A completeness theorem for intuitionistic predicate logic. An intuitionistic proof*)

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It is well known that the completeness theorem for predicate logic play the important part for modern model theory in classic logic as well as in intuitionistic one. So it is understandable the aspiration for constructive treating of this theorem. Unfortunately, usual proofs of this theorem are not constructive (see, for example, expositions in [1], [2], [3]).

In 1973 W. VELDMAN [4] proposed an intuitionistic proof of the completeness theorem for the intuitionistic predicate calculus with respect to *modified* Kripke models. The modification was in admitting the so-called *strange worlds* (or *exploding worlds*), i.e. such moments in which every sentence is true.

Veldman's theorem has the following form: a modified Kripke (or Beth) model M can be constructed, such that if a sentence A is true in M, then A necessary is deducible in the intuitionistic predicate calculus (IPC). However, the distinguished Veldman's model M has continual power, the worlds in this model are intuitionistic free choice sequences, M has a nondiscrete ordering, so it looks rather strange from point of view of usual constructive reasoning.

A *countable* distinguished model with analogous properties is constructed in [3] chapter 5 for higher order intuitionistic logic. But its semantic is abstract algebraic one rather than intuitionistically plausible semantic of Kripke or Beth models.

H. DE SWART [5], [6] gave somewhat other form of the completeness theorem for IPC. He constructs the whole fan S of modified models, such that if a sentence A is true in every model from S, then A is deducible in IPC. Every model of S has already a discrete ordering, but certainly the whole family of models is continual. Moreover, the truth-definition of formulas in de Swart meaning has an important disadvantage: if a model from S has at least one strange world, then all worlds of this model turn out to be strange. This fact destroys the monotonocity property in intuitionistic model theory. In this point de Swart's truth-definition distinguishes from the Veldman-like truth-definition.

Essential point in Veldman and de Swart intuitionistic proofs is the using of the intuitionistic fan theorem. This theorem is genuine for classical nonconstructive understanding of sequences as well as for specific intuitionistic understanding of free choice sequences, but it is not appropriate for mu^h other directions in con-

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structive mathematics (see, for example, discussion in [7]). So we are interested in avoiding the using of this theorem.

The main result of this paper can be formulated in the following form: can be constructed the modified Beth model M such that: (i) the set of worlds of M is the set of all finite 0—1-sequences with natural discrete ordering; (ii) M possible contains strange worlds, but truth-definition in M is monotone in style of Veldman; (iii) if a sentence A is true in M, then A is deducible in IPC.

The proof of this theorem is essentially neutral, it is valid from classical and from intuitionistic point of view. We do not use the fan theorem. Practically we don't use the properties of choice sequences using instead some sorts of inductive definitions.

The sign $\div$ below means "is by definition". By $\rhd$ we mark the *beginning* of a proof and by $\square$ mark its *end*. We use logical symbols simultaneously in formal and metamathematical contexts with the exception of an implication, where we use $\Rightarrow$ for metamathematical contexts rather than $\supset$.

1. Modified Beth-models

1.1. We consider formulas in an usual first-order language, so they are built from atomic formulas with help of logical connectivies $\wedge$, $\vee$, $\supset$, $\neg$, logical constant $\perp$ ("false") and quantifiers $\forall$, $\exists$. For simplicity we suppose, that our language is one-sorted, without equality, without constants and functional symbols. The generalization of the main results for these more complicate situations is rather straightforward and we shall not deal with them.

If d is a nonempty set, then *d-valued* formula is an expression A', obtained from the formula A of our language by substitution of free variables of A by elements of d; therefore d-valued formula has no free variables, but possible has elements of d as constants.

1.2. Let Σ be a set ω of all natural numbers or the finite subset of this set. Let us denote by Σ^* the set of all finite sequences of elements Σ, including the empty sequence Λ. By $p*q$ we denote the concatenation p and q, so if $p=\langle i_0, \ldots, i_{m-1}\rangle$ and $q=\langle j_0, \ldots, j_{n-1}\rangle$, then $p*q=\langle i_0, \ldots, i_{m-1}, j_0, \ldots, j_{n-1}\rangle$. The number of members of p we denote by ∂p; for example, $\partial\langle i_0, \ldots, i_{m-1}\rangle=m$, $\partial\Lambda=0$. The one element sequence is denoted by $\langle i\rangle$. Instead of $p*\langle i\rangle$ we shall write sometimes $p*i$ simply.

For $p, q\in\Sigma^*$, q is said to be an *extension* of p (in symbols $p\leqq q$) iff p is an initial segment of q, i.e. iff $\exists r(p*r=q)$. The *strict order* relation is introduced by definition: $p<q\div(p\leqq q)\wedge\neg(q\leqq p)$. *One-step order* relation is defined by

$$p \mathrel{<\!\!\cdot} q \div (\exists i\in\Sigma)(p*\langle i\rangle = q).$$

1.3. A *tree* is a subset $T\subseteqq\Sigma^*$, such that

(i) there is an element $p_0\in T$ (a *root* of the tree), $(\forall q\in T)(p_0\leqq q)$;

(ii) $p_0\leqq q\leqq r,\ r\in T\Rightarrow q\in T$;

(iii) $(\forall p\in T)(\exists q\in T)\ (p \mathrel{<\!\!\cdot} q)$;

(iv) T is decidable subset of Σ^*, i.e.

$$(\forall p\in\Sigma^*)(p\in T\vee p\notin T).$$

The last condition is important only from intuitionistic point of view, certainly.

For example, Σ^* itself is a tree.

A function $\alpha: \omega \to T$ is said to be a *path,* if $\alpha(0)=p_0$ and $\forall n(\alpha(n) \prec\!\cdot\, \alpha(n+1))$. A path α is said *to pass through* $p \in T$ iff $\exists n(\alpha(n)=p)$.

1.4. A set $x \subseteq T$ is *(order) open* iff for all $p, q \in T$

$$p \in x, \quad p \leqq q \Rightarrow q \in x.$$

Let $\mathcal{O}$ be the family of all open subsets of T.

A set $x \subseteq T$ is said to be *complete* iff

$$(\forall p \in T)(\forall q(p \prec\!\cdot\, q \Rightarrow q \in x) \Rightarrow p \in x),$$

$\mathfrak{C}$ denotes the family of all complete subsets of T.

Let now $x \subseteq T$ is an arbitrary subset of T, let us define a *completion* of x: $\mathbf{D}x = \bigcap \{y \in \mathfrak{C} | x \subseteq y\}$, i.e. $\mathbf{D}x$ is an intersection of all complete subsets of T, containing x. Evidently:

(i) $$\mathbf{D}x \in \mathfrak{C};$$

(ii) $$x \subseteq \mathbf{D}x;$$

(iii) $$(\forall y \in \mathfrak{C})(x \subseteq y \Rightarrow \mathbf{D}x \subseteq y).$$

From intuitionistic point of view it is possible to perceive (i)—(iii) as an independent "generalized inductive" definition of an operator $\mathbf{D}x$ and do not use the original set theoretic definition.

1.5. Definition. A *modified Beth-model* for IPC is a structure

$$M = \langle T, \boldsymbol{o}, d, V \rangle,$$

where:

(i) T is a tree;

(ii) $\boldsymbol{o} \in \mathfrak{C} \cap \mathcal{O}$ (a *zero* of M);

(iii) d is a nonempty set (an *individ domain* of M);

(iv) V is a *valuation function* of M, namely, V is defined for every d-valued atomic formula P and $V(P) \in \mathfrak{C} \cap \mathcal{O}$, $\boldsymbol{o} \subseteq V(P)$.

Now for every d-valued formula A we can naturally define a *truthvalue* $\|A\|$ of A in M by induction, $\|A\| \in \mathfrak{C} \cap \mathcal{O}$, $\boldsymbol{o} \subseteq \|A\|$:

1. $$\|A\| = V(A) \quad \text{for atomic } A;$$

2. $$\|A \wedge B\| = \|A\| \cap \|B\|;$$

3. $$\|A \vee B\| = \mathbf{D}(\|A\| \cup \|B\|);$$

4. $$\|A \supset B\| = \{p \in T | (\forall q \geqq p)(q \in \|A\| \Rightarrow q \in \|B\|)\};$$

5. $$\|\neg A\| = \{p \in T | (\forall q \geqq p)(q \in \|A\| \Rightarrow q \in \boldsymbol{o})\};$$

6. $$\|\perp\| = \boldsymbol{o};$$

7. $$\|\forall x A(x)\| = \bigcap_{a \in d} \|A(a)\|;$$

8. $$\|\exists x A(x)\| = \mathbf{D}\big(\bigcup_{a \in d} \|A(a)\|\big).$$

Elements of T we call *worlds* or *moments* of T. The world $p \in T$ is said to be *strange* or *exploded* iff $p \in \boldsymbol{o}$.

We say that a d-valued formula A is *true in the moment* $p \in T$ (in symbols $p \Vdash A$) iff $p \in \|A\|$. We say that a formula A is *true in a model* M (in symbols $M \Vdash A$) iff for every d-valued formula A', obtained from A by substitution and for every moment p we have $p \Vdash A'$.

It is a straightforward exercise to prove that $M \Vdash A$ for every deducible in IPC formula A.

Remark. We modify a traditional Beth-model notion in two aspects:

1. in traditional notion $\boldsymbol{o} = \varnothing$, i.e. there are no strange worlds;
2. one uses the following completion operator

$$\mathbf{D}'x \div \{p \in T | \forall \alpha (\exists n (\alpha(n) = p) \supset (\exists m \geqq n)(\alpha(m) \in x))\}$$

("every path, passing through p, pass also through some element of x") rather than our operator $\mathbf{D}x$. *Classically* it is not difficult to prove that $\mathbf{D}'x = \mathbf{D}x$ for every $x \subseteqq T$. From intuitionistic point of view using $\mathbf{D}x$ has some important advantages and, in particular, allows to avoid employment of the fan theorem.

2. General semantic constructions

Here we include our previous considerations in some more general context.

2.1. Let T is an arbitrary set. Let us denote by $\mathscr{P}$ the family of all subsets of T.

Fact. The structure $\langle \mathscr{P}, \subseteqq \rangle$ is a complete Heyting algebra (about main properties of Heyting algebras consult, for example, [8], where these algebras are called pseudo-Boolean algebras).

In this algebra

$$\mathbf{1} = T, \quad \boldsymbol{o} = \varnothing,$$

$$a \wedge b = a \cap b, \quad a \vee b = a \cup b,$$

$$a \Rightarrow b = (a \supset_t b) = \{p \in T | p \in a \Rightarrow p \in b\},$$

$$\mathbf{1}a = (\neg_t a) = \{p \in T | p \notin a\}.$$

Further, if $Q \subseteqq \mathscr{P}$, then $\bigwedge Q = \bigcap Q$, $\bigvee Q = \bigcup Q$.

Here and below in analogous cases

$$\bigcap Q = \{p \in T | (\forall a \in Q)(p \in a)\},$$

so $\bigcap Q = T$, if $Q = \varnothing$.

Remark. Classically $\langle \mathscr{P}, \subseteqq \rangle$ is even *a Boolean algebra* and $(a \supset_t b) = (T \setminus a) \cup b$, but intuitionistically we can prove only

$$(T \setminus a) \cup b \subseteqq (a \supset_t b).$$

2.2. Let now $(T, \leqq)$ is an arbitrary (partially) ordered set. Put $p<q \div (p\leqq q)\wedge \wedge \neg(q\leqq p)$,

$$x\in\mathcal{O}\div(x \subseteqq T)\wedge(\forall pq\in T)(p\in x\wedge p \leqq q \Rightarrow q\in x).$$

Fact. The structure $\langle\mathcal{O}, \subseteqq\rangle$ is a complete Heyting algebra.
In this algebra

$$\mathbf{1} = T, \quad \boldsymbol{o} = \varnothing,$$

$$b\boldsymbol{\wedge} b = a\cap b, \quad a\boldsymbol{\vee} b = a\cup b,$$

$$a \to b = (a\supset_0 b) = \{p\in T|(\forall q \geqq p)(q\in a \Rightarrow q\in b)\},$$

$$\neg a = (\neg_0 a) = \{p\in T|(\forall q \geqq p)(q\notin a)\}.$$

Further, if $Q\subseteqq\mathcal{O}$, then $\boldsymbol{\wedge} Q=\bigcap Q$, $\boldsymbol{\vee} Q=\bigcup Q$.

2.3. Let $(T, \leqq)$ again an ordered set.

2.3.1. Definition. A completion structure on T is a function J defined on T and such that for every $p\in T$, $J(p)$ is a family subsets of T. Additionally we demand, that a condition

$$q\in a, \quad a\in J(p) \Rightarrow p \leqq q$$

is fulfilled for all $p, q\in T$, $a\subseteqq T$.
A set $x\subseteqq T$ is said to be *complete* (relatively J) iff

$$\big(\forall a\in J(p)\big)(a \subseteqq x \Rightarrow p\in x).$$

Let $\mathfrak{C}$ denotes the family of all complete (relatively J) subsets of T.

2.3.2. Fact. $Q\subseteqq\mathfrak{C}\Rightarrow\bigcap Q\in\mathfrak{C}$.
Every completion structure J generates *a completion operator*. Namely, if $x\subseteqq T$, then

$$\mathbf{D}x = \bigcap\{b\in\mathfrak{C}|x \subseteqq b\}.$$

2.3.3. Fact. For all $a, b\subseteqq T$, we have:

(i) $$a \subseteqq \mathbf{D}a;$$

(ii) $$\mathbf{D}a\in\mathfrak{C};$$

(iii) $$a \subseteqq b\in\mathfrak{C} \Rightarrow \mathbf{D}a \subseteqq b;$$

(iv) $$a \subseteqq b \Rightarrow \mathbf{D}a \subseteqq \mathbf{D}b;$$

(v) $$\mathbf{D}\mathbf{D}a = \mathbf{D}a;$$

(vi) $$a\in\mathfrak{C} \Leftrightarrow \mathbf{D}a = a;$$

(vii) $$\mathbf{D}(a\cup b) = \mathbf{D}(a\cup \mathbf{D}b).$$

2.3.4. Lemma. $a\in\mathcal{O}$, $x\subseteqq T\Rightarrow a\cap\mathbf{D}x\subseteqq\mathbf{D}(a\cap x)$.
⊳ Let $a\in\mathcal{O}$ and $x\subseteqq T$, we consider the set

$$c = \{p\in T|p\in a \Rightarrow p\in\mathbf{D}(a\cap x)\}.$$

Now it should be checked that $x \subseteq c$ and $c \in \mathfrak{C}$ (hint: use the condition in 2.3.1. and assumption $a \in \mathcal{O}$). Therefore $\mathbf{D}x \subseteq c$, hence $a \cap \mathbf{D}x \subseteq \mathbf{D}(a \cap x)$. □

2.4. For an ordered set $(T, \leqq)$ we define

$$a \geqq b \div (\forall p \in b)(\exists q \in a)(q \leqq p)$$

for $a, b \subseteq T$. If $a \geqq b$ we say that *a shades b*.

2.4.1. Definition. A completion structure J is said to be *ordered* iff for all $p, q \in T$

$$p \leqq q \Rightarrow (\forall a \in J(p))(\exists b \in J(q))(a \geqq b).$$

Sometimes we use a sufficient condition for the structure to be ordered. Namely, J is *discrete below* iff:

a) $$(\forall pq \in T)(p \leqq q \vee \neg p \leqq q);$$

b) $$(\forall p \in T)(\exists a \in J(p));$$

c) $$p < q, \quad a \in J(p) \Rightarrow (\exists r \in a)(r \leqq q).$$

If J is discrete below, then J is ordered. Certainly, the condition a) is essential only from intuitionistic point of view.

2.4.2. Lemma. *Let J is an ordered completion structure. Then*

$$x \in \mathcal{O} \Rightarrow \mathbf{D}x \in \mathfrak{C} \cap \mathcal{O}.$$

⊳ Let $x \in \mathcal{O}$, we show $\mathbf{D}x \in \mathcal{O}$ ($\mathbf{D}x \in \mathfrak{C}$ follows from 2.3.3.). Let us consider the set

$$c = \{p \in T | (\forall q \geqq p)(q \in \mathbf{D}x)\}.$$

Now it should be checked, that $x \subseteq c$ and $c \in \mathfrak{C}$. In checking $c \in \mathfrak{C}$ we use the condition, that J is ordered. Hence $\mathbf{D}x \subseteq c$ and therefore $\mathbf{D}x \in \mathcal{O}$. □

2.4.3. Lemma. *Let J is an ordered completion structure. Then*

$$x \in \mathcal{O}, \quad y \in \mathfrak{C} \Rightarrow (x \supset_0 y) \in \mathfrak{C} \cap \mathcal{O}.$$

⊳ $(x \supset_0 y) = \{p \in T | (\forall q \geqq p)(q \in x \Rightarrow q \in y)\}$, so from definition $(x \supset_0 y) \in \mathcal{O}$ for all $x, y \subseteq T$. Suppose, that $x \in \mathcal{O}$, $y \in \mathfrak{C}$. We show, that $(x \supset_0 y) \in \mathfrak{C}$. Let $a \in J(p)$, $a \subseteq (x \supset_0 y)$, it is necessary to prove $p \in (x \supset_0 y)$. Let us consider $q \geqq p$, $q \in x$ and conclude, that $q \in y$. As J is an ordered structure for a given $a \in J(p)$ there exists $b \in J(q)$, $a \geqq b$. We claim $b \subseteq y$. Indeed, let $r \in b$. As $a \geqq b$, there exists $s \leqq r$, $s \in a$. From $a \subseteq (x \supset_0 y) \in \mathcal{O}$ follows $s \in (x \supset_0 y)$ and $r \in (x \supset_0 y)$. Further, $r \in b \in J(q)$, so $q \leqq r$. But $q \in x \in \mathcal{O}$, hence $r \in x$. Thus $r \in x$, $r \in (x \supset_0 y)$, hence $r \in y$. From $b \subseteq y \in \mathfrak{C}$ and $b \in J(q)$ we conclude $q \in y$. □

2.4.4. Lemma. *Let J is an ordered completion structure. Then*

$$x, y \in \mathcal{O} \Rightarrow \mathbf{D}x \cap \mathbf{D}y = \mathbf{D}(x \cap y).$$

⊳ Nontrivial is only the inclusion

$$\mathbf{D}x \cap \mathbf{D}y \subseteq \mathbf{D}(x \cap y).$$

We prove it with help 2.4.2., 2.4.3. using some inclusions in the algebra $\mathcal{O}$ 2.2. as follows:

$$x \cap y \subseteqq \mathbf{D}(x \cap y);$$

$$x \subseteqq (y \supset_0 \mathbf{D}(x \cap y));$$

$$\mathbf{D}x \subseteqq \mathbf{D}(y \supset_0 \mathbf{D}(x \cap y)) = (y \supset_0 \mathbf{D}(x \cap y));$$

$$y \cap \mathbf{D}x \subseteqq \mathbf{D}(x \cap y);$$

$$y \subseteqq (\mathbf{D}x \supset_0 \mathbf{D}(x \cap y));$$

$$\mathbf{D}y \subseteqq \mathbf{D}(\mathbf{D}x \supset_0 \mathbf{D}(x \cap y)) = (\mathbf{D}x \supset_0 \mathbf{D}(x \cap y));$$

$$\mathbf{D}x \cap \mathbf{D}y \subseteqq \mathbf{D}(x \cap y). \quad \square$$

2.5. Theorem. *Let J is an ordered completion structure. Then the structure $\langle \mathfrak{E} \cap \mathcal{O}, \subseteqq \rangle$ is a complete Heyting algebra. In this algebra*

$$\mathbf{1} = T, \quad \boldsymbol{o} = \mathbf{D}(\varnothing),$$

$$a \wedge b = a \cap b, \quad a \vee b = \mathbf{D}(a \cup b),$$

$$a \rightarrow b = (a \supset_0 b),$$

$$\neg a = (a \supset_0 \boldsymbol{o}).$$

Further, if $Q \subseteqq \mathfrak{E} \cap \mathcal{O}$, then $\bigwedge Q = \bigcap Q$, $\bigvee Q = \mathbf{D}(\bigcup Q)$.

⊳ It is a corollary of 2.4.2., 2.4.3., 2.4.4. The completion operators for creating Heyting algebras in more general situation can be found in [3]. □

2.6. Let us call a set $x \subseteqq T$ *weak-open* if

$$a \in J(p), \quad p \in x \Rightarrow a \subseteqq x.$$

The family of all weak-open subsets of T let us denote by $\overline{\mathcal{O}}$.

2.6.1. *Fact.* (i) $\mathcal{O} \subseteqq \overline{\mathcal{O}}$;

(ii) $x, y \in \overline{\mathcal{O}} \Rightarrow x \cap y,\ x \cup y \in \overline{\mathcal{O}}$;

(iii) $Q \subseteqq \overline{\mathcal{O}} \Rightarrow \bigcup Q \in \overline{\mathcal{O}}$.

2.6.2. Lemma. $x \in \overline{\mathcal{O}},\ y \in \mathfrak{E} \Rightarrow (x \supset_t y) \in \mathfrak{E}$.

⊳ Is similar to 2.4.3. □

2.6.3. A completion structure is said to be *monadic* iff for every $p \in T$, $J(p)$ is at the most one-element set, more precisely: $(\forall ab \in J(p))\ (a = b)$.

2.6.4. Lemma. *Let J is a monadic completion structure. Then*

$$x \in \overline{\mathcal{O}} \Rightarrow \mathscr{D}x \in \mathfrak{E} \cap \overline{\mathcal{O}}.$$

⊳ Let $x \in \bar{\mathcal{O}}$, we show $\mathbf{D}x \in \bar{\mathcal{O}}$ ($\mathbf{D}x \in \mathfrak{E}$ follows from 2.3.3.). Let us consider the set

$$c = \{p \in \mathbf{D}x \mid (\forall a \in J(p))(a \subseteq \mathbf{D}x)\}.$$

Now it should be checked, that $x \subseteq c \in \mathfrak{E}$.

Suppose, for example, $a \in J(p)$, $a \subseteq c$ and let us show $p \in c$ (proving $c \in \mathfrak{E}$). If $b \in J(p)$, then, as J is monadic, $b = a \subseteq c \subseteq \mathbf{D}x$. So $(\forall b \in J(p))(b \subseteq \mathbf{D}x)$. Further, from $b \subseteq \mathbf{D}x \in \mathfrak{E}$ it follows $p \in \mathbf{D}x$. Hence $p \in c$.

From $x \subseteq c \in \mathfrak{E}$ it follows $\mathbf{D}x \subseteq c$, so $\mathbf{D}x \in \bar{\mathcal{O}}$. □

2.6.5. Lemma. *Let J is a monadic completion structure. Then*

$$x, y \in \bar{\mathcal{O}} \Rightarrow \mathbf{D}x \cap \mathbf{D}y = \mathbf{D}(x \cap y).$$

⊳ Is similar to 2.4.4. We use 2.6.2., 2.6.4. and some inclusions in algebra $\mathscr{P}$ (2.1.). □

2.7. Let J is a completion structure and $z \subseteq T$. Let us define a new completion structure:

$$J_0(p) = \{a \subseteq T \mid a \in J(p) \vee (p \in z \wedge a = \varnothing)\}.$$

Let $\mathfrak{E}_0$, $\mathbf{D}_0$ are respective notions related to J_0.

Fact. (i) J_0 is a completion structure on T;

(ii) $x \in \mathfrak{E}_0 \Leftrightarrow x \in \mathfrak{E} \wedge (z \subseteq x)$;

(iii) $\mathbf{D}_0 x = \mathbf{D}(x \cup z) = \mathbf{D}(x \cup \mathbf{D}z)$;

(iv) if J is an ordered structure and $z \in \mathcal{O}$, then J_0 is also an ordered structure.

3. Branching of sequents, sequent trees

In this point we develop an apparatus essentially equivalent to deducibility in IPC, but more appropriate in our considerations.

3.1. A *collection* of formulas is by definition a finite (maybe empty) set of formulas, in which a repetitions of some formulas are admitted. The order of formulas in a collection is not essential, but for every member of a collection it is indicated, how many copies of this member there are in a collection considered. Traditionally we write $\Gamma\Delta$ instead of $\Gamma \cup \Delta$, so $\Gamma\Delta$ and $\Delta\Gamma$ is the same collection. The collection $A\Gamma$ is obtained from Γ by adjoining one copy of A.

A *sequent* is a formal expression of the form $(\Gamma \to \Delta)$, where Γ and Δ are collections. If S is $\Gamma \to \Delta$, then let us denote $(S)^0 = \Gamma$ and $(S)^1 = \Delta$.

A sequent $\Gamma \to \Delta$ is said to be *intuitionistic* iff Δ is empty or one element collection. Further, $\Gamma \to \Delta$ is said to be *strong deducible* (in symbols $\vdash^+ \Gamma \to \Delta$) iff there exists Δ' $\Delta' \subseteq \Delta$, such that $\Gamma \to \Delta'$ is an intuitionistic sequent and $\Gamma \to \Delta'$ is deducible in the intuitionistic sequent calculus *without cuts*. Certainly, for intuitionistic $\Gamma \to \Delta$ the notion of strong deducibility coincides with the usual notion of cut-free deducibility (about the intuitionistic sequent calculus and the cut rule consult, for example, [3], addition B, or [9] § 77).

A sequent is said to be *primitive* iff it has one of the following form: $A\Gamma \to \Delta A$ or $\bot \Gamma \to \Delta$, where A is an arbitrary formula. Of course, if S is primitive, then $\vdash^{+} S$.

3.2. If S_1, S_2, S_3 are sequents, then let us define two three place relations:

$$S_1 \prec_0 S_2, S_3 \quad \text{and} \quad S_1 \prec_1 S_2, S_3$$

(in words: S_1 *branches* into S_2 and S_3 in *reversible* manner and, respectively, S_1 branches into S_2 and S_3 in *nonreversible* manner).

To begin with, for an arbitrary sequent S we define:

$$S \prec_0 S, S.$$

Further we list all the rest cases of both relations $S_1 \prec_i S_2$, S_3. Every case will have a special symbolic name depending on the construction of S_1. The notation $S_1 \prec_i S_2$ is an abbreviation for $S_1 \prec_i S_2, S_2$.

1. $(\wedge \to)$ $\quad (A\wedge B)\Gamma \to \Delta \prec_0 AB(A\wedge B)\Gamma \to \Delta;$

2. $(\to \wedge)$ $\quad \Gamma \to \Delta(A\wedge B) \prec_0 \Gamma \to \Delta(A\wedge B)A,$
$\qquad \Gamma \to \Delta(A\wedge B)B;$

3 $(\vee \to)$ $\quad (A\vee B)\Gamma \to \Delta \prec_0 A(A\vee B)\Gamma \to \Delta,$
$\qquad B(A\vee B)\Gamma \to \Delta;$

4. $(\to \vee)$ $\quad \Gamma \to \Delta(A\vee B) \prec_0 \Gamma \to \Delta(A\vee B)AB;$

5. $(\supset \to)$ $\quad (A\supset B)\Gamma \to \Delta \prec_0 (A\supset B)\Gamma \to \Delta A,$
$\qquad B(A\supset B)\Gamma \to \Delta;$

6. $(\to \supset)$ $\quad \Gamma \to \Delta(A\supset B) \prec_1 \Gamma \to \Delta(A\supset B),$
$\qquad A\Gamma \to B;$

7. $(\neg \to)$ $\quad \neg A\Gamma \to \Delta \prec_0 \neg A\Gamma \to \Delta A;$

8. $(\to \neg)$ $\quad \Gamma \to \Delta\neg A \prec_1 \Gamma \to \Delta\neg A, \quad A\Gamma \to;$

9. $(\forall \to)$ $\quad \forall x A(x)\Gamma \to \Delta \prec_0 A(z)\forall x A(x)\Gamma \to \Delta;$

10. $(\to \forall)$ $\quad \Gamma \to \Delta\forall x A(x) \prec_1 \Gamma \to \Delta\forall x A(x), \quad \Gamma \to A(y),$

where y is not free in Γ;

11. $(\exists \to)$ $\quad \exists x A(x)\Gamma \to \Delta \prec_0 A(y)\exists x A(x)\Gamma \to \Delta,$

where y is not free in $\exists x A(x)\Gamma \to \Delta$;

12. $(\to \exists)$ $\quad \Gamma \to \Delta\exists x A(x) \prec_0 \Gamma \to \Delta\exists x A(x)A(z).$

Therefore $\prec_1$ relation is true only in cases $(\to \supset), (\to \neg), (\to \forall)$.

3.2.1. *Fact*. (i) If $S_1 \prec_i S_2, S_3$, then $(S_1)^0 \subseteq (S_2)^0$, $(S_1)^0 \subseteq (S_3)^0$, $(S_1)^1 \subseteq (S_2)^1$.
(ii) If $S_1 \prec_0 S_2, S_3$, then $(S_1)^1 \subseteq (S_3)^1$.

3.2.2. *Fact*. (i) If $S_1 \prec_0 S_2, S_3$ and *simultaneously* $\vdash^+ S_2$ and $\vdash^+ S_3$, then $\vdash^+ S_1$.
(ii) If $S_1 \prec_1 S_2, S_3$ and *at least* $\vdash^+ S_2$ or $\vdash^+ S_3$, then $\vdash^+ S_1$.

3.3. A *binary tree* T is a tree (see 1.3.) such that for every $p \in T$, a set $\{q | p \prec\cdot q\}$ consists precisely of two elements. For example, $\{0, 1\}^*$ is a binary tree.

A *sequent tree* on a binary tree T is a couple of functions (τ, h) defined on T, such that for every $p \in T$, $\tau(p)$ equals 0 or 1, $h(p)$ is a sequent and, moreover

$$h(p) \prec_{\tau(p)} h(p * \langle i_0 \rangle), \quad h(p * \langle i_1 \rangle),$$

where

$$i_0 < i_1, \quad p * \langle i_0 \rangle, \quad p * \langle i_1 \rangle \in T.$$

For simplicity below we shall write $p * 0$, $p * 1$ instead of $p * \langle i_0 \rangle$, $p * \langle i_1 \rangle$, so we shall deal mainly with $T = \{0, 1\}^*$. The case of a general binary tree is quite similar.

3.4. A *zero* of a sequent tree (τ, h) is a set

$$\boldsymbol{o} = \{p \in T | (\exists q \leqq p)(\vdash^+ h(q))\}$$

Evidently $\boldsymbol{o} \in \mathcal{O}$.

3.5. Let (τ, h) is a sequent tree on T. Let us define some completion structures on T. First of all, put

$$J(p) = \{\{p * 0, p * 1\}\}.$$

This structure is discrete below and, hence, ordered (2.4.1.). Moreover, it is monadic (2.6.3.). The notions, corresponding J we denote as $\mathfrak{C}$, $\bar{\mathcal{O}}$, $\mathbf{D}$ etc.

Further:

$$J_0(p) = \{a \subseteq T | a \in J(p) \vee (p \in \boldsymbol{o} \wedge a = \varnothing)\}.$$

This structure is ordered either (2.7.). Corresponding notions are $\mathfrak{C}_0$, $\bar{\mathcal{O}}_0$, $\mathbf{D}_0$ etc. According 2.7.:

$$\mathbf{D}_0(x) = \mathbf{D}(x \cup \boldsymbol{o}), \quad \mathfrak{C}_0 = \{x \in \mathfrak{C} | \boldsymbol{o} \subseteq x\}.$$

Moreover, in view 2.5. $\langle \mathfrak{C}_0 \cap \mathcal{O}, \subseteq \rangle$ is a complete Heyting algebra.

At last, let us define two further completion structures:

$$J_1(p) = \begin{cases} \{\{p * 0, p * 1\}\}, & \text{if } \tau(p) = 0; \\ \{\{p * 0\}\}, & \text{if } \tau(p) = 1. \end{cases}$$

$$J_2(p) = \begin{cases} \{\{p * 0, p * 1\}\}, & \text{if } \tau(p) = 0; \\ \{\{p * 0\}, \{p * 1\}\}, & \text{if } \tau(p) = 1. \end{cases}$$

J_1, J_2 are not ordered. Structures J, J_1 are monadic, but J_0, J_2 are not. Respectively arise notions $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathbf{D}_2$, $\bar{\mathcal{O}}_2$ etc.

3.5.1. Lemma. (i) $o\in\mathcal{O}$, $o\in\overline{\mathcal{O}}_1$;
(ii) $o\in\mathfrak{E}$, $o\in\mathfrak{E}_i$, for $i=0, 1, 2$.

$\rhd$ 2.6.1., 3.2.2. $\square$

3.5.2. *Fact*. If $x\subseteqq T$, then
(i) $x\in\mathfrak{E}_2\Rightarrow x\in\mathfrak{E}_1\Rightarrow x\in\mathfrak{E}$;
(ii) $\mathbf{D}x\subseteqq\mathbf{D}_1x\subseteqq\mathbf{D}_2x$.

3.6. Let again (τ, h) is a sequent tree on binary tree T. For every formula A we define two sets $L(A)$ and $R(A)$:

$$L(A) = \{p\in T|A\in(h(p))^0\},$$

$$R(A) = \{p\in T|A\in(h(p))^1\}.$$

3.6.1. Lemma. $L(A)\in\mathcal{O}$, $R(A)\in\overline{\mathcal{O}}_1$.
$\rhd$ See 3.2.1. $\square$

3.6.2. Lemma. (i) *Let* $n\in\omega$ *is a natural number,* $p\in L(A)$ *and*

$$x = \{q|q \geqq p,\ q\in L(A),\ \partial q = \partial p+n\}.$$

Then $p\in\mathbf{D}x$.

(ii) Let $n\in\omega$, $p\in R(A)$ and

$$x = \{q|q \geqq p,\ q\in R(A),\ \partial q = \partial p+n\}.$$

Then $p\in\mathbf{D}_1x$ *and* $p\in\mathbf{D}_2x$.

$\rhd$ Induction on n. Let us consider (ii) and operator $\mathbf{D}_1$. If $n=0$, then $x=\{p\}$, $p\in x$ and, hence, $p\in\mathbf{D}_1x$.

Let now $n>0$. For $i=0, 1$ we define

$$x_i = \{q \geqq p*i|q\in R(A),\ \partial q = \partial(p*i)+(n-1\}.$$

Let us consider two cases:

1. $\tau(p) = 0$. Then $p*0\in R(A)$ and $p*1\in R(A)$ (3.2.1.), so on inductive supposition $p*i\in\mathbf{D}_1x_i$ for $i=0, 1$. But $x_i\subseteqq x$, therefore $p*i\in\mathbf{D}_1x$. If $a=\{p*0, p*1\}$, then $a\in J_1(p)$, $a\subseteqq\mathbf{D}_1x$ (note $\tau(p)=0$). Hence, $p\in\mathbf{D}_1x$.

2. $\tau(p)=1$. In this case $p*0\in R(A)$ (3.2.1.). On inductive supposition $p*0\in\mathbf{D}_1x_0$. But $x_0\subseteqq x$, so $p*0\in\mathbf{D}_1x$. If $a=\{p*0\}$, then $a\in J_1(p)$, $a\subseteqq\mathbf{D}_1x$ (note $\tau(p)=1$), hence $p\in\mathbf{D}_1x$. $\square$

4. Systematic sequent trees. A Beth-model, associated with a sequent tree

4.1. A sequent tree (τ, h) on binary tree T is said to be *systematic* if the following conditions are fulfilled:

1. $L(\perp) \subseteq \boldsymbol{o}$;
2. $L(A) \cap R(A) \subseteq \boldsymbol{o}$;
3. $L(A \wedge B) \subseteq \mathbf{D}(L(A) \cap L(B))$;
4. $R(A \wedge B) \subseteq \mathbf{D}_1(R(A) \cup R(B))$;
5. $L(A \vee B) \subseteq \mathbf{D}(L(A) \cup L(B))$;
6. $R(A \vee B) \subseteq \mathbf{D}_1(R(A) \cap R(B))$;
7. $L(A \supset B) \subseteq \mathbf{D}(R(A) \cup L(B))$;
8. $R(A \supset B) \subseteq \mathbf{D}_2(L(A) \cap R(B))$;
9. $L(\neg A) \subseteq \mathbf{D}(R(A))$;
10. $R(\neg A) \subseteq \mathbf{D}_2(L(A))$;
11. $L(\forall x A(x)) \subseteq \mathbf{D}(L(A(z)))$, $z \in \mathrm{Var}$;
12. $R(\forall x A(x)) \subseteq \mathbf{D}_2(\bigcup_{y \in Var} R(A(y)))$;
13. $L(\exists x A(x)) \subseteq \mathbf{D}(\bigcup_{y \in Var} L(A(y)))$;
14. $R(\exists x A(x)) \subseteq \mathbf{D}_1(R(A(z)))$, $z \in \mathrm{Var}$;

Var is a set of all variables of our language.

4.2. Theorem. *Let S is a sequent and T is a binary tree. Then a systematic sequent tree (τ, h) on T can be constructed, such that $h(p_0)=S$. Here p_0 is a root of T.*

$\rhd$ We define $\tau(p)$ and $h(p)$ by induction on ∂p.

If $\partial p=0$, i.e. $p=p_0$, then we define $h(p_0)=S$.

Let now $\partial p>0$, $h(p)$ already is defined, and $\tau(p')$, $h(p')$ are defined for every $p' \in T$, $\partial p' < \partial p$. Note, that $\tau(p)$ is not defined yet.

In this situation we define $\tau(p)$ and $h(p*0)$, $h(p*1)$.

Let us represent ∂p in the form

$$\partial p = 2^{m_0} \cdot 3^{m_1} \cdot 5^{m_2} \cdot m_3,$$

where 2, 3, 5 do not divide m_3.

The moment p is said to be *expressive* iff

(i) m_0 equals 0 or 1;

(ii) m_1 is a (Goedel) number of some formula A, such that A is nonatomic and A differs from $\perp$; moreover, if $m_0=0$, then A occurs in $(h(p))^0$, and if $m_0=1$, then A occurs in $(h(p))^1$;

(iii) if $m_0=0$ and A begins from quantifier $\forall$, then m_2 is a (Goedel) number of some variable z, if $m=1$ and A begins from quantifier $\exists$, then m_2 also is a (Goedel) number of some variable z.

Now, if p is not expressive, we put $\tau(p)=0$ and

$$h(p*0) = h(p*1) = h(p).$$

If p is expressive we define

$$\tau(p) = i, \quad h(p*0) = S_2, \quad h(p*1) = S_3,$$

in such a way, that $h(p) \prec_i S_2, S_3$ with a given formula A with number m_1, corresponding cases 1.—12. in 3.2. Moreover, in the cases 9. $(\forall \to)$ and 12. $(\to\exists)$ we use a given variable z with number m_2.

The definition of functions τ, h is finished. It is clear, that (τ, h) is a sequent tree on T.

Let us check the systematic conditions 4.1.

The conditions 1., 2. are true for arbitrary sequent tree. Indeed, if $p\in L(\perp)$ or $p\in L(A)\cap R(A)$, then $h(p)$ is primitive and so $\vdash^+ h(p)$, hence, $p\in \boldsymbol{o}$.

Let us check several of conditions 3.—14.

7. $$L(A\supset B) \subseteqq \mathbf{D}(R(A)\cup L(B)).$$

Let $p\in L(A\supset B)$. Let us consider a natural $m=2^0\cdot 3^{m_1}\cdot 5^{m_2}\cdot m_3$, $m>\partial p$, such that m_1 is a (Goedel) number of $(A\supset B)$. Let

$$x = \{q \geqq p \mid \partial q = m,\ q\in L(A\supset B)\}.$$

Then $p\in \mathbf{D}x$ (3.6.2.).

In this situation every $q\in x$ is expressive and in accordance with the construction (τ, h) (cf. 3.2., 5. $(\supset\to)$), $\tau(q)=0$, we have $x\subseteqq \mathbf{D}(R(A)\cup L(B))$. Hence it follows

$$p\in \mathbf{D}x \subseteqq \mathbf{D}(R(A)\cup L(B)).$$

8. $$R(A\supset B) \subseteqq \mathbf{D}_2(L(A)\cap R(B)).$$

Let $p\in R(A\supset B)$. Let us consider a natural $m=2^1\cdot 3^{m_1}\cdot 5^{m_2}\cdot m_3$, $m>\partial p$, such that m_1 is a number of $(A\supset B)$. Let

$$x = \{q \geqq p \mid \partial q = m,\ q\in R(A\supset B)\}.$$

Then $p\in \mathbf{D}_2 x$ (3.6.2.).

In this situation every $q\in x$ is expressive and in accordance with the construction (τ, h), $\tau(q)=1$ and the branching 3.2., 6. $(\to\supset)$ is used. Hence $x\subseteqq \mathbf{D}_2(L(A)\cap R(B))$, so

$$p\in \mathbf{D}_2 x \subseteqq \mathbf{D}_2(L(A)\cap R(B)).$$

12. $$R(\forall x A(x)) \subseteqq \mathbf{D}_2\Big(\bigcup_{y\in Var} R(A(y))\Big).$$

Let $p\in R(\forall x A(x))$. Let us consider a natural $m=2^1\cdot 3^{m_1}\cdot 5^{m_2}\cdot m_3$, $m>\partial p$, where m_1 is a number of $\forall x A(x)$. Let

$$x = \{q \geqq p \mid \partial q = m,\ q\in R(\forall x A(x))\}.$$

Then $p\in \mathbf{D}_2 x$ (3.6.2.).

In this situation every $q \in x$ is expressible $\tau(q)=1$, and the branching 3.2., 10. $(\rightarrow\forall)$ is used. Hence,

$$x \subseteq \mathbf{D}_2\Big(\bigcup_{y\in Var} R(A(y))\Big)$$

(because for every $q \in x$ there exists a variable y, such that $q*1 \in R(A(y)))$. Therefore $p \in \mathbf{D}_2 x \subseteq \mathbf{D}_2\big(\bigcup_{y\in Var} R(A(y))\big)$.

14. $$R(\exists x A(x)) \subseteq \mathbf{D}_1(R(A(z))), \quad z \in \mathrm{Var}.$$

Let us fix some variable z, let $p \in R(\exists x A(x))$. Let us consider a natural $m=2^1\cdot 3^{m_1}\cdot 5^{m_2}\cdot m_3$, $m > \partial p$, where m_1 is a number of formula $\exists x A(x)$ and m_2 is a number of z. Let, further,

$$x = \{q \geqq p \mid \partial q = m,\ q \in R(\exists x A(x))\}.$$

Then $p \in \mathbf{D}_1 x$ (3.6.2.).

In this situation all $q \in x$ are expressible, $\tau(q)=0$, and the branching 3.2., 12. $(\rightarrow\exists)$ is used with the fixed variable z. Therefore for all $q \in x$ we have $q*0, q*1 \in R(A(z))$, so $x \subseteq \mathbf{D}_1(R(A(z)))$, hence, $p \in \mathbf{D}_1 x \subseteq \mathbf{D}_1(R(A(z)))$. □

4.3. With every sequent tree (τ, h) on T we associate some modified Beth-model

$$M = \langle T, \boldsymbol{o}, \mathrm{Var}, V\rangle$$

(1.5.) in the following way:

1. a zero of M is a zero of the sequent tree (3.4.);
2. an individ domain of M coincides with the set Var of all variables of our language, so every formula is automatically Var-valued;
3. for every atomic formula P we put $V(P)=\mathbf{D}(L(P)\cup \boldsymbol{o})$.

According 1.5. for every formula A can be defined a truthvalue $\|A\|$ in M. Note (3.5.) that $\|A\| \in \mathfrak{E}_0 \cap \mathcal{O}$ is an element of complete Heyting algebra $\mathfrak{E}_0 \cap \mathcal{O}$ and logical connectivies are calculated in M in accordance with operations in this algebra.

4.4. Theorem. *If* (τ, h) *is a systematic sequent tree, then for associated Beth-model and for every formula A we have:*

(i) $$L(A) \subseteq \|A\|;$$

(ii) $$R(A) \cap \|A\| \subseteq \boldsymbol{o}.$$

⊳ The both points of the theorem we prove by induction on the construction of A.

1. A-atomic, $L(A) \subseteq \|A\|$.

Indeed, $L(A) \subseteq \mathbf{D}_0(L(A)) = \|A\|$ (4.3., 3.5.).

2. A-atomic, $R(A) \cap \|A\| \subseteq \boldsymbol{o}$.

Indeed, $R(A) \cap L(A) \subseteq \boldsymbol{o}$ (4.1,2.), so

$$R(A) \cap (\boldsymbol{o} \cup L(A)) \subseteq \boldsymbol{o}.$$

Hence, $\mathbf{D}_1(R(A)\cap(\boldsymbol{o}\cup L(A)))\subseteq\mathbf{D}_1\boldsymbol{o}$. But $\boldsymbol{o}\in\mathfrak{E}_1$ (3.5.1.), $R(A)\in\overline{\mathcal{O}_1}$ (3.6.1.), $\boldsymbol{o}\cup L(A)\in\overline{\mathcal{O}_1}$ (3.5.1., 3.6.1., 2.6.1.), so using 2.6.5. we get $\mathbf{D}_1(R(A))\cap\mathbf{D}_1(\boldsymbol{o}\cup L(A))\subseteq\boldsymbol{o}$.

But $R(A)\subseteq\mathbf{D}_1(R(A))$, and $\|A\|=\mathbf{D}(\boldsymbol{o}\cup L(A))\subseteq\mathbf{D}_1(\boldsymbol{o}\cup L(A))$ (3.5.2.). Hence, $R(A)\cap\|A\|\subseteq\boldsymbol{o}$.

3. $$L(\bot)\subseteq\|\bot\|.$$

See (4.1., 1.)).

4. $$R(\bot)\cap\|\bot\|\subseteq\boldsymbol{o}.$$

It is trivial, because $\|\bot\|\subseteq\boldsymbol{o}$.

5. $$L(A\wedge B)\subseteq\|A\wedge B\|.$$

We have $L(A)\subseteq\|A\|$ and $L(B)\subseteq\|B\|$, hence,

$$L(A)\cap L(B)\subseteq\|A\|\cap\|B\| = \|A\wedge B\|.$$

Further, $\mathbf{D}(L(A)\cap L(B))\subseteq\mathbf{D}(\|A\wedge B\|)=\|A\wedge B\|$. But $L(A\wedge B)\subseteq\mathbf{D}(L(A)\cap L(B))$ (4.1., 3.), so we get the desired result.

6. $$R(A\wedge B)\cap\|A\wedge B\|\subseteq\boldsymbol{o}.$$

We have $R(A)\cap\|A\|\subseteq\boldsymbol{o}$ and $R(B)\cap\|B\|\subseteq\boldsymbol{o}$, so much the more $R(A)\cap\|A\wedge B\|\subseteq\boldsymbol{o}$ and $R(B)\cap\|A\wedge B\|\subseteq\boldsymbol{o}$. Hence, $(R(A)\cup R(B))\cap\|A\wedge B\|\subseteq\boldsymbol{o}$. Further (cf. 3.5.1.)

$$\mathbf{D}_1((R(A)\cup R(B))\cap\|A\wedge B\|)\subseteq\mathbf{D}_1\boldsymbol{o}\subseteq\boldsymbol{o}.$$

But $\|A\wedge B\|\in\mathcal{O}$, so using 2.3.4.

$$\mathbf{D}_1(R(A)\cup R(B))\cap\|A\wedge B\|\subseteq\boldsymbol{o}.$$

But $R(A\wedge B)\subseteq\mathbf{D}_1(R(A)\cup R(B))$ (4.1., 4.), so we get the result.

7. $$L(A\vee B)\subseteq\|A\vee B\|.$$

We have $L(A)\subseteq\|A\|$ and $L(B)\subseteq\|B\|$, hence, $(L(A)\cup L(B))\subseteq\|A\|\cup\|B\|$. Further,

$$\mathbf{D}(L(A)\cup L(B))\subseteq\mathbf{D}(\|A\|\cup\|B\|)=\|A\vee B\|.$$

But $L(A\vee B)\subseteq\mathbf{D}(L(A)\cup L(B))$ (4.1., 5.).

8. $$R(A\vee B)\cap\|A\vee B\|\subseteq\boldsymbol{o}.$$

We have $R(A)\cap\|A\|\subseteq\boldsymbol{o}$ and $R(B)\cap\|B\|\subseteq\boldsymbol{o}$, so much the more $R(A)\cap R(B)\cap\|A\|\subseteq\boldsymbol{o}$ and $R(A)\cap R(B)\cap\|B\|\subseteq\boldsymbol{o}$. Hence, $(R(A)\cap R(B))\cap(\|A\|\cup\|B\|)\subseteq\subseteq\boldsymbol{o}$. Further,

$$\mathbf{D}_1(R(A)\cap R(B))\cap(\|A\|\cup\|B\|)\subseteq\mathbf{D}_1\boldsymbol{o}.$$

But $\boldsymbol{o}\in\mathfrak{E}_1$ (3.5.1.), $R(A)\cap R(B)\in\overline{\mathcal{O}_1}$ (3.6.1., 2.6.1.), $(\|A\|\cup\|B\|)\in\overline{\mathcal{O}_1}$ (2.6.1.). Using 2.6.5.

$$\mathbf{D}_1(R(A)\cap R(B))\cap\mathbf{D}_1(\|A\|\cup\|B\|)\subseteq\boldsymbol{o}.$$

But $\|A \vee B\| = \mathbf{D}(\|A\| \cup \|B\|) \subseteq \mathbf{D}_1(\|A\| \cup \|B\|)$ (3.5.2.) and, moreover, $R(A \vee B) \subseteq \mathbf{D}_1(R(A) \cap R(B))$ (4.1., 6), so we get $R(A \vee B) \cap \|A \vee B\| \subseteq \boldsymbol{o}$.

9. $$L(A \supset B) \subseteq \|A \supset B\|.$$

We have $R(A) \cap \|A\| \subseteq \boldsymbol{o}$ and $L(B) \subseteq \|B\|$. Hence, $R(A) \cap \|A\| \subseteq \|B\|$ and $L(B) \cap \|A\| \subseteq \|B\|$, so $(R(A) \cup L(B)) \cap \|A\| \subseteq \|B\|$. Further,

$$\mathbf{D}((R(A) \cup L(B)) \cap \|A\|) \subseteq \mathbf{D}(\|B\|) = \|B\|.$$

As $\|A\| \in \mathcal{O}$, using 2.3.4.

$$\mathbf{D}(R(A) \cup L(B)) \cap \|A\| \subseteq \|B\|.$$

But $L(A \supset B) \subseteq \mathbf{D}(R(A) \cup L(B))$ (4.1., 7.). Therefore $L(A \supset B) \cap \|A\| \subseteq \|B\|$. Now acting in Heyting algebra $\mathcal{O}$ (2.2.) we get

$$L(A \supset B) \subseteq (\|A\| \supset_0 \|B\|) = \|A \supset B\|$$

(note $L(A \supset B) \in \mathcal{O}$, 3.6.1.).

10. $$R(A \supset B) \cap \|A \supset B\| \subseteq \boldsymbol{o}.$$

We have $R(B) \cap \|B\| \subseteq \boldsymbol{o}$ and $L(A) \subseteq \|A\|$. In Heyting algebra $\mathfrak{E}_0 \cap \mathcal{O}$ moreover $\|A\| \cap \|A \supset B\| \subseteq \|B\|$. In view $R(B) \cap \|B\| \subseteq \boldsymbol{o}$ hence $\|A\| \cap R(B) \cap \|A \supset B\| \subseteq \boldsymbol{o}$. From $L(A) \subseteq \|A\|$ we get $L(A) \cap R(B) \cap \|A \supset B\| \subseteq \boldsymbol{o}$. Further,

$$\mathbf{D}_2(L(A) \cap R(B) \cap \|A \subset B\|) \subseteq \mathbf{D}_2 \boldsymbol{o}.$$

But $\boldsymbol{o} \in \mathfrak{E}_2$ (3.5.1.) and $\|A \supset B\| \in \mathcal{O}$, so using 2.3.4.

$$\mathbf{D}_2(L(A) \cap R(B)) \cap \|A \supset B\| \subseteq \mathbf{D}_2 \boldsymbol{o}.$$

But $R(A \supset B) \subseteq \mathbf{D}_2(L(A) \cap R(B))$ (4.1., 8.), so we get the result.

11. $$L(\neg A) \subseteq \|\neg A\|.$$

We have $R(A) \cap \|A\| \subseteq \boldsymbol{o}$, so $\mathbf{D}(R(A) \cap \|A\|) \subseteq \mathbf{D}\boldsymbol{o} = \boldsymbol{o}$. But $\|A\| \in \mathcal{O}$ and using 2.3.4. $\mathbf{D}(R(A)) \cap \|A\| \subseteq \boldsymbol{o}$. Moreover, $L(\neg A) \subseteq \mathbf{D}(R(A))$ (4.1., 9.), so $L(\neg A) \cap \|A\| \subseteq \boldsymbol{o}$. Further, $L(\neg A) \in \mathcal{O}$ (3.6.1.). Now acting in Heyting algebra $\mathcal{O}$ (2.2.) we get

$$L(\neg A) \subseteq (\|A\| \supset_0 \boldsymbol{o}) = \|\neg A\|.$$

12. $$R(\neg A) \cap \|\neg A\| \subseteq \boldsymbol{o}.$$

We have $L(A) \subseteq \|A\|$. Additionally in Heyting algebra $\mathfrak{E}_0 \cap \mathcal{O}$ $\|A\| \cap \|\neg A\| \subseteq \boldsymbol{o}$. Hence, $L(A) \cap \|\neg A\| \subseteq \boldsymbol{o}$. Further,

$$\mathbf{D}_2(L(A) \cap \|\neg A\|) \subseteq \mathbf{D}_2 \boldsymbol{o}.$$

But $\boldsymbol{o} \in \mathfrak{E}_2$ (3.5.1.) and $\|\neg A\| \in \mathcal{O}$, so using 2.3.4.

$$\mathbf{D}_2(L(A)) \cap \|\neg A\| \subseteq \boldsymbol{o}.$$

But $R(\neg A) \subseteq \mathbf{D}_2(L(A))$ (4.1., 10.) and, hence $R(\neg A) \cap \|\neg A\| \subseteq \boldsymbol{o}$.

13. $$L(\forall x A(x)) \subseteq \|\forall x A(x)\|.$$

We have $L(A(z))\subseteq\|A(z)\|$ for all $z\in Var$. Further,

$$\mathbf{D}(L(A(z)))\subseteq\mathbf{D}(\|A(z)\|)=\|A(z)\|.$$

But $L(\forall xA(x))\subseteq\mathbf{D}(L(A(z)))$ (4.1., 11.), so $L(\forall xA(x))\subseteq\|A(z)\|$. Let us take an intersection for all $z\in Var$ on the right side of the last inclusion. Then $L(\forall xA(x))\subseteq\bigcap_{z\in Var}\|A(z)\|=\|\forall xA(x)\|$.

14. $$R(\forall xA(x))\cap\|\forall xA(x)\|\subseteq \boldsymbol{o}.$$

We have $R(A(y))\cap\|A(y)\|\subseteq\boldsymbol{o}$, so much the more $R(A(y))\cap\|\forall xA(x)\|\subseteq\boldsymbol{o}$. Taking a join on the left for all $y\in Var$ and using distributivity we get

$$\Big(\bigcup_{y\in Var}R(A(y))\Big)\cap\|\forall xA(x)\|\subseteq\boldsymbol{o}.$$

Further,

$$\mathbf{D}_2\Big(\Big(\bigcup_{y\in Var}R(A(y))\Big)\cap\|\forall xA(x)\|\Big)\subseteq\mathbf{D}_2\boldsymbol{o}=\boldsymbol{o}.$$

But $\|\forall xA(x)\|\in\mathcal{O}$, so using 2.3.4. $\mathbf{D}_2\Big(\bigcup_{y\in Var}R(A(y))\Big)\cap\|\forall xA(x)\|\subseteq\boldsymbol{o}$. Moreover, $R(\forall xA(x))\subseteq\mathbf{D}_2\Big(\bigcup_{y\in Var}R(A(y))\Big)$ (4.1., 12.), so we get the result.

15. $$L(\exists xA(x))\subseteq\|\exists xA(x)\|.$$

We have $L(A(y))\subseteq\|A(y)\|$ for all $y\in Var$, so much the more $L(A(y))\subseteq\|\exists xA(x)\|$. Hence, $\bigcup_{y\in Var}L(A(y))\subseteq\|\exists xA(x)\|$. Further, $\mathbf{D}\Big(\bigcup_{y\in Var}L(A(y))\Big)\subseteq\mathbf{D}(\|\exists xA(x)\|)=\|\exists xA(x)\|$. But $L(\exists xA(x))\subseteq\mathbf{D}\Big(\bigcup_{y\in Var}L(A(y))\Big)$ (4.1., 13.), so we get the result.

16. $$R(\exists xA(x))\cap\|\exists xA(x)\|\subseteq\boldsymbol{o}.$$

We have $R(A(z))\cap\|A(z)\|\subseteq\boldsymbol{o}$. Further,

$$\mathbf{D}_1(R(A(z))\cap\|A(z)\|)\subseteq\mathbf{D}_1\boldsymbol{o}.$$

But $\boldsymbol{o}\in\mathfrak{E}_1$ (3.5.1.), $\|A(z)\|\in\mathcal{O}$, so using 2.3.4.

$$\mathbf{D}_1(R(A(z)))\cap\|A(z)\|\subseteq\boldsymbol{o}.$$

Moreover, $R(\exists xA(x))\subseteq\mathbf{D}_1(R(A(z)))$ (4.1., 14.), so

$$R(\exists xA(x))\cap\|A(z)\|\subseteq\boldsymbol{o}.$$

Let us take a join on the left for all $z\in Var$ and use distributivity: $R(\exists xA(x))\cap\bigcup_{z\in Var}\|A(z)\|\subseteq\boldsymbol{o}$. Further, $\mathbf{D}_1\Big(R(\exists xA(x))\cap\bigcup_{z\in Var}\|A(z)\|\Big)\subseteq\mathbf{D}_1\boldsymbol{o}$.

We have $R(\exists xA(x))\in\bar{\mathcal{O}}_1$ (3.6.1.) and $\bigcup_{z\in Var}\|A(z)\|\in\bar{\mathcal{O}}_1$ (2.6.1.), so using 2.6.5.

$$\mathbf{D}_1(R(\exists xA(x)))\cap\mathbf{D}_1\Big(\bigcup_{z\in Var}\|A(z)\|\Big)\subseteq\boldsymbol{o}.$$

But $R(\exists xA(x))\subseteq\mathbf{D}_1(R(\exists xA(x)))$ (2.3.3.) and $\|\exists xA(x)\|=\mathbf{D}\Big(\bigcup_{z\in Var}\|A(z)\|\Big)\subseteq\mathbf{D}_1\Big(\bigcup_{z\in Var}\|A(z)\|\Big)$ (3.5.2.).

Hence, $R(\exists xA(x))\cap\|\exists xA(x)\|\subseteq\boldsymbol{o}$. □

5. Main results

5.1. Theorem. *Let* (τ, h) *be a systematic sequent tree on* T *and* M *an associated modified Beth-model.*

Let a sequent $A_1 \dots A_n \to B$ *be true in* M. *Then*

$$L(A_1) \cap \dots \cap L(A_n) \cap R(B) \subseteq \boldsymbol{o}.$$

$\rhd$ In this case $\|A_1 \dots A_n \to B\| = T$, which is equivalent $\|A_1\| \cap \dots \cap \|A_n\| \subseteq \|B\|$. But according 4.4. $L(A_i) \subseteq \|A_i\|$, so

$$L(A_1) \cap \dots \cap L(A_n) \subseteq \|B\|.$$

From the other hand, according 4.4. $\|B\| \cap R(B) \subseteq \boldsymbol{o}$. Hence, $L(A_1) \cap \dots$ $\dots L(A_n) \cap R(B) \subseteq \boldsymbol{o}$. □

5.2. Theorem. *For every binary tree* T *and every intuitionistic sequent* S *can be constructed a modified Beth-model* M_S *on* T, *such that if* S *is true in* M_S, *then* $\vdash^+ S$.

$\rhd$ For a given S we construct a systematic sequent tree (τ, h) on the binary tree T with the root p_0 such that $h(p_0) = S$ (4.2.). Let M_S be an associated with (τ, h) model. Let S be $A_1 \dots A_n \to B$. If S is true in M, then

$$L(A_1) \cap \dots \cap L(A_n) \cap R(B) \subseteq \boldsymbol{o} \quad (5.1.).$$

Because $h(p_0) = (A_1 \dots A_n \to B)$ we have

$$p_0 \in L(A_1) \cap \dots \cap L(A_n) \cap R(B), \quad \text{so} \quad p_0 \in \boldsymbol{o}.$$

By definition of zero (3.4.), then $\vdash^+ h(p_0)$. □

5.3. Corollary. *(Cut-elimination theorem, model-theoretic proof.)*

If an intuitionistic sequent S *is deducible in intuitionistic sequent calculus, then it is deducible also without cuts.*

$\rhd$ If S is deducible, then it is true in *every* modified Beth-model and, in particular, in M_S. Use 5.2. □

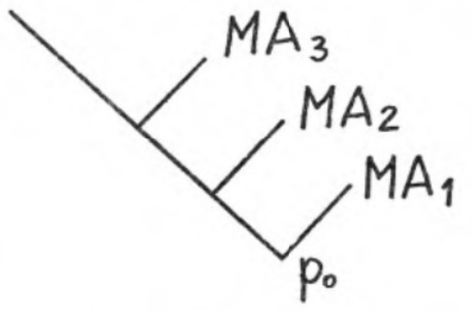
5.4. Corollary. *For every formula* A *and binary tree* T *a modified Beth-model* M_A *on* T *can be constructed, such that if* A *is true in* M_A, *then* A *is deducible in IPC. Additionally, we can suppose that individ domain of* M_A *is the set* Var *of all variables of our language.*

$\rhd$ Use 5.2. with S to be $\to A$. □

5.5. Theorem. *(Universal model property.)*

A modified Beth-model M *on a binary tree* T *can be constructed, such that for every formula* A, *if* A *is true in* M, *then* A *is deducible in IPC. Additionally it can be supposed, that the individ domain of* M *is the set* Var *of all variables of our language.*

⊳ Let $A_1, A_2, A_3, \ldots$ be an enumeration of all formulas. According 5.4. for every A_i we construct corresponding model M_{A_i}. Then we construct M the such way:



Now if A_i is true in M then A_i is true in M_{A_i} also, therefore A_i is deducible. □

5.6. Remark. An algebraic nature of our constructions allows to generalize our main results for higher order logics, for example, for intuitionistic type theory with or without extensionality (for example, for theories IT and ITE of TAKAHASHI [10], cf. also [3], chapter 5) but it is out of scope of this paper and will be done some where else.

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