On dilation functions and some applications

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Abstract. The aim of this paper is to study some properties of the so called dilation functions ([7]), and applications of these to questions on Orlicz Spaces and linear bounded operators on them. Some results are part of a Ph. D, dissertation presented by the author at Chelsea College, London yet unpublished.

1. *Introduction*

Let $\varphi(u)$, $u\in[0,\infty)$, be a real, increasing function, right continuous on $(0,\infty)$. The function $\Phi(u)$, $u\geqq 0$, defined by

$$\Phi(u)=\int_0^u \varphi(t)\,dt$$

is called a *Young function.*

The function $\Psi(v)$, $u\geqq 0$, defined by

$$\Psi(v)=\sup_{u\geqq 0}\{uv-\Phi(u)\},$$

where sup can be replaced by max if $\Psi(v)$ is finite for finite v, is called the complementary function to $\Phi(u)$. One also has that

$$\Phi(u)=\max_{v\geqq 0}\{uv-\Psi(v)\}.$$

A Young function satisfies the $\delta_2(\Delta_2)$ condition if there is some $u_0\geqq 0$ and $M>0$ such that

$$\Phi(2u)\leqq M\Phi(u),$$

for all u in $[0,u_0]$ (in $[u_0,\infty)$). If this inequality holds for all $u\geqq 0$, then it is said that Φ satisfies the (δ_2,Δ_2) condition ([9]).

A Young function satisfies the $\delta'(\Delta')$ condition if there are $u_0\geqq 0$, $M>0$ such that

$$\Phi(uv)\leqq M\Phi(u)\Phi(v)$$

for all u,v in $[0,u_0]$ (in $[u_0,\infty)$). If Φ satisfies both conditions, then it is said that Φ is submultiplicative.

Whenever these inequalities hold in reverse we say that Φ satisfies the $\varrho'(\nabla')$ condition and that Φ is supermultiplicative respectively.

The Young functions $\Phi_1(u)$, $\Phi_2(u)$ are said to be equivalent on the set A if for some positive constants k_1, k_2 we have

$$\Phi_1(k_1 u) \leqq \Phi_2(u) \leqq \Phi_1(k_2 u)$$

for all u in A.

A Young function $\Phi(u)$ with representation

$$\Phi(u) = \int_0^u \varphi(t)\,dt,$$

is called an N-function ([6]) if $\varphi(t)$ is positive for positive t, and satisfies the conditions $\varphi(0)=0$, $\lim_{t\to\infty} \varphi(t)=\infty$.

One can easily see that the following hold for $\Phi(u)$:

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty.$$

Let $\Phi(u)$ be a Young function that satisfies the (δ_2, Δ_2) condition. Let μ be a totally σ-finite measure on $\mathbf{R}^n$. The Orlicz space $L_\Phi(\mathbf{R}^n, \mu)$ consists of all μ-measurable functions f, such that

$$\int_{\mathbf{R}^n} \Phi(|f|)\,d\mu < \infty.$$

By l_Φ we mean, as usual, the space of all scalar sequences $\{a_n\}_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty \Phi(|a_n|) < \infty.$$

Conditions for these spaces to be reflexive are known since long ago. Here we give yet another such condition which seems to be new.

Consider the spaces $L_\Phi(\mathbf{R}^n, \mu)$, where μ is a positive Radón measure. For any $h \in \mathbf{R}^n$, the operation of translation is defined by

$$\tau(h)f(x) = f(x-h),$$

for any μ-measurable function f. In this paper we generalize a result in [2] which gives necessary and sufficient conditions for $\tau(h)$ to be defined as an operator on $L_\Phi(\mathbf{R}^n, \mu)$. We also obtain a necessary condition for there to exist a translation invariant operator T,

$$T\colon L_{\Phi_1}(\mathbf{R}^n, \mu) \to L_{\Phi_2}(\mathbf{R}^n, \gamma).$$

When restricted to L_p spaces this condition gives those in [2] and [5].

Moreover, a necessary and sufficient condition for there to exist a linear bounded translation invariant operator T,

$$T\colon l_{\Phi_1} \to l_{\Phi_2}$$

is obtained.

Let X, Y be normed spaces. A linear bounded operator $T\colon X \to Y$ is said to be strictly singular if for any subspace A of X, the restriction of T to A is not an isomorphism. For a submultiplicative function Φ_1 a sufficient condition for every linear bounded operator $T: l_{\Phi_1} \to l_{\Phi_2}$ to be strictly singular, is given in this paper.

The following theorem can be easily deduced from [7] (Th. 1.2. p. 52).

Theorem 1. *Let Φ be a submultiplicative Young function. Then, there exist real numbers α, β such that $1 \leqq \alpha \leqq \beta < \infty$ and*

$$\Phi(t) \geqq t^{\beta} \quad for \quad t \in [1, \infty), \quad \Phi(t) \geqq t^{\alpha} \quad for \quad t \in [0, 1].$$

Moreover, given $\varepsilon > 0$ there exist real numbers a_ε and b_ε such that

$$\Phi(t) \leqq t^{\beta+\varepsilon} \quad for \quad t \in [b_\varepsilon, \infty) \quad and \quad \Phi(t) \leqq t^{\alpha-\varepsilon} \quad for \quad t \in [0, a_\varepsilon].$$

[2.]

Let $\Phi(u)$ be a non negative, increasing, left continuous real function defined on the interval $[0, \infty)$. Let u_0 be a non negative number fixed throughout. Define the function $n(\Phi, u_0; x)$ by

$$n(\Phi, u_0; x) = \sup \{s \geqq 0;\ \Phi(su) \leqq x\Phi(u),\ u \geqq u_0\}.$$

The function $n(\Phi, u_0; x)$ is manifestly increasing and the inequality

$$\Phi\big(n(\Phi, u_0; x)u\big) \leqq x\Phi(u), \quad u \geqq u_0,$$

holds whenever $n(\Phi, u_0; x)$ is finite.

The basic idea behind the function $n(\Phi, u_0; x)$, with $u_0 = 0$, seems to go back to D. W. Boyd [1]. The less restrictive definition we use here is taken from [3]. These appear named dilation functions in [7]; and are also considered in in [4].

The following properties of $n(\Phi, u_0; x)$ are easy consequences of the definition.

Let $\Phi(u)$, be as above, then

a) if $n(\Phi, u_0; x)$ is finite on $[0, a)$, then it is right continuous on $[0, a)$.

b) The inequality $n(\Phi, u_0; x) \geqq x$, for any $x \in (0, 1)$, holds true if and only if

$$\Phi(xu) \leqq x\Phi(u)$$

for any $u \geqq u_0$ and $x \in (0, 1)$.

c) For any $x \geqq 0$, and $y \geqq 1$, we have that

$$n(\Phi, u_0; x)\, n(\Phi, u_0; y) \leqq n(\Phi, u_0; xy).$$

Lemma 1. *Let $\Phi(u), u > 0$, be an increasing left continuous real function such that $\Phi(0) = 0$ and $\Phi(u) > 0$ for $u > 0$. If for any $y \in (0, 1)$ we have*

$$\Phi(yu) \leqq y\Phi(u),$$

for all $u > u_0$; then $n(\Phi, u_0; x)$ is continuous for any $x \geqq 1$.

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of real positive numbers whose limit is one, then

$$\Phi(x_k u) \leqq \Phi\big(n(\Phi, u_0; x_k)u\big) \leqq x_k \Phi(u)$$

for $k \in \mathbf{N}$ and any $u \geqq u_0$. By passing to the limit as $k \to \infty$, we get

$$\Phi(u) \leqq \Phi\big(n(\Phi, u_0; 1^-)u\big) \leqq \Phi(u),$$

that is, $n(\Phi, u_0; 1^-) = 1$; so that $n(\Phi, u_0; x)$ is continuous at 1.

Let $x_0 \geqq 1$, and $\{x_k\}_{k=1}^{\infty}$ be as above, then $n(\Phi, u_0, x_0^+) =$

$$= \lim_{n\to\infty} n(\Phi, u_0; x_0^+) n(\Phi, u_0; x_k) \leqq \lim_{n\to\infty} n(\Phi, u_0; x_0 x_k) = n(\Phi, u_0, x_0^-),$$

that is $n(\Phi, u_0; x_0^+) = n(\Phi, u_0; x_0^-)$.

If $n(\Phi, u_0; x)$ is supermultiplicative then we also get that, in the conditions of the previous Lemma, it is continuous for all $x \geqq 0$.

Lemma 2. *Let $\Phi(u)$, $u \geqq 0$, be an increasing, left continuous real function such that $\Phi(0)=0$. A necessary and sufficient condition that $n(\Phi, u_0; x)$ tend to infinity as x tends to infinity and be finite for finite values of the argument x, is that $\Phi(u)$ satisfy the Δ_2 condition for $u \geqq u_0$, and that*

$$\lim_{n\to\infty} \Phi(u) = \infty.$$

Proof. If $\Phi(2u) \leqq M\Phi(u)$, $u \geqq u_0$ then

$$\Phi(2^k u) \leqq M^k \Phi(u), \quad u \geqq u_0, \quad k \in \mathbf{N},$$

and consequently

$$n(\Phi, u_0; M^k) \geqq 2^k;$$

so that $n(\Phi, u_0; x) \to \infty$ as $x \to \infty$.

Suppose by absurd that, for some $x < \infty$, we have that $n(\Phi, u_0; x) = +\infty$, then for a fixed $u \geqq u_0$ and any $y > 0$, we have

$$\Phi(yu) < x\Phi(u).$$

However, this contradicts the fact that $\Phi(u) \to \infty$ as $u \to \infty$.

Hence, $n(\Phi, u_0; x)$ must be finite for finite x.

Conversely, if $n(\Phi, u_0; x) \to \infty$ as $x \to \infty$ and is finite for finite x, then, given $\lambda > 1$, there is some x_λ such that $n(\Phi, u_0; x_\lambda) > \lambda$. Thus,

$$\Phi(\lambda u) \leqq \Phi\big(n(\Phi, u_0; x_\lambda) u\big) \leqq x_\lambda \Phi(u),$$

for all $u \geqq u_0$. In particular x_λ must be larger than one.

Finally, if $\Phi(u)$ is bounded, say $\Phi(u) < K$ for all $u \geqq u_0$, with $K > 1$, then, taking some $\hat{x} > \dfrac{K}{\Phi(u_0)}$ we would have that

$$\Phi(yu) \leqq \hat{x}\Phi(u), \quad u \geqq u_0$$

for all $y > 1$. However, this contradicts the fact that $n(\Phi, u_0; \hat{x})$ is finite.

If in the previous Lemma we assume further that $\Phi(u)$ is a Young function then $n(\Phi; u_0; x)$ is positive for positive x. Also, $n(\Phi, u_0; x)$ is a concave function of x.

Definition. The function $N(\Phi, u_0; x)$, inverse to the function $n(\Phi, u_0; x)$, will be called the right dilation function of Φ.

For any Young function Φ, which satisfies the Δ_2 condition for $u \geqq u_0$, we have that $N(\Phi, u_0; x)$ is a convex function such that

$$N(\Phi, u_0; xy) \leqq N(\Phi, u_0; x) N(\Phi, u_0; y)$$

for any $x \geqq 0$, and $y \geqq 1$. Also, $N(\Phi, u_0; x)$ satisfies the $(\delta_2; \Delta_2)$ condition.

For $u_0=0$ $N(\Phi, u_0; x)=N(\Phi, x)$ is submultiplicative, that is

$$N(\Phi, xy) \leqq N(\Phi; x)\, N(\Phi; y),$$

for any $x, y\geqq 0$. Also,

$$\Phi(xu) \leqq N(\Phi, u_0; x)\, \Phi(u),$$

for all $u\geqq u_0$.

The following proposition gives an answer to an elementary question posed by KRASNOSELSKII and RUTITSKII [6] p. 30.

Proposition 1. *In each class of functions which satisfy the Δ' condition there is a submultiplicative function.*

PROOF. Let $\Phi(u)$ be a Young function which satisfies the Δ' condition for $u\geqq u_0$. We assume, as we may, that $\Phi(u)$ satisfies the (δ_2, Δ_2) condition.

The function $\hat{\Phi}(u)$, defined by

$$\hat{\Phi}(u) = \Phi(u_0 u), \quad u \geqq 0,$$

is equivalent to Φ and satisfies the Δ' condition for $u\geqq 1$. Then, the function $N(\hat{\Phi}; u)$ is equal to $\hat{\Phi}(u)$ in $[1, \infty)$; and

$$N(\hat{\Phi}; xy) \leqq N(\hat{\Phi}, x)\, N(\hat{\Phi}, y)$$

for all $x, y\geqq 0$.

One can also see that

$$N(N(\Phi; x); u) = N(\Phi; u)$$

Definition. Let Φ be a Young function that satisfies the (δ_2, Δ_2) condition. The function $K(\Phi; x)$ defined by

$$K(\Phi; x) = \inf_{0<u<\infty} \frac{\Phi(xu)}{\Phi(u)}$$

will be called the left dilation function of Φ.

It is easy to see that $K(\Phi; x)=\dfrac{1}{N(\Phi; 1/x)}$ for all $x>0$; so that

$$\frac{K(\Phi; x)}{x} = \frac{1/x}{N(\Phi;\, 1/x)}$$

is strictly increasing and

$$\int_0^x \frac{K(\Phi; t)}{t}\, dt$$

is convex. That is, $K(\Phi; x)$ is equivalent to a Young function that satisfies the (δ_2, Δ_2) condition. Also, $K(\Phi; x)$ is supermultiplicative, that is $K(\Phi; xy)\geqq K(\Phi, x)\times \times K(\Phi; y)$ for all $x, y\geqq 0$. Moreover, $K(K(\Phi, u), x)=K(\Phi, x)$.

One can also see that

$$\Phi(xu) \geqq K(\Phi; x)\, \Phi(u)$$

for all $x, u\geqq 0$.

For a Young function Φ not satisfying the (δ_2, Δ_2) condition, the study of the function $K(\Phi; x)$ is more complicate as the example of $K(e^u-1; x)$ shows. This is a concave function discontinuous at zero.

There is no mention of this function in [1]. However, it is safe to think that this author already studied this function.

Lemma 3. *Let $\Phi(u)$, $u \geqq 0$, be a Young function. If Φ satisfies the δ' and ϱ' conditions, then it is equivalent to x^p for some $P \geqq 1$.*

Proof. We have that $N(\Phi; x)$, $K(\Phi; x)$ and $\Phi(x)$ are all equivalent. It now follows from Theorem 1. That, for some $k \geqq 1, \alpha \geqq 1$ and all $x \geqq 0$.

$$x^\alpha \leqq N(\Phi, x) \leqq kx^\alpha.$$

Proposition 2. *Let $\Phi(u)$, $u \geqq 0$, be an N-function which satisfies the (δ_2, Δ_2) condition. A necessary and sufficient condition that the complementary function Ψ of Φ satisfy the (δ_2, Δ_2) condition is that for some $x > 1$, $K(\Phi; x) > x$.*

Proof. If, for some $x > 1$, $K(\Phi, x) > x$, then $K(\Phi, x) > \alpha x$, for some $\alpha > 1$; so that

$$\Phi(xu) > \alpha x \Phi(u), \quad u > 0,$$

Thus,

$$\Psi(2v) = \sup_{0<u} \{\alpha xuv - \Phi(xu)\} = \alpha x \Psi(v), \quad v \geqq 0.$$

If, on the other hand, ψ satisfies the (δ_2, Δ_2) condition then, there exist $\alpha > 1$, $x > 1$ such that

$$N(\Psi; \alpha) < \alpha x;$$

consequently

$$\Phi(xu) = \sup_{0<v} \{\alpha vxu - \Psi(\alpha v)\} = \alpha x \sup_{0<v} \left\{uv - \frac{\Psi(\alpha v)}{\alpha x}\right\} >$$

$$> \alpha x \sup_{0<v} \left\{uv - \frac{N(\Psi; \alpha)}{\alpha x} \Psi(v)\right\} > \alpha x \sup_{0<v} \{uv - \Psi(v)\} = \alpha x \Phi(u).$$

Therefore $K(\Phi, x) > \alpha x$.

Corollary. *The complementary Ψ to the function Φ satisfies the (δ_2, Δ_2) condition if and only if, for some $x < 1$, $N(\Phi; x) < x$.*

In terms of Orlicz spaces this result can be restated as follows:

Theorem 2. *Let l_Φ be separable space. We have that l_Φ is reflexive if and only if $N(\Phi; x) < x$ for some $x < 1$.*

In some instances the following theorem may also be of interest. We asume, as we may, that $\Phi_1(1) = \Phi_2(1) = 1$.

Theorem 3. *Let l_{Φ_1}, be a separable space. Assume that $K(\Phi_1; x)$ is convex. Then a necessary and sufficient condition that l_{Φ_1} be reflexive is that there exist a Young function Φ_2 which satisfies the (δ_2, Δ_2) condition and such that*

$$\Phi_1(u) \leqq \Phi_2(u), \quad u \in [0, 1].$$

Proof. If the property holds, then for some $x \in (0, 1)$ $\Phi_1(x) < \Phi_2(x)$; so that $K(\Phi_1; x) < N(\Phi_2; x) \leqq x$, for this x. Since $K(\Phi_1; x)$ is convex and $K(\Phi_1; 1) = 1$, we must have that $K(\Phi_1; x) < x$, for all x in $(0, 1)$. This in turn implies that $K(\Phi_1; x) > x$ for $x > 1$.

Thus, Ψ_1, the complementary to Φ_1, satisfies the (δ_2, Δ_2) condition and L_{Φ_1} is reflexive.

If, on the other hand, l_{Φ_1} is reflexive, then Ψ_1 satisfies the (δ_2, Δ_2) condition and this implies that $N(\Phi_1, x) < x$ for all x in $(0, 1)$. We see thus that $\Phi_1(x) < x$, $x \in [0, 1]$.

The case when the N-function Φ is submultiplicative is particulary simple.

Proposition 3. *If $\Phi(x)$, $x \geqq 0$, is a submultiplicative N-function; then l_Φ is reflexive.*

Proof. Since Φ is submultiplicative, then the complementary function Ψ is supermultiplicative, so that $\overline{\Psi} = 1/\Psi(1/x)$ satisfies the (δ_2, Δ_2) condition, that is $\overline{\Psi}(2x) \leqq M\overline{\Psi}(x)$, all $x \geqq 0$.

Therefore $\Psi(2x) = \dfrac{1}{\overline{\Psi}(1/2x)} \leqq \dfrac{1}{1/M(\overline{\Psi}(1/x))} = M\Psi(x)$, for all $x \geqq 0$.

From now on let us write K_i and N_i for the dilation functions of the Young function Φ_i.

Theorem 4. *Let Φ_1, Φ_2 be non equivalent Young functions that satisfy the (δ_2, Δ_2) condition and such that l_{K_2} is continuously embedded in l_{Φ_1}. If Φ_1 is submultiplicative, then every linear bounded operator T,*

$$T\colon\, l_{\Phi_2} \to l_{\Phi_1}$$

is strictly singular.

Proof. According to Theorem 2 the space l_{Φ_1} happens to be reflexive. Let T be a linear bounded operator

$$T\colon\, l_{\Phi_1} \to l_{\Phi_2}$$

and suppose that there exist subspaces $X \subset l_{\Phi_1}$, $Y \subset l_{\Phi_2}$ such that

$$T\colon\, X \to Y$$

is an isomorphism, then there exist normalized block basic sequences $\{B_k\}_{k=1}^\infty$, $\{A_k\}_{k=1}^\infty$

$$B_k = \sum_{i=P_k+1}^{i=P_{k+1}} t_i e_i,\ \{A_k\}_{k=1},\quad A_k = \sum_{j=q_k+1}^{j=q_{k+1}} r_j e_j$$

in X and Y respectively, where $\{e_i\}_{i=1}^\infty$ is the unit basis, such that

$$T(B_k) = A_k, \quad k \in \mathbf{N}.$$

Since Φ_1, Φ_2 are non equivalent, then there is a sequence $a = \{a_n\}_{n=1}^\infty$ such that $\sum_{k=1}^\infty K_2(|a_k|)$ converges and $\sum_{n=1}^\infty \Phi_1(|a_n|)$ diverges.

Let $x = \sum_{k=1}^\infty a_k \sum_{i=P_k+1}^{i=P_{k+1}} t_i e_i$, then we have

$$\sum_{k=1}^\infty \sum_{i=P_k+1}^{i=P_{k+1}} \Phi_2(|a_k|\,|t_i|) \geqq \sum_{k=1}^\infty K_2(|a_k|) \sum_{i=P_k+1}^{i=P_{k+1}} \Phi_2(|t_i|) = \sum_{k=1}^\infty K_2(|a_k|);$$

that is $\sum_{k=1}^{\infty}\sum_{i=P_k+1}^{i=P_{k+1}} \Phi_2(|a_k|\,|t_i|)$ diverges. On the other hand $T(x)$ is in Y. Indeed,

$$\sum_{k=1}^{\infty}\sum_{i=q_k+1}^{i=q_{k+1}} \Phi_1(|a_k r_j|) < \sum_{k=1}^{\infty}\Phi_1(|a_k|)\sum_{j=q_k+1}^{j=q_{k+1}}\Phi_1(|r_j|) = \sum_{k=1}^{\infty}\Phi_1(|a_k|) < \infty.$$

Contradiction.

Related results can be found in [8] and [10].

Theorem 5. *Let μ be a positive Radon measure defined on $\mathbf{R}^n$. Let $\Phi(u)$ be a Young function that satisfies the (δ_2, Δ_2) condition. Then the following conditions are equivalent.*

a) *if $f\in L_\Phi(\mathbf{R}^n,\mu)$ then $\tau(h)f(x)\in L_\Phi(\mathbf{R}^n,\mu)$ for all h in $\mathbf{R}^n$,*

b) *$\tau(h)$ is a continuous map of $L_\Phi(\mathbf{R}^n,\mu)$ to itself for any h,*

c) *there is a positive Lebesgue measurable function $\lambda(x)$ bounded with $\lambda(x)^{-1}$ over any compact set of values of x such that $\lambda(x)\,dx=d\mu$ and*

$$K^{-1}(\|\tau(h)\|) \leqq \sup\frac{\lambda(x+h)}{\lambda(x)} \leqq N^{-1}(\|\tau(h)\|).$$

Proof. If (a) holds, then $\mu(E)=0$ implies that $\mu(E+h)=0$ for all h. For, let $\mu(E)=0$ and let $f(x)=\infty$ for $x\in E$, $f(x)=0$ otherwise, so that $\int\Phi(|f(x)|)d\mu=0$; and since $\tau(h)f(x)=f(x-h)$ is infinity on $E+h$, then we must have that $\mu(E+h)=0$.

Let us now write $\tau(h)\mu=\mu_h$, that is

$$\int_{\mathbf{R}^n} f(x)\,d\mu_h = \int f(x+h)\,d\mu.$$

We see that μ_h is absolutely continuous with respect to μ and that μ is absolutely continuous with respect to μ_h; whence $d\mu_h=\varphi(x,h)\,d\mu$ with $\varphi(x,h)$ and $\varphi(x,h)^{-1}$ locally summable.

Therefore

$$\int f(x+h)\,d\mu = \int f(x)\,\varphi(x,h)\,d\mu.$$

Let us define

$$\varphi_n(x,h) = \min\{\varphi(x,h), 2^n\},$$

and

$$F_{h,n}(f) = f\big(x(N^{-1}(\varphi_n(x,h)))\big),\quad f\in L_\Phi(\mathbf{R}^n,\mu).$$

Then

$$\int\Phi\big(|F_{h,n}(f)|\big)\,d\mu \leqq \int\Phi\big(2^n|f(x)|\big)\,d\mu \leqq M^n\int\Phi\big(|f(x)|\big)\,d\mu,$$

so that $F_{h,n}$ is a linear bounded transformation of $L_\Phi(\mathbf{R}^n,\mu)$ to itself for any fixed n and h. Moreover $\|F_{h,n}\|\leqq 2^n$.

It now follows that $\sup\|\varphi_n(x,h)\|_\infty<\infty$ and $\varphi(x,h)$ is bounded for each h; whence $\tau(h)$ is bounded for each h.

We have thus proved (a)$\Rightarrow$(b). The converse is inmediate.

Let us now assume that (b) holds. Then for any $f \in L_\Phi$ with $\|f\| = \|f\|_{L_\Phi} > 0$, we have that

$$1 = \int_{\mathbf{R}^n} \Phi\left(\frac{|\tau(h)f|}{\|\tau(h)f\|}\right) d\mu = \int_{\mathbf{R}^n} \Phi\left(\frac{|f|}{\|\tau(h)f\|}\right) \varphi(x, h)\, d\mu \leqq$$

$$\leqq \int_{\mathbf{R}^n} \Phi\left(\frac{K^{-1}(\|\varphi(x, h)\|_\infty)\,|f|}{\|\tau(h)f\|}\right) d\mu,$$

so that $\|\tau(h)f\| \leqq K^{-1}(\|\varphi(x, h)\|_\infty)\,\|f\|$.

Given $\varepsilon > 0$ there is a set E such that

$$\int_{\mathbf{R}^n} \Phi(|\tau(h)\chi_E|)\, d\mu = \int_{\mathbf{R}^n} \Phi(|\chi_E|)\, \varphi(x, h)\, d\mu \geqq \int_{\mathbf{R}^n} \Phi\big(N^{-1}(\|\varphi(x, h)\|_\infty - \varepsilon)\,|\chi_E|\big)\, d\mu,$$

thus

$$\|\tau(h)\chi_E\| \geqq \big|N^{-1}(\|\varphi(x, h)\|_\infty - \varepsilon)\big|\,\|\chi_E\|.$$

We have thus proved that

$$N^{-1}(\|\varphi(x, h)\|_\infty) \leqq \|\tau(h)\| \leqq K^{-1}(\|\varphi(x, h)\|_\infty),$$

hence $\|\tau(h)\|$ is bounded or unbounded over any compact set of values of h together with $\|\varphi(x, h)\|_\infty$.

It now follows from the previous Lemma and the fact that $\log \|\tau(h)\|$ is subadditive that $\|\varphi(x, h)\|_\infty$ is bounded over any compact set of values of h.

Since μ is a Radon measure, it follows from the Radon—Nikodym theorem that

$$d\mu = \lambda(x)\, dx$$

with λ bounded over any compact. Thus

$$d\mu_h = \lambda(x+h)\, dx,$$

so that $\varphi(x, h) = \dfrac{\lambda(x+h)}{\lambda(x)}$, and

$$K(\|\tau(h)\|) \leqq \sup_x \frac{\lambda(x+h)}{\lambda(x)} \leqq N(\|\tau(h)\|).$$

This proves that (b) implies (c). It is easy to see that (c) implies (b).

Necessary conditions for the existence of non-trivial, linear, translation invariant operators acting on L_p spaces with general Radon measures subject to some conditions of regularity have been studied by J. L. B. Cooper [2].

We now pass on to examine the existence of operators acting on Orlicz spaces $L_{\Phi_1}(\mathbf{R}^n, \mu)$ and $L_{\Phi_2}(\mathbf{R}^2, \nu)$ where Φ_1 and Φ_2 satisfy the (δ_2, Δ_2) condition, $\mu = e^{a\|x\|}$ and $\nu = e^{b\|x\|}$.

In some important particular instances the condition that Φ_1 and Φ_2 satisfy the (δ_2, Δ_2) condition is necessary. For example, D. Boyd [1] has proved that, a necessary condition that the Hilbert transform be a map of the space of Lebesgue measurable functions $L_\Phi(\mathbf{R}^n)$ to itself, is that Φ satisfy the (δ_2, Δ_2) condition. This condition turns out to be sufficient.

Let $I\left(0, \frac{m}{2}\right)$ be the closed cube in $\mathbf{R}^n$ centred at 0 and having side m. Let $h(k, r, m)$ be the element in $\mathbf{R}^n$ whose components are all equal to $\frac{m|kr-k-1|}{r-1}$, where k is a natural number greater than or equal to one and $r>1$. We also write $H(k, r, m)= =\|h(h, r, m)\|$.

Lemma 4. a) *For any* $x\in I\left(0, \frac{m}{2}\right)$ *we have that*

$$\|x+h(k, r, m)\| \geqq \|x\|+\frac{H(k, r, m)}{r}.$$

b) *For any* $y\in I\left(0, \frac{m}{2}\right)+h(k, r, m)$, *we have that*

$$\|x+h(k+1, r, m)\| \geqq \|y\|,$$

holds for any $x\in I\left(0, \frac{m}{2}\right)$.

PROOF. a) The minimum of $\|x+h(k, r, m)\|$ with $x\in I\left(0, \frac{m}{2}\right)$ is attained at $x_m=\left(-\frac{m}{2}, \ldots, \frac{m}{2}\right)$ and its value is

$$\|x_m+h(k, r, m)\| = \sqrt{n}\left\{-\frac{m}{2}+m\frac{|kr-(k-1)|}{r-1}\right\}.$$

On the other hand, the maximum of $\|x\|+\frac{H(k, r, m)}{r}$ is attained at $x=x_m$ and at $x=-x_m$ and its value is

$$\|x_m\|+\frac{H(k, r, m)}{r} = \sqrt{n}\left\{\frac{m}{2}+\frac{m|kr-(k-1)|}{r(r-1)}\right\}.$$

Thus

$$\|x_m+h(k, r, m)\|-\|x_m\|-\frac{H(k, r, m)}{r} = \sqrt{n}\,\frac{m(r-1)(k-1)}{2} \geqq 0.$$

b) The maximum of $\|y\|$ is attained at $y=-x_m+h(k, r, m)$ and

$$\|-x_m+h(k, r, m)\| = \sqrt{n}\left\{\frac{m}{2}+m\frac{|kr-(k-1)|}{r-1}\right\}.$$

The minimum of $\|x+h(k+1, r, m)\|$ is attained at $x=x_m$ and,

$$\|x_m+h(k+1, r, m)\| = \sqrt{n}\left\{-\frac{m}{2}+m\frac{|(k-1)r-k|}{r-1}\right\}.$$

Therefore,

$$\|x_m+h(k+1, r, m)\|-\|-x_m+h(k, r, m)\| = \sqrt{n}\left\{-m+\frac{m(r-1)}{r-1}\right\} = 0.$$

The same results follow if we replace h by $-h$ throughout.

Theorem 6. *Let* $L_{\Phi_1}(\mathbf{R}^n, \mu)$ *and* $L_{\Phi_2}(\mathbf{R}^n, \nu)$ *be Orlicz spaces defined by the Young functions* $\Phi_1(u)$, $\Phi_2(u)$ *that satisfy the* (δ_2, Δ_2) *condition, where* $\mu = e^{a\|x\|}$ *and* $\nu = e^{b\|x\|}$. *Then, in order that there should exist a nonzero, translation invariant, bounded operator*

$$T: L_{\Phi_1}(\mathbf{R}^n, \mu) \to L_{\Phi_2}(\mathbf{R}^n, \nu),$$

it is necessary that, for any natural number $s \geqq 1$ *and any real number* $r, r > 1$,

$$\liminf_{m\to\infty} \frac{K_1\big(1+\sum_{k=1}^{s} e^{aH(k,r,m)}\big)}{N_2\big(1+\sum_{k=1}^{s} e^{b/rH(k,r,m)}\big)} \geqq 1,$$

where the expression $H(k, r, m)$ *is defined as in Lemma 4.*

Proof. Let $I\left(0, \frac{m}{2}\right)$ and $h(k, r, m)$ be as in the previous Lemma. For any function $f(x)$ we write $f_m(x)$ for $\chi_{m(x)/2}\, f(x)$, where $\chi_{m/2}(x)$ stands for the characteristic function of $I\left(0, \frac{m}{2}\right)$.

On account of part (b) of the previous Lemma we have that, for any $f \in L_{\Phi_1}(\mathbf{R}^n, \mu)$

$$\int_{\mathbf{R}^n} \Phi_1\Big(\Big|f_m(x) + \sum_{k=1}^{s} \tau\big(h(k, r, m)\big) f_m(x)\Big|\Big)\, d\mu =$$

$$(1) \qquad = \int_{\mathbf{R}^n} \Phi_1\big(|f_m(x)|\big)\, d\mu + \sum_{k=1}^{s} \Phi_1\big(|f_m(x)|\big)\, e^{a\|x+h(k,r,m)\|}\, d\mu \leqq$$

$$\leqq \big(1+\sum_{k=1}^{s} e^{aH(k,r,m)}\big) \int_{\mathbf{R}^n} \Phi_1(|f_m|)\, d\mu.$$

Similarly,

$$\int_{\mathbf{R}^n} \Phi_2\Big(\Big|(Tf_m)_m + \sum_{k=1}^{s} \tau\big(h(k, r, m)\big)(Tf_m)_m\Big|\Big)\, d\nu =$$

$$= \int_{\mathbf{R}^n} \Phi_2\big(|(Tf_m)_m|\big)\, d\nu + \sum_{k=1}^{s} \int_{\mathbf{R}^n} \Phi_2\big(|\tau(h(k, r, m))(Tf_m)_m|\big)\, d\nu,$$

and, by virtue of part (a) of the previous Lemma, this expression is greater than or equal to

$$(2) \qquad \big(1+\sum_{k=1}^{s} e^{(b/r)H(k,r,m)}\big) \int_{\mathbf{R}^n} \Phi_2\big(|(Tf_m)_m|\big)\, d\nu.$$

From (1) and (2) above, it follows that

$$N_2\big(1+\sum_{k=1}^{s} e^{(b/r)H(k,r,m)}\big) \|(Tf_m)_m\|_{L_{\Phi_2}} \leqq \Big\|(Tf_m)_m + \sum_{k=1}^{s} \tau\big(h(k, r, m)\big)(Tf_m)_m\Big\|_{L_{\Phi_2}} \leqq$$

$$\leqq \Big\|Tf_m + \sum_{k=1}^{s} \tau\big(h(k, r, m)\big)\, Tf_m\Big\|_{L_{\Phi_2}} \leqq \|T\|\, \Big\|f_m + \sum_{k=1}^{s} \tau\big(h(k, r, m)\big) f_m\Big\|_{L_{\Phi_1}},$$

so that

$$\|(Tf_m)_m\|_{L_{\Phi_2}} \leqq \|T\| \frac{K_1\left(1+\sum_{k=1}^{s} e^{aH(k,r,m)}\right)}{N_2\left(1+\sum_{k=1}^{s} e^{(b/r)H(k,r,m)}\right)} \|f_m\|_{\Phi}. \tag{3}$$

We prove next that $\|(Tf_m)_m\|_{L_{\Phi_2}} \to \|Tf\|_{L_{\Phi_2}}$ as $m\to\infty$. In fact

$$\|(Tf_m)_m - Tf\|_{L_{\Phi_2}} \leqq \|(Tf_m)_m-(Tf)_m\|_{L_{\Phi_2}} + \|(Tf)_m - Tf\|_{L_{\Phi_2}} \geqq$$

$$\leqq \|Tf_m - Tf\|_{L_{\Phi_2}} + \|(Tf)_m - Tf\|_{L_{\Phi_2}} \leqq \|T\|\,\|f_m - f\|_{L_{\Phi_1}} + \|(Tf)_m - Tf\|_{L_{\Phi_2}}$$

and this expression tends to 0 as $m\to\infty$.

Therefore, by passing to the limit as $m\to\infty$ on both sides of the expression (3) above, we see that, if for some natural number s and a real number $r>1$,

$$\liminf_{m\to\infty} \frac{K_1\left(1+\sum_{k=1}^{s} e^{aH(k,r,m)}\right)}{N_2\left(1+\sum_{k=1}^{s} e^{(b/r)H(k,r,m)}\right)} < 1,$$

then $T=0$.

Thus, in order that T be different from zero it is necessary that

$$\liminf_{m\to\infty} \frac{K_1\left(1+\sum_{k=1}^{s} e^{aH(k,r,m)}\right)}{N_2\left(1+\sum_{k=1}^{s} e^{(b/r)H(k,r,m)}\right)} \geqq 1$$

for any natural number s and any real $r>1$.

In particular, if $a=b=0$, then the condition above becomes

$$\frac{K_1(1+s)}{N_2(1+s)} \geqq 1.$$

If $\Phi_1(u)=u^p, p>1$ and $\Phi_2(u)=u^q, q>1$ the condition is $(1+s)^{1/p}>(1+s)^{1/q}$, that is $q\geqq p$. (HÖRMANDER [5] p. 96.)

If $a\neq 0, b\neq 0, \Phi_1(u)=u^p, p>1$ and $\Phi_2(u)=u^q, q>1$, then the condition that T be different from zero is $\frac{a}{p}-\frac{b}{rq}\geqq 0$ for any $r>1$, as becomes apparent from writing out explicitly the condition found in the theorem above. In this case we see that $\frac{a}{p}\leqq\frac{b}{q}$ (COOPER [2], p. 44).

A more clear picture emerges when we consider the same problem by replacing the spaces L_Φ with Orlicz spaces l_Φ.

Let us recall that, given a Young function Φ, the indices α_Φ an β_Φ are defined as follows (see [8])

$$\alpha_\Phi = \sup\left\{p > 0;\ \sup_{0<x,t\leqq 1} \frac{\Phi(tx)}{\Phi(t)x^p} < \infty\right\}$$

$$\beta_\Phi = \inf\left\{p > 0;\ \inf_{0<x,t\leqq 1} \frac{\Phi(tx)}{\Phi(t)x^p} > 0\right\}.$$

We now prove:

Lemma 5. *Let us write $F(t)$ for any of the functions N, K, Φ. Let α, β be as in Theorem* 1. *Then, the interval* $[\alpha_F, \beta_F]$ *is contained in* $[\alpha, \beta]$.

Proof. Let $\varepsilon>0$, then for some $a_\varepsilon>0$.

$$\frac{F(\lambda t)}{F(t)\lambda^{\alpha-\varepsilon}} \leqq \frac{F(t)N(\lambda)}{F(t)\lambda^{\alpha-\varepsilon}} \leqq \frac{\lambda^{\alpha-\varepsilon}}{\lambda^{\alpha-\varepsilon}}, \quad \lambda\in[0, \alpha_\varepsilon],$$

that is $\alpha-\varepsilon\leqq\alpha_F$ and so $\alpha\leqq\alpha_F$.

Also,

$$\frac{F(\lambda t)}{F(t)\lambda^{\beta+\varepsilon}} \geqq \frac{F(t)K(\lambda)}{F(t)\lambda^{\beta+\varepsilon}} \geqq \frac{\lambda^{\beta+\varepsilon}}{\lambda^{\beta+\varepsilon}}, \quad \lambda\in\left(0, \frac{1}{b_\varepsilon}\right).$$

It follows that $\beta\geqq\beta_F$.

Let $\Phi(u)$, $u\geqq 0$, be an N-function such that $\sup\limits_{0<u<\infty} \dfrac{\Phi(u\lambda)}{\Phi(u)}$ is attained on the interval [0, 1]. Then

$$\sup_{0<\lambda_1 t} \frac{\Phi(\lambda t)}{\Phi(\lambda)t^{\alpha+\varepsilon}} \geqq \sup_{0<x\leqq 1}\left\{\sup_{0<x\leqq 1} \frac{\Phi(t\lambda)}{\Phi(\lambda)t^{\alpha+\varepsilon}}\right\} = \sup_{0<t\leqq 1} \frac{N(t)}{t^{\alpha+\varepsilon}} \geqq \sup_{0<t\leqq 1} \frac{t^\alpha}{t^{\alpha+\varepsilon}} = \infty,$$

that is, $\alpha\geqq\alpha_\Phi$. It now follows from the above Lemma that $\alpha=\alpha_\Phi$.

A similar calculation shows us that, if $\inf\limits_{0<u<\infty} \dfrac{\Phi(u\lambda)}{\Phi(u)}$ is attained on [0, 1], then $\beta=\beta_\Phi$.

These conditions hold for $N(\Phi; x)$ and $K(\Phi, x)$ respectively. We thus have that $\alpha=\alpha_N$ and $\beta=\beta_K$ also

$$\lim_{t\to 0} \frac{LN(t)}{Lt} = \alpha_N,$$

and

$$\lim_{t\to\infty} \frac{LN(t)}{Lt} = \beta = \lim_{t\to\infty} \frac{L\dfrac{1}{K(1/t)}}{Lt} = \lim_{t\to 0} \frac{LK(t)}{Lt} = \beta_K.$$

Lemma 6. *Let $N(t)$ be submultiplicative and $K(t)$ supermultiplicative functions defined on* [0, 1] *such that*

$$N(0) = K(0) = 0,$$

$$N(1) = K(1) = 1.$$

If $N(t)$ and $K(t)$ are not equivalent in any set $[0, \delta]$ with $\delta \leqq 1$, then we must have that either $N(t) < K(t), \quad x \in (0, 1)$ *or* $K(t) \leqq N(t), \quad x \in [0, 1]$.

PROOF. Assume that neither case hold; then we have that, for some decreasing sequence $\{t_n\}_{n=1}^{\infty}$, with $t_1=1$ and $\lim_{n\to\infty} t_n=0$,

$$N(t_n) = K(t_n), \quad n \in N.$$

The function $F(t)=\dfrac{tN(t)}{K(t)}$, $t>0$, is submultiplicative and

$$\alpha_F = \liminf_{t\to 0} \frac{tF'(t)}{F(t)} \geqq 1+\liminf_{t\to 0} \frac{tN'(t)}{N(t)} - \limsup_{t\to 0} \frac{tK'(t)}{K(t)} = 1+\alpha_N-\beta_K.$$

From $N(t_n)=K(t_n)$ we deduce that

$$\alpha_N = \lim_{t\to 0} \frac{LN(t)}{Lt} = \lim_{t\to 0} \frac{LK(t)}{Lt} = \beta_K;$$

that is $\alpha_F \geqq 1$. Assume $\alpha_F > 1$, then we deduce from Theorem 1 that given $\varepsilon>0$ there is $\delta>0$ such that

$$t^{\alpha_F} \leqq \frac{tN(t)}{K(t)} \leqq t^{\alpha_F-\varepsilon} < t, \quad t \in (0, \delta).$$

By placing $t=t_n$, we get

$$t_n \leqq t_n^{\alpha_F-\varepsilon} < t_n.$$

Contradiction. We must have that $\alpha_F=1$. This in turn implies that

$$\frac{tN(t)}{K(t)} \geqq t, \quad t<1,$$

and so $N(t) \geqq K(t)$, $t<1$.

Proceeding exactly as in Theorem 6, we see that a necessary condition that there exist a linear bounded, translation invariant operator $T\colon l_{\Phi_1} \to l_{\Phi_2}$ is that

$$\liminf_{x\to\infty} \frac{K_1(x)}{N_2(x)} \geqq 1$$

and since $\dfrac{K_1(x)}{N_2(x)} = \dfrac{N_1\left(\frac{1}{x}\right)^{-1}}{K_2\left(\frac{1}{x}\right)^{-1}} = \dfrac{K_2\left(\frac{1}{x}\right)}{N_1\left(\frac{1}{x}\right)}$, then this condition is equivalent to the condition

$$\liminf_{x\to 0} \frac{K_2(x)}{N_1(x)} \geqq 1.$$

In the following Theorem, by l_Φ we mean the Banach space of all sequences $\{a_n\}_{n=-\infty}^{n=+\infty}$ such that $\sum_{n=-\infty}^{n=+\infty} \Phi(|a_n|) < \infty$.

Theorem 7. A necessary and sufficient condition that there exist a linear bounded, translation invariant operator

$$T\colon l_{\Phi_1} \to l_{\Phi_2}$$

is that $N_1(x) \prec K_2(x)$, for all x in $(0, 1)$.

PROOF. If N_1 and K_2 are equivalent in $[0, 1]$ then there is nothing to prove. Otherwise, according to the previous Lemma we have that

$$\text{either} \quad K_2(x) \leqq N_1(x) \quad \text{or} \quad K_2(x) > N_1(x) \quad \text{on} \quad [0, 1].$$

Assume the first case. If

$$\liminf_{x \to 0} \frac{K_2(x)}{N_1(x)} = 1,$$

then, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\frac{K_2(x)}{N_1(x)} > 1 - \varepsilon, \quad x \in (0, \delta);$$

that is $(1-\varepsilon) N_1(x) < K_2(x) \leqq N_1(x)$, $x \in (0, \delta)$, and since $\dfrac{K_2(x)}{N_1(x)}$ is bounded, and bounded away from zero on $[\delta, 1]$, we see that $K_2(x)$ and $N_1(x)$ are equivalent on $[0, 1]$. Contradiction; we must have then that in this case

$$\liminf_{x \to 0} \frac{K_2(x)}{N_1(x)} < 1.$$

In the second case it is apparent that

$$\liminf_{x \to 0} \frac{K_2(x)}{N_1(x)} \geqq 1.$$

Also, since $N_1(x) \prec K_2(x)$, $x \in (0, 1)$ implies that $\Phi_1(x) \prec \Phi_2(x)$, $x \in (0, 1)$; we can see that the identity $I\colon l_{\Phi_2}, \to L_{\Phi_1}$ is continuous.

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