)$, $H(2n+1)\to -H(2)$ $(n\to\infty)$, and so that $H(m)=0$. So $H(2n+1)=0$ $(\forall n)$, and so $H(2)=0$.

4. In a recent joint paper [5] we proved the following assertion.

Let G be a metrically compact Abelian group. Let us assume that the completely additive function $\varphi\colon \mathbf{N}\to G$ satisfies the following requirement: if $n_1<n_2<\ldots$ is such an infinite subsequence in $\mathbf{N}$ for which $\lim_k \varphi(n_k)$ exists, then there exists $\lim_k \varphi(n_k+1)$ as well. Then (1.8) *holds.*

4. *Additive functions defined over the Gaussian integers*

Let $\mathbf{F}$ be the nonzero Gaussian integers, $\mathbf{K}$ be the multiplicative group of nonzero Gaussian rationals, $\mathbf{C}_x$ be the multiplicative group of nonzero complex numbers, $\mathbf{C}$ be the additive group of complex numbers.

Let G be an Abelian group for the addition. A mapping $\varphi\colon \mathbf{F}\to G$ is said to be an additive function, if

$$\varphi(ab)=\varphi(a)+\varphi(b) \quad (\forall a,b\in\mathbf{F}) \tag{4.1}$$

holds.

Let $\beta\in\mathbf{F}$ be fixed,

$$\Delta_\beta\varphi(\alpha):=\varphi(\alpha+\beta)-\varphi(\alpha) \quad (\alpha,\alpha+\beta\in\mathbf{F}). \tag{4.2}$$

We should like to determine those φ for which

$$\Delta_\beta\varphi(\alpha)\to 0 \quad (|\alpha|\to\infty) \tag{4.3}$$

holds.

Since

$$\Delta_\beta \varphi(\alpha\beta) = \Delta_1 \varphi(\alpha),$$

therefore we may restrict ourselves to the case $\beta=1$,

$$\Delta\varphi(\alpha) := \varphi(\alpha+1)-\varphi(\alpha) \to 0 \quad (|\alpha| \to \infty). \tag{4.4}$$

It is easy to prove that for $G=\mathbf{R}$ (4.4) implies that

$$\varphi(\alpha) = c \log |\alpha|.$$

Let us consider now the case $G=T$, written multiplicatively. E. WIRSING proved [3] the next theorem, which we quote now as

Lemma 4. *Let $V\colon \mathbf{F}\to T$ be a completely multiplicative function, such that*

$$V(\alpha+1)V^{-1}(\alpha) \to 1 \quad \text{as} \quad |\alpha| \to \infty, \quad \alpha\in\mathbf{F}. \tag{4.5}$$

Then $V(\alpha):=e^{i\tau\log|\alpha|}\cdot e^{ik\arg\alpha}$, where τ a real constant, k is a rational integer.

By repeating the argument used in Section 1 and 2 from Lemma 4 we can deduce easily

Theorem 4. *Let G be a metrically compact Abelian group, $\varphi\colon \mathbf{F}\to G$ be a completely additive function, satisfying* (4.4). *Then the domain of φ can be extended on $\mathbf{K}$ by*

$$\varphi\left(\frac{\alpha}{\beta}\right) := \varphi(\alpha)-\varphi(\beta) \quad (\alpha, \beta\in\mathbf{F}),$$

and on $\mathbf{C}_x$ by

$$\varphi(\gamma) = \lim_{\substack{P_n\to\gamma \\ P_n\in\mathbf{K}}} \varphi(P_n), \quad (\gamma\in\mathbf{C}_x) \tag{4.6}$$

uniquely. The mapping $\varphi\colon \mathbf{C}_x\to G$ is a continuous homomorphism.

Inversely, let $\varphi\colon \mathbf{C}_x\to G$ be a continuous homomorphism. Then the restriction of ψ on $\mathbf{F}$ defines a completely additive function, for which (4.4) holds.

References

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