Convolution of temperate distributions

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Let $\|\cdot\|_0$ be the norm in the space $L^2(R^n)$ of square integrable functions. We put $L_q=\{\varphi:R^n\to C;\ \|\varphi\|_q^2=\sum_{|\alpha+\beta|\leqq q}\|x^\alpha D^\beta\varphi\|_0^2<+\infty\}$, $q\in N$. Here $D^\beta\varphi$ is the generalized derivative defined by Sobolev in [1]. Each space L_q, and its dual L_{-q}, is Hilbert. The $\operatorname{proj\,lim}_{q\to\infty} L_q=\mathscr{S}$ is the space of rapidly decreasing functions, and the $\operatorname{ind\,lim}_{q\to\infty} L_{-q}=\mathscr{S}'$ is the space of temperate distributions. The Fourier transformation $\mathscr{F}:L_k\to L_k$ is a topological isomorphism for any integer k.

For $\varphi\in\mathscr{S}$, $f\in\mathscr{S}'$, we use the dilation $d_\lambda\varphi(x)=\varphi(\lambda x)$, $x\in R^n$, $\lambda\in R$, $\lambda\neq 0$, $(d_\lambda f)\varphi=|\lambda|^{-n}f(d_{\lambda^{-1}}\varphi)$ and translation $(\tau_h\varphi)(x)=\varphi(x-h)$, $x,h\in R^n$, $(\tau_h f)\varphi=$ $=f(\tau_{-h}\varphi)$. It is convenient to introduce the weight-function $W(x)=(1+|x|^2)^{1/2}$, $x\in R^n$.

Definition. Let $f\in\mathscr{S}'$ and $\varphi\in\mathscr{S}$. The C^∞-function $x\mapsto f(\tau_x d_{-1}\varphi)$ is called convolution of f and φ and is denoted by $f*\varphi$. For each $p,q\in N$, we define

$$0_{p,q}^*=\{f\in\mathscr{S}';\ \|f\|_{p,q}^*=\sup\{\|f*\varphi\|_q;\ \varphi\in\mathscr{S},\ \|\varphi\|_p\leqq 1\}<+\infty\}.$$

The space $\mathscr{S}$ is dense in L_p. Hence for each $f\in 0_{p,q}^*$ the continuous map $\varphi\mapsto f*\varphi$: $(\mathscr{S},\|\cdot\|_p)\to L_q$ can be continuously extended to L_p. Therefore $0_{p,q}^*$ is a subspace of $\mathscr{L}(L_p,L_q)$. We denote the norm of $\mathscr{L}(L_p,L_q)$ restricted to $0_{p,q}^*$ by $\|\cdot\|_{p,q}^*$.

The identity maps $\mathrm{id}: 0_{p,q+1}^*\to 0_{p,q}^*\to 0_{p+1,q}^*$, $p,q\in N$ are continuous. Hence the limits $0_q^*=\operatorname{ind\,lim}_{p\to\infty} 0_{p,q}^*$ and $0^*=\operatorname{proj\,lim}_{q\to\infty} 0_q^*$ make sense.

Definition. Let $f\in 0_{p,q}^*$ and $g\in L_{-q}$. The functional $\varphi\mapsto g(d_{-1}f*\varphi):L_p\to C$ is called the convolution of f and g. We denote it by $f*g$.

For $f\in 0_{p,q}^*$ and $g\in\mathscr{S}$ we have two definitions of $f*g$. They coincide. Also, if f and g are both functions, our convolution $f*g$ is the same as the convolution defined by an integral.

Theorem 1. *A linear map $L:L_{-q}\to L_{-p}$, resp. $L:L_{-q}\to\mathscr{S}'$, $p,q\in N$, is translation invariant and continuous iff there exists a unique $f\in 0_{p,q}^*$, resp. $f\in 0_q^*$, such that $Lg=f*g$, $g\in L_{-q}$.*

PROOF. The only if part is evident. Let $L:L_{-q}\to L_{-p}$ be translation invariant and continuous. Its conjugate L^*: $L_p\to L_q$ is then also translation invariant and continuous. Hence $L^*\varphi\in C^\infty(R^n)$ for any $\varphi\in\mathscr{S}$. The continuous functional $\varphi\mapsto(L^*d_{-1}\varphi)(0)$: $\mathscr{S}\to C$ defines a distribution $f\in\mathscr{S}'$.

For any $\varphi\in\mathcal{S}$, $x\in R^n$, we have $(f*\varphi)(x)=f(\tau_x d_{-1}\varphi)=(L^* d_{-1}\tau_x d_{-1}\varphi)(0)=$ $=(L^*\tau_{-x}\varphi)(0)=\tau_{-x}(L^*\varphi)(0)=(L^*\varphi)(x)$. Further,

$$\|f\|^*_{p,q} = \sup\{\|L^*\varphi\|_q, \varphi\in\mathcal{S}, \|\varphi\|_p \leqq 1\} = \|L^*\|, \quad \text{i.e.,} \quad f\in 0^*_{p,q}.$$

The uniqueness of f is evident and the respective part of the theorem immediately follows.

Definition. Let $\mathcal{F}$ be the Fourier transformation with the kernel $\exp(-2\pi ix, y)$ and Δ the Laplace operator. Each map $f\mapsto(1-\Delta)^{-k}f=\mathcal{F}^{-1}(d_{2\pi}W^{-2k}\cdot\mathcal{F}f)\colon \mathcal{S}'\to\mathcal{S}'$, $k\in N$, is injective. We define spaces $(1-\Delta)^{-k}L_{-q}=\{(1-\Delta)^{-k}f; f\in L_{-q}\}$ and $(1-\Delta)^k L_q=\{f\in\mathcal{S}'; (1-\Delta)^{-k}f\in L_q\}$, $k, q\in N$, and provide them with the topologies which make the operator $(1-\Delta)^{-k}$ a topological isomorphism.

For each $k, g\in N$, the spaces $(1-\Delta)^{-k}L_{-q}$ and $(1-\Delta)^k L_q$ are Hilbert and mutually dual with the duality form:

$$\langle u, v\rangle\mapsto((1-\Delta)^k u), ((1-\Delta)^{-k}v)\colon (1-\Delta)^{-k}L_{-q}\times(1-\Delta)^k L_q\to C.$$

Lemma. *$\mathcal{S}$ is dense in each $(1-\Delta)^{-k}L_{-q}$.*

Proof. The operator $(1-\Delta)^{-k}\colon \mathcal{S}\to\mathcal{S}$ is bijective. Since $\mathcal{S}$ is dense in L_{-q}, $\mathcal{S}=(1-\Delta)^{-k}\ \mathcal{S}$ is dense in $(1-\Delta)^{-k}L_{-q}$.

Proposition. *For any $k, q\in N$, we have:*

(1) $L_q\subset(1-\Delta)L_q\subset(1-\Delta)^2L_q\subset\ldots$ *with all maps* $\mathrm{id}\colon(1-\Delta)^k L_q\to(1-\Delta)^{k+1}L_q$ *continuous.*

(2) $L_{-q}\supset(1-\Delta)^{-1}L_{-q}\supset(1-\Delta)^{-2}L_{-q}\supset\ldots$ *with all maps* $\mathrm{id}\colon(1-\Delta)^{-k-1}L_{-q}\to(1-\Delta)^{-k}L_{-q}$ *continuous.*

(3) $\operatorname{ind}\lim_{k\to\infty}(1-\Delta)^k L_q$ *is the strong dual of the Fréchet space* $\operatorname{proj}\lim_{k\to\infty}(1-\Delta)^{-k}L_{-q}$.

Proof. (1) & (2) are evident.

(3) follows from [3, CH IV, Th. 4.4] since, by lemma,

$$\operatorname{proj}\lim_{k\to\infty}(1-\Delta)^{-k}L_{-q} \quad \text{is dense in each} \quad (1-\Delta)^{-k}L_{-q}.$$

Theorem 2. $\operatorname{ind}\lim_{k\to\infty}(1-\Delta)^k L_q=0^*_q$, $q\in N$.

Proof. Take $f\in 0^*_{p,q}$ and put $r=1+\left[\frac{1}{2}n\right]$, $k=1+\left[\frac{1}{2}(p+r)\right]$. Then $W^{-2k}\in L_p$, $\varphi=\mathcal{F}d_{2\pi}W^{-2k}\in L_p$, and $g=f*\varphi\in L_q$. Since $\mathcal{F}g=\mathcal{F}f\cdot d_{2\pi}W^{-2k}$, we have $g=(1-\Delta)^{-k}f$ and $f\in(1-\Delta)^k L_q$. The maps

$$f\mapsto f*\varphi\mapsto(1-\Delta)^k(f*\varphi)=f\colon\ 0^*_{p,q}\to L_q\to(1-\Delta)^k L_q$$

are both continuous, hence $\mathrm{id}\colon 0^*_{p,q}\to(1-\Delta)^k L_q$ and $\mathrm{id}\colon 0^*_q\to\operatorname{ind}\lim_{k\to\infty}(1-\Delta)^k L_q$ are continuous too.

In the sequel we use the inequality $|x| \leqq W(x-y) \cdot W(y)$, $x, y \in R^n$. Take $k, q \in N$, $g \in L_q$, $\varphi \in \mathscr{S}$, and put $\psi = (1-\Delta)^k \varphi \in \mathscr{S}$. Then there are constants A, B, C such that

$$\|(1-\Delta)^k g * \varphi\|_q^2 = \|g * \psi\|_q^2 = \sum_{|\alpha+\beta| \leqq q} \int_{R^n} x^{2\alpha} |g * D^\beta \psi|^2 \, dx =$$

$$= \sum_{|\alpha+\beta| \leqq q} \int_{R^{3n}} x^{2\alpha} g(x-y) D^\beta \psi(y) \overline{g(x-z)} D^\beta \overline{\psi(z)} \, dx \, dy \, dz \leqq$$

$$\leqq \sum_{|\alpha+\beta| \leqq q} \int_{R^{2n}} W^{|\alpha|}(y) D^\beta \psi(y) W^{|\alpha|}(z) D^\beta \overline{\psi(z)} \int_{R^n} W^{|\alpha|}(x-y) \, g(x-y)$$

$$W^{|\alpha|}(x-z) \overline{g(x-z)} \, dx \, dy \, dz \leqq A \|g\|_q^2 \sum_{|\alpha+\beta| \leqq q} \|W^{|\alpha|} D^\beta \psi\|_0^2 \leqq B \|\psi\|_q^2 \leqq C \|\varphi\|_{q+2k}^2.$$

Hence $(1-\Delta)^k g \in 0^*_{q+2k, q}$ and

$$\text{id}\colon (1-\Delta)^k L_q \to 0^*_{q+2k, q} \quad \text{as well as} \quad \text{id}\colon \operatorname{ind}\lim_{k \to \infty} (1-\Delta)^k L_q \to 0^*_q$$

are continuous.

Consequences:

(1) id: $0^*_q \to \mathscr{S}'$ is continuous,

(2) 0^*_q is strong dual of the Fréchet space $\operatorname{proj}\lim_{k \to \infty} (1-\Delta)^{-k} L_{-q}$.

Theorem 3. *Each 0^*_q is a complete, reflexive, and bornological space.*

Proof. The space $\operatorname{proj}\lim_{k \to \infty} (1-\Delta)^{-k} L_{-q}$ is reflexive, see [3, CH IV, §§ 5.5, 5.6, 5.8]. The strong dual 0^*_q is reflexive and complete. 0^*_q is bornological as an inductive limit of Hilbert spaces $(1-\Delta)^k L_q$.

Theorem 4. *Let $f \in \mathscr{S}'$. Then*

(1) $f \in 0^*_q$ iff $f = \sum_{\alpha \in A} D^\alpha f_\alpha$, where $f_\alpha \in L_q$, $\alpha \in A$, and $A \subset N^n$ is finite.

(2) $f \in 0^*_q$ iff the map $\varphi \mapsto W^q(f * \varphi)\colon \mathscr{S} \to L^2(R^n)$ is continuous.

Proof. (1) Take $\alpha \in N^n$, integer $k \geqq \frac{1}{2}|\alpha|$, and $f \in L_q$. We have $\mathscr{F} f \in L_q$ and $g = \mathscr{F}^{-1}\left((2\pi i x)^\alpha W^{-2k}(2\pi x) \mathscr{F} f\right) \in L_q$. Then $(1-\Delta)^{-k} D^\alpha f = \mathscr{F}^{-1}\left(d_{2\pi} W^{-2k} \mathscr{F}(D^\alpha f)\right) = g \in L_q$ and $D^\alpha f \in (1-\Delta)^k L_q \subset 0^*_q$. The inverse implication follows from Theorem 2.

(2) Let $f \in 0^*_q$. Then the maps $\varphi \mapsto f * \varphi \mapsto W^q(f * \varphi)\colon \mathscr{S} \to L_q \to L^2(R^n)$ are continuous.

Assume $\varphi \mapsto W^q(f * \varphi)\colon \mathscr{S} \to L^2(R^n)$ to be continuous. For each $\alpha, \beta \in N^n$, $|\alpha+\beta| \leqq q$, the maps

$$\varphi \mapsto D^\beta \varphi \mapsto x^\alpha (f * D^\beta \varphi) = x^\alpha D^\beta (f * \varphi)\colon \mathscr{S} \to \mathscr{S} \to L^2(R^n)$$

are continuous. Hence $\varphi \to f * \varphi\colon \mathscr{S} \to L_q$ is continuous too and $f \in 0^*_q$.

Theorem 5. *The inductive topology of 0_q^* is generated by the family of seminorms $f \mapsto \|f * \varphi\|_q$, $\varphi \in \mathscr{S}$.*

Proof. Denote by T the topology generated by these seminorms and by T_i the inductive topology of 0_q^*. Since each seminorm $f \mapsto \|f * \varphi\|_q : (0_q^*, T_i) \to R$ is continuous, T is weaker than T_i.

Take $U \in T_i$, $0 \in U$. By Theorem 2 there is a bounded set $B \subset \operatorname{proj}\lim_{k\to\infty} (1-\Delta)^{-k} L_{-q}$ such that the polar $B^0 \subset U$. By [6, Lemma 4] there exists $\varphi \in \mathscr{S}$ such that $B \subset \varphi * D$, where D is the unit ball in L_{-q}.

For $v \in (1-\Delta)^k L_q$ and $\psi \in \mathscr{S}$, we have $\langle \varphi * \psi, v\rangle = (1-\Delta)^k \varphi * \psi)((1-\Delta)^{-k} v) = \psi((1-\Delta)^k d_{-1}\varphi * (1-\Delta)^{-k} v) = \psi(d_{-1}\varphi * v)$. Since $\mathscr{S}$ is dense in L_{-q} we can, for any $f \in 0_q^*$, write

$$\sup\{|\langle g, f\rangle|;\ g \in B\} \leqq$$

$$\leqq \sup\{|\langle \varphi * h, f\rangle|;\ h \in D\} = \sup\{|\langle \varphi * \psi, f\rangle|;\ \psi \in \mathscr{S},\ \|\psi\|_{-q} \leqq 1\} =$$

$$= \sup\{|\psi(d_{-1}\varphi * f)|;\ \psi \in \mathscr{S},\ \|\psi\|_{-q} \leqq 1\} = \|d_{-1}\varphi * f\|_q.$$

Hence U contains $\{f \in 0_q^*;\ \|d_{-1}\varphi * f\|_q \leqq 1\} \in T$.

For $p, q \in N$, we put $0_{p,q} = \{f : R^n \to C;\ \varphi \mapsto f\varphi : L_q \to L_q \text{ continuous}\}$. Each $0_{p,q}$ with the topology of $\mathscr{L}(L_p, L_q)$ is a Banach space. We denote its norm by $\|\cdot\|_{p,q}$.

Theorem 6. *The Fourier transformation $\mathscr{F} : 0_{p,q}^* \to 0_{p,q}$, $p, q \in N$, is a topological isomorphism.*

Proof. For convenience, put $A_k = \sup\{\|\mathscr{F} f\|_k;\ f \in L_k,\ \|f\|_k \leqq 1\}$, $k \in N$. Take $f \in 0_{p,q}^*$, $\lambda > 0$, and choose $\psi \in \mathscr{S}$ such that $\mathscr{F}\psi(x) = 1$ for $|x| < \lambda$. Then $f * \psi \in L_q$ and $\mathscr{F}(f * \psi) = \mathscr{F} f \cdot \mathscr{F}\psi \in L_q$ are both functions. Thus $\mathscr{F} f$, restricted to $\{x \in R^n;\ |x| < \lambda\}$, is a function too. Since λ was arbitrary, $\mathscr{F} f$ is a function.

For any $\varphi \in \mathscr{S}$, $\|\mathscr{F} f \cdot \mathscr{F}\varphi\|_q \leqq A_q \|f * \varphi\|_q \leqq A_q \|f\|_{p,q}^* \|\varphi\|_p \leqq A_p A_q \|f\|_{p,q}^* \|\mathscr{F}\varphi\|_p$. Hence $\mathscr{F} f \in 0_{p,q}$ and $\mathscr{F} : 0_{p,q}^* \to 0_{p,q}$ is continuous.

Let $g \in 0_{p,q}$. By [5, Lemma 3], $0_{p,q} \subset L_{q-p-r}$, $r = 1 + \left[\frac{1}{2} n\right]$. This implies $0_{p,q} \subset \mathscr{S}'$ and $g \in \mathscr{S}'$. For any $\varphi \in \mathscr{S}$, we have $\varphi W^n \in L_p$. Then $g\varphi W^n \in L_q$ and the integral in the next formula is absolutely convergent.

$$\mathscr{F}^{-1}(g\varphi)(x) = \int_{R^n} g(t)\,\varphi(t)\,W^n(t) W^{-n}(t) \exp(2\pi i x, t)\,dt =$$

$$= g_t(\varphi(t)\exp(2\pi i x, t)) = \mathscr{F}^{-1} g(\tau_x \mathscr{F}\varphi) = \mathscr{F}^{-1} g(\tau_x d_{-1} \mathscr{F}^{-1}\varphi) =$$

$$= (\mathscr{F}^{-1} g * \mathscr{F}^{-1}\varphi)(x).$$

Further

$$\|\mathscr{F}^{-1} g * \mathscr{F}^{-1} g\|_q = \|\mathscr{F}^{-1}(g\varphi)\|_q \leqq$$

$$\leqq A_q \|g\varphi\|_q \leqq A_q \|g\|_{p,q} \|\varphi\|_p \leqq A_p A_q \|g\|_{p,q} \|\mathscr{F}^{-1}\varphi\|_p.$$

Hence $\mathscr{F}^{-1} g \in 0_{p,q}^*$ and $\mathscr{F}^{-1} : 0_{p,q} \to 0_{p,q}^*$ is continuous.

Theorem 7. The convolution is continuous on $0_q^* \times L_{-q}$ for any $q \in N$, and not continuous on $0^* \times \mathscr{S}'$.

PROOF. Let U be an absolutely convex 0-neighborhood in $\mathscr{S}'$ and B the unit ball in L_{-q}. Then $U \cap L_{-p}$ is a 0-neighborhood in L_{-q} for any $p \in N$. Since the convolution is continuous on $0_{p,q}^* \times L_{-q}$, there exists a ball B_p in $0_{p,q}^*$ such that $B_p * B \subset U \cap L_{-p}$. The absolutely convex hull A of $U\{B_p; p \in N\}$ is a 0-neighborhood in 0_q^* and $A * B \subset U$.

Denote by 0 the $\operatorname{proj\,lim}_{q \to \infty} \operatorname{ind\,lim}_{p \to \infty} 0_{p,q}$. It is proved in [7] that the multiplication $(f, g) \mapsto fg: 0 \times \mathscr{S}' \to \mathscr{S}'$ is not continuous. Hence the convolution $(f, g) \mapsto f * g$: $0^* \times \mathscr{S}' \to \mathscr{S}'$ is not continuous, either.

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