On one-sided invertibility of linear coercive Fourier series operators

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Abstract

Let C_π^∞ be the linear space of all smooth periodic functions. The paper considers the linear operators L from C_π^∞ into itself which possess the formal transpose L': $C_\pi^\infty \to C_\pi^\infty$, that is, there exists a linear operator L': $C_\pi^\infty \to C_\pi^\infty$ such that

$$\int_W (L\varphi)(x)\,\psi(x)\,dx = \int_W \varphi(x)(L'\psi)(x)\,dx$$

for all $\varphi, \psi \in C_\pi^\infty$, where W is the (2π)-cube in $\mathbf{R}^n$. One shows that these operators can always be expressed in the form

$$(1) \qquad (L\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} L(x, l)\varphi_l e^{i(l,x)} \quad \text{for} \quad \varphi \in C_\pi^\infty,$$

where the mapping $x \to L(x, l)$ lies in C_π^∞ for each $l \in \mathbf{Z}^n$ and the derivative $D_x^\alpha L(x, l)$ is tempered with a polynomial. On the other hand supposing that for a given function $L(\cdot, \cdot)$: $\mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ the mapping $x \to L(x, l)$ lies in C_π^∞ and that there exist constants $\mu \in \mathbf{R}$ and $0 \leqq \delta < 1$ such that

$$(2) \qquad |D_x^\alpha L(x, l)| \leqq (1+|l|^2)^{(\mu+\delta|\alpha|)/2} \quad \text{for all} \quad l \in \mathbf{Z}^n, x \in \mathbf{R}^n,$$

one ver fies that the operator L defined by (1) maps C_π^∞ into itself and that the formal transpose L' of L exists.

For the operators L: $C_\pi^\infty \to C_\pi^\infty$ which own the formal transpose L', the existence of the one-sided inverse K of the maximal extension $L_k'^{\#}$: $H_k^\pi \to H_k^\pi$ is proved under the following coercivity condition (posed on L')

$$(3) \qquad \|L'\varphi\|_{1/k^\vee} \geqq C_1 \|\varphi\|_{k^\sim/k^\vee} - C_2 \|\varphi\|_{1/k^\vee} \quad \text{for all} \quad \varphi \in C_\pi^\infty,$$

where $C_1 > 0$ and $C_2 \geqq 0$. The spaces H_k^π are certain weighted subspaces of the space D_π' of all periodic distributions and the norm in the space H_k^π is denoted by $\|\cdot\|_k$. The weight function $k^\sim$ is assumed to obey the condition that $k^\sim(l) \to \infty$ with $|l| \to \infty$, and $k^\vee$ is defined by $k^\vee(l) = k(-l)$. Furthermore, sufficient criteria under which the one-sided inverse K can be expressed as the extension of a Fourier series operator are revealed.

1. Introduction

Let $L(x, D)$ be a linear partial differential operator with smooth periodic coefficients. Then $L(x, D)\varphi$ can be expressed in the form

$$(1.1) \qquad \big(L(x, D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} L(x, l)\,\varphi_l e^{i(l,x)}$$

for all φ lying in the space C_π^∞ of all smooth periodic functions. With the help of the symbol $L(x, l)$ one is able to show existence, uniqueness and regularity results for the distributional equation $L(x, D)u=f$, where u and f lie in the space D'_π of all periodic distributions. The exposition is usually done in the frame of the Hilbert spaces H_π^s of generalized trigonometric polynomials. For the elliptic case we refer to [2], pp. 131—299 and [4], pp. 95—124. In the contribution [6] one has considered (global) hypoelliptic operators and in [7] one has exposed the existence and uniqueness theory for t-coercive operators. For the generalizations we refer also to [3]. there the operators of the following form

$$(1.2)\qquad \big(A(D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} A(l)\,\varphi_l e^{i(l,x)}$$

have been studied, where A is a mapping $\mathbf{Z}^n \to \mathbf{C}$.

This paper generalizes the notion of a partial differential operator by examing the operators defined by

$$(1.3)\qquad \big(L(x, D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} L(x, l)\,\varphi_l e^{i(l,x)}$$

for φ lying in C_π^∞. Here $L(\cdot,\cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n \to \mathbf{C}$ is a mapping satisfying some temperating criteria (cf. Theorem 2.3). We call these operators Fourier series operators.

For the operators (1.3) which own the formal transpose $L'(x, D)\colon C_\pi^\infty \to C_\pi^\infty$, we will construct a one-sided continuous inverse K of the maximal operator $L_k'^{\#}$ under the assumption that a certain a priori estimate holds for $L'(x, D)$ (cf. Theorem 3.5 and Corollary 3.6). In addition we prove that the operator K can be expressed as the extension of a Fourier series operator in the case when "k is large enough" (cf. Corollary 4.3) and in the case when $L(x, D)$ is hypoelliptic (cf. Theorem 5.1).

2. Spaces H_k^π and Fourier series operators

2.1. Let W be a cube of $\mathbf{R}^n$ such that

$$(2.1)\qquad W = \{x\in\mathbf{R}^n \mid -\pi < x_j < \pi \text{ for } j\in\{1, \dots, n\}\}.$$

Furthermore, denote by C_π^∞ the linear subspace of all those $C^\infty(\mathbf{R}^n)$-functions which are periodic with respect to W. In the space C_π^∞ we set a locally convex topology defined by the semi-norms $q_\sigma\colon C_\pi^\infty \to \mathbf{R}$ such that

$$(2.2)\qquad q_\sigma(\psi) = \sup_{x\in W} |(D^\sigma\psi)(x)|, \quad \sigma\in\mathbf{N}_0^n.$$

It is well-known that C_π^∞ is a Frechet space and that ψ belongs to C_π^∞ if and only if it owns the form

$$(2.3)\qquad \psi(x) = (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} \psi_l e^{i(l,x)},$$

where the scalars $\psi_l\in\mathbf{C}$ satisfy the condition

$$(2.4)\qquad \sup_{l\in\mathbf{Z}^n} |\psi_l|(1+|l|^2)^{s/2} \leqq C_{s,\psi}$$

for every $s\in\mathbf{R}$ (cf. [1], p. 131).

The periodic distribution is a continuous linear form $C_\pi^\infty \to \mathbf{C}$. Let D'_π denote the linear space of all periodic distributions. Then T lies in D'_π if and only if there exist $t \in \mathbf{N}_0$ and $C > 0$ such that

$$|T\psi| \leqq C \sum_{|\sigma| \leqq t} q_\sigma(\psi) \quad \text{for all} \quad \psi \in C_\pi^\infty. \tag{2.5}$$

We use weak dual topology in D'_π.

Let $k\colon \mathbf{Z}^n \to \mathbf{R}$ be a positive function such that there exist constants $C > 0$ and $N > C$ with which

$$k(l+z) \leqq C(1+|l|^2)^{N/2} k(z) \quad \text{for all} \quad l, z \in \mathbf{Z}^n. \tag{2.6}$$

Denote by K_π the family of these mappings. Clearly one has for all $l \in \mathbf{Z}^n$

$$k(0)C^{-1}(1+|l|^2)^{-N/2} \leqq k(l) \leqq k(0)C(1+|l|^2)^{N/2}. \tag{2.7}$$

Furthermore one sees that the functions k_1+k_2, k_1k_2 and k^s are lying in K_π for all k_1 and $k_2 \in K_\pi$ and $s \in \mathbf{R}$. The basic example about the elements of K_π is the function $k_s\colon \mathbf{Z}^n \to \mathbf{R}$ defined by $k_s(l) = (1+|l|^2)^{s/2}$, $s \in \mathbf{R}$.

Definition 2.1. A distribution $T \in D'_\pi$ belongs to the space H_k^π if and only if

$$\big(\lambda_n \sum_{l \in \mathbf{Z}^n} |T_l k(l)|^2\big)^{1/2} < \infty, \tag{2.8}$$

where $\lambda_n := (2\pi)^{-n}$ and

$$T_l := T(e^{-i(l,x)}). \tag{2.9}$$

The mapping $T \to \|T\|_k := \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |T_l k(l)|^2\big)^{1/2}$ is clearly a norm in H_k^π. By using the properties of l_2-spaces one sees that the linear space H_k^π is a separable and reflexive Banach space. Furthermore it can be equipped with a scalar product defined by

$$(u, v)_k = \lambda_n \sum_{l \in \mathbf{Z}^n} \bar{u}_l v_l k(l)^2. \tag{2.10}$$

The space C_π^∞ is a dense subspace of H_k^π for each $k \in K_\pi$ and then the space H_k^π can be interpreted as a completion of C_π^∞ with respect to the scalar product $(\cdot,\cdot)_k\colon C_\pi^\infty \times C_\pi^\infty \to \mathbf{C}$ such that

$$(\varphi, \psi) = \lambda_n \sum_{l \in \mathbf{Z}^n} \bar{\varphi}_l \psi_l k(l)^2,$$

where φ_l is the Fourier coefficient of φ defined by

$$\varphi_l := \varphi(e^{-i(l,x)}) := \int_W \varphi(x) e^{-i(l,x)}\, dx.$$

Let $k^\vee \in K_\pi$ be defined through $k^\vee(l) = k(-l)$. One has for all φ and $\psi \in C_\pi^\infty$

$$\varphi(\psi) := \int_W \varphi(x)\psi(x)\, dx = \lambda_n \sum_{l \in \mathbf{Z}^n} \varphi_l \psi_{-l} \tag{2.11}$$

and then by Hölder's inequality we obtain

(2.12) $$|\varphi(\psi)| \leqq \|\varphi\|_k \|\psi\|_{1/k^\vee},$$

where $\|\varphi\|_k$ is the norm induced by the scalar product (2.10). In view of the inequality (2.12) one sees that

(2.13) $$|T(\varphi)| \leqq \|T\|_k \|\varphi\|_{1/k^\vee} \quad \text{for all} \quad T\in H_k^\pi \quad \text{and} \quad \varphi\in C_\pi^\infty.$$

This means that the topology of H_k^π is stronger than the topology induced by D'_π. Thus we have established that

(2.14) $$C_\pi^\infty \subset H_k^\pi \subset D'_\pi \quad \text{for each} \quad k\in K_\pi,$$

where both inclusions are topological.

Let $H_k^{\pi *}$ be the dual space of H_k^π. Then due to the Frechet—Riesz Theorem one has

Theorem 2.2. *Assume that k lies in K_π. Then for every $L\in H_k^{\pi *}$ there exists $V\in H_{1/k^\vee}^\pi$ such that*

(2.15) $$L\varphi = V(\varphi) \quad \textit{for all} \quad \varphi\in C_\pi^\infty$$

and $\|L\| = \|V\|_{1/k^\vee}$. Conversely the linear form $L\colon C_\pi^\infty \to C_\pi^\infty$ defined by $L\varphi = V(\varphi)$ with $V\in H_{1/k^\vee}^\pi$ can be (uniquely) extended to a continuous linear form on H_k^π.

We finally formulate an algebraic criterion for the compactness of the imbedding $\lambda\colon H_{k^\sim}^\pi \to H_k^\pi$, where k and $k^\sim$ lie in K_π (cf. [4], pp. 111—112).

Theorem 2.3. *The imbedding λ is compact if and only if the weight functions k and $k^\sim$ obey*

(2.16) $$k(l)/k^\sim(l) \to 0 \quad \text{for} \quad |l| \to \infty.$$

Proof. At first we suppose that (2.16) holds. Let $\{T_n\} \subset H_{k^\sim}^\pi$ be a sequence such that

(2.17) $$\|T_n\|_{k^\sim} \leqq M \quad \text{for all} \quad n\in \mathbf{N}.$$

Then one has for all $l\in \mathbf{Z}^n$ and $n\in \mathbf{N}$

(2.18) $$(\lambda_n)^{1/2} |(T_n)_l k^\sim(l)| \leqq \|T_n\|_{k^\sim} \leqq M.$$

The inequality (2.18) implies that one is able to find a subsequence $\{T_{n_j}\}$ such that

(2.19) $$\sum_{|l|\leqq\varrho} |(T_{n_j} - T_{n_k})_l k(l)|^2 \to 0 \quad \text{with} \quad j, k \to \infty$$

for each $\varrho \geqq 0$ (cf. [4], pp. 111—112).

Let ε be an arbitrary positive number and let $\varrho\in\mathbf{R}$ such that $k(l)/k^\sim(l) \leqq \varepsilon$ for $|l| \geqq \varrho$. Then we obtain

(2.20) $$\|T_{n_j} - T_{n_k}\|_k \leqq 2^{1/2}\big(2M\varepsilon + \sum_{|l|\leqq\varrho} |T_{n_j} - T_{n_k})_l k(l)|^2\big)^{1/2}.$$

Hence due to (2.19) one sees that $\{T_{n_j}\}$ is a Cauchy sequence in H_k^π.

The converse is easy to see by applying the assumption to the sequence $\{u_j\}\subset C_\pi^\infty$ defined through

$$u_j(x) = \lambda_n e^{i(l_j, x)}/k^{\sim}(l_j), \tag{2.21}$$

where $\{l_j\}\subset \mathbf{Z}^n$ is an (arbitrary) sequence such that $|l_j|\to\infty$ with $j\to\infty$. ∎

2.2. Suppose that L is a linear operator $C_\pi^\infty\to C_\pi^\infty$ such that its formal transpose $L'\colon C_\pi^\infty\to C_\pi^\infty$ exists, in other words, one can find a linear operator $L'\colon C_\pi^\infty\to C_\pi^\infty$ satisfying the relation

$$(L\varphi)(\psi) := \int_W (L\varphi)(x)\psi(x)\,dx = (\varphi)(L'\psi). \tag{2.22}$$

In the sequel we show that this kind of operator L can always be expressed in the form

$$\big(L(x, D)\varphi\big)(x) = \lambda_n \sum_{l\in\mathbf{Z}^n} L(x, l)\,\varphi_l\, e^{i(l,x)} \quad \text{for} \quad x\in\mathbf{R}^n, \tag{2.23}$$

where $L(\cdot,\cdot)$ is a mapping $\mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ such that the function $x\to L(x, l)$ belongs to C_π^∞ for each $l\in\mathbf{Z}^n$ and that the derivative $D_x^\alpha L(x, l)$ is tempered with a polynomial in l.

Lemma 2.4. *Suppose that L is a continuous linear operator $C_\pi^\infty\to C_\pi^\infty$. Then L can be expressed as a Fourier series operator defined by*

$$(L\varphi)(x) = \lambda_n \sum_{l\in\mathbf{Z}^n} L(x, l)\,\varphi_l\, e^{i(l,x)} =: \big(L(x, D)\varphi\big)(x), \tag{2.24}$$

where

1^0 *the mapping $x\to L(x, l)$ lies in C_π^∞ for each $l\in\mathbf{Z}^n$*

and

2^0 *for each $\alpha\in\mathbf{N}_0^n$ one can find constants $C_\alpha>0$ and $\mu_\alpha\in\mathbf{R}$ such that*

$$|D_x^\alpha L(x, l)| \leqq C_\alpha(1+|l|^2)^{\mu_\alpha/2} \quad \text{for all} \quad x\in W \quad \text{and} \quad l\in\mathbf{Z}^n. \tag{2.25}$$

Conversely, suppose that $L(\cdot,\cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ is a mapping satisfying 1^0 and 2^0. Then the relation (2.24) introduces a linear continuous operator $L\colon C_\pi^\infty\to C_\pi^\infty$.

Proof. *A*. Let L be a continuous linear mapping $C_\pi^\infty\to C_\pi^\infty$. Then for each $\alpha\in\mathbf{N}_0^n$ there exists $N_\alpha\in\mathbf{N}$ such that for all $\varphi\in C_\pi^\infty$

$$\sup_{x\in W}|D_x^\alpha(L\varphi)(x)| \leqq C_\alpha \sum_{|\beta|\leqq N_\alpha}\sup_{x\in W}|D_x^\beta(\varphi)(x)| \tag{2.26}$$

(cf. [8], p. 42). Applying this inequality with $\varphi=e^{i(l,x)}$ we obtain

$$\begin{aligned} &\sup_{x\in W}\big|D_x^\alpha\big(L(e^{i(l,x)})(x)e^{-i(l,x)}\big)\big| \leqq \\ &\leqq \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}\sup_{x\in W}\big|D_x^\gamma\big(L(e^{i(l,x)})\big)(x)\big|\,|l|^{|\alpha-\gamma|} \leqq \\ &\leqq \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma} C_\gamma \sum_{|\beta|\leqq N_\gamma}|l|^{|\beta|}|l|^{|\alpha-\gamma|} \leqq C_\alpha k_{\mu_\alpha}(l), \end{aligned} \tag{2.27}$$

where we denoted $(1+|l|^2)^{\mu_\alpha/2}=k_{\mu_\alpha}(l)$.

Define now a relation through

$$(L\varphi)(x) = g(x)(\varphi), \quad \text{for} \quad \varphi \in C_\pi^\infty \quad \text{and} \quad x \in \mathbf{R}^n. \tag{2.28}$$

Then $g(x)$ lies in D'_π (for a fixed $x \in \mathbf{R}^n$): Clearly $g(x)$ is a well-defined linear form $C_\pi^\infty \to \mathbf{C}$. Furthermore, the convergence

$$\varphi_n \to \varphi \quad \text{in} \quad C_\pi^\infty$$

implies (in view of (2.26)) that

$$\sup_{x \in W} |(L\varphi_n)(x) - (L\varphi)(x)| \to 0 \tag{2.29}$$

and then

$$g(x)(\varphi_n) = (L\varphi_n)(x) \to (L\varphi)(x) = g(x)(\varphi).$$

This says that $g(x)$ belongs to D'_π.

The expression (2.28) yields us

$$(L\varphi)(x) = \sum_{l \in \mathbf{Z}^n} \big(g(x)\big)_{-l}\varphi_l = \lambda_n \sum_{l \in \mathbf{Z}^n} \big(\lambda_n^{-1}(g(x))_{-l}e^{-i(l,x)}\big)\varphi_l e^{i(l,x)} =: \tag{2.30}$$

$$=: \lambda_n \sum_{l \in \mathbf{Z}^n} L(x, l)\varphi_l e^{i(l,x)},$$

where we used the fact that

$$T\varphi - \sum_{l \in \mathbf{Z}^n} T_{-l}\varphi_l \quad \text{for} \quad T \in D'_\pi \quad \text{and} \quad \varphi \in C_\pi^\infty. \tag{2.31}$$

The mapping $L(\cdot, \cdot)\colon \mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ such that

$$L(x, l) = \lambda_n^{-1}\big(g(x)\big)_{-l}e^{-i(l,x)} = \tag{2.32}$$

$$= \lambda_n^{-1} g(x)(e^{i(l,\cdot)})e^{-i(l,\cdot)} = \lambda_n^{-1} L(e^{i(l,\cdot)})(x)e^{-i(l,x)}$$

is well-defined and the mapping $x \to L(x, l)$ lies in C_π^∞. By taking into account relations (2.27) and (2.32) we obtain the validity of (2.25). Hence the first part of the proof is ready.

B. We now assume that the mapping $L(\cdot, \cdot)\colon \mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ satisfies 1^0 and 2^0. The distribution $\varphi \in D'_\pi$ lies in C_π^∞ if and only if for each $N \in \mathbf{N}$ there exists $C_N > 0$ such that

$$|\varphi_l| \leqq C_N(1+|l|^2)^{-N/2} \quad \text{for all} \quad l \in \mathbf{Z}^n \tag{2.33}$$

(cf. the subsection 2.1). Hence one sees in view of (2.25) that the sum (with $\varphi \in C_\pi^\infty$)

$$\lambda_n \sum_{l \in \mathbf{Z}^n} D_x^\alpha\big(L(x, l)\varphi_l e^{i(l,x)}\big)$$

is (absolutely) uniformly convergent for each $\alpha \in \mathbf{N}_0^n$. This yields that $D_x^\alpha(L\varphi)(x)$

exists and that

$$|D_x^\alpha(L\varphi)(x)| = \lambda_n \Big| \sum_{l\in \mathbf{Z}^n} D_x^\alpha\big(L(x,l)\,e^{i(l,x)}\big)\varphi_l\Big| \leqq \tag{2.34}$$

$$\leqq \lambda_n \sum_{l\in\mathbf{Z}^n} \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} D_x^\beta L(x,l)\,|l|^{|\alpha-\beta|}\,|\varphi_l| \leqq$$

$$\leqq \lambda_n \sum_{l\in\mathbf{Z}^n} \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} C_\beta\, k_{\mu_\beta}(l)\, k_{|\alpha-\beta|}(l)\,|\varphi_l| \leqq$$

$$\leqq C'_\alpha \sum_{l\in\mathbf{Z}^n} (1+|l|^2)^{N'_\alpha}\,|\varphi_l|/k_{n+1}(l) \leqq$$

$$\leqq C''_\alpha \sum_{l\in\mathbf{Z}^n} \sum_{|\gamma|\leqq N'_\alpha} l^{2\gamma}\,|\varphi_l|/k_{n+1}(l) \leqq$$

$$\leqq C''_\alpha \sum_{l\in\mathbf{Z}^n} \sum_{|\gamma|\leqq N'_\alpha} |(D_x^{2\gamma}\varphi)_l|/k_{n+1}(l) =$$

$$\leqq C''_\alpha \lambda_n \sum_{|\gamma|\leqq N'_\alpha} \sup_{x\in W} |(D_x^{2\gamma}\varphi)(x)| \sum_{l\in\mathbf{Z}^n} 1/k_{n+1}(l),$$

where N'_α is a sufficiently large natural number. This completes the proof. ∎

We are now ready to verify

Theorem 2.5. *Suppose that the linear operator L defined on C_π^∞ obeys*
(i) *L maps C_π into itself,*
(ii) *there exists the formal transpose $L'\colon C_\pi^\infty \to C_\pi^\infty$ of L.*
Then $L\colon C_\pi^\infty \to C_\pi^\infty$ is continuous so that it can be expressed as a Fourier series operator (2.24), *where the mapping $L(\cdot,\cdot)$ satisfies* 1^0 *and* 2^0.

Proof. We show that $L\colon C_\pi^\infty \to C_\pi^\infty$ is a closed operator. Let $\{\varphi_n\}\subset C_\pi^\infty$ be a sequence such that

$$\varphi_n \to \varphi \quad \text{in} \quad C_\pi^\infty \tag{2.35}$$

and

$$L\varphi_n \to \psi \quad \text{in} \quad C_\pi^\infty. \tag{2.36}$$

In virtue of the assumption (ii) for each $\Phi\in C_\pi^\infty$

$$(L\varphi_n)(\Phi) = \varphi_n(L'\Phi) \to \varphi(L'\Phi) = L\varphi(\Phi) \tag{2.37}$$

and then by (2.36) $L\varphi=\psi$. The closedness of L proves (by the Closed Graph Theorem) that L is continuous. The last assertions follow from Lemma 2.4 and then the proof is complete. ∎

A sufficient criterion under which the assumptions (i) and (ii) of Theorem 2.5 hold is given by the following

Theorem 2.6. *Assume that the mapping $x\to L(x,l)$ lies in C_π^∞ for each $l\in\mathbf{Z}^n$ and that one can find numbers $\mu\in\mathbf{R}$ and $0\leqq\delta<1$ such that for each $\alpha\in\mathbf{N}_0^n$ there exists a constant $C_\alpha>0$ with which the inequality*

$$|D_x^\alpha L(x,l)| \leqq C_\alpha(1+|l|^2)^{(\mu+\delta|\alpha|)/2} \tag{2.38}$$

for all $l\in\mathbf{Z}^n$, $x\in W$ holds. Then we have

3° *the linear operator given through* (2.23) *maps* C_π^∞ *into itself,*
4° *the formal transpose* $L'(x, D)\colon C_\pi^\infty \to C_\pi^\infty$ *of* $L(x, D)$ *exists and*

$$(2.39)\qquad \big(L'(x, D)\varphi\big)(x) = \lambda_n \sum_{l\in \mathbf{Z}^n} \Big(\int_W L(y, -l)\,\varphi(y)\,e^{-i(l,y)}\,dy\Big)\,e^{i(l,x)}$$

for every $\varphi \in C_\pi^\infty$.

Proof. In virtue of (2.38) the sum $\sum_{l\in\mathbf{Z}^n} L(y, l)\,\varphi_l e^{i(l,y)}$ is uniformly convergent. Hence we get via a direct computation

$$(2.40)\qquad \big(L(x, D)\psi\big)(\varphi) = \int_W \big(L(y, D)\psi\big)(y)\,\varphi(y)\,dy =$$

$$= \int_W \big(\lambda_n \sum_{l\in\mathbf{Z}^n} L(y, l)\,\psi_l e^{i(l,y)}\,\varphi(y)\,dy =$$

$$= \lambda_n \sum_{l\in\mathbf{Z}^n} \psi_l \Big(\int_W L(y, l)\,\varphi(y)\,e^{i(y,l)}\,dy\Big) =$$

$$= \lambda_n \sum_{l\in\mathbf{Z}^n} \psi_l \big(L'(x, D)\varphi\big)_{-l} = \psi\big(L'(x, D)\varphi\big).$$

We now check that $L'(x, D)\varphi$ defined through (2.39) lies in C_π^∞. Employing (2.33) our only task is to show that for every $\tau\in\mathbf{N}_0^n$

$$\big|l^\tau\big(L'(x, D)\varphi\big)_l\big| \leqq C_{\tau,\varphi} \quad \text{for every} \quad l\in\mathbf{Z}^n.$$

The condition (2.38) yields

$$(2.41)\qquad \big|l^\alpha\big(L'(x, D)\varphi\big)_l\big| = \Big|\int_W L(y, l)\,\varphi(y)\,D_y^\alpha(e^{i(l,y)})\,dy\Big| \leqq$$

$$\leqq \sum_{\gamma\leqq\alpha} \binom{\alpha}{\gamma} \int_W |D_y^\gamma L(y, l)(D_y^{\alpha-\gamma}\varphi)(y)| \leqq$$

$$\leqq \sum_{\gamma\leqq\alpha} \sup_{y\in W} |D_y^\gamma L(y, l)|\,\|D_y^{\alpha-\gamma}\|_{L^1(W)} \leqq$$

$$\leqq \sum_{\gamma\leqq\alpha} \binom{\alpha}{\gamma} C_\gamma (1+|l|^2)^{(\mu+\delta|\gamma|)/2}\,\|D_y^{\alpha-\gamma}\|_{L^1(W)} \leqq$$

$$\leqq C_{\alpha,\varphi}(1+|l|^2)^{(\mu+\delta|\alpha|)/2}.$$

The inequality (2.41) tells us that for every $m\in\mathbf{N}$ one is able to find a constant $C_{m,\varphi}>0$ such that

$$(2.42)\qquad \big|l^\tau\big(L'(x, D)\varphi\big)_l\big| \leqq C_{m,\varphi}(1+|l|^2)^{(\mu+\delta|\tau|+(\delta-1)m)/2}.$$

Since $\delta<1$ our assertion follows from (2.42) by choosing m large enough. ∎

2.3. Define a linear operator $L_k\colon H_k^\pi \to H_k^\pi$ via the requirement (here L is further linear operator $C_\pi^\infty \to C_\pi^\infty$ satisfying the condition (ii) of Theorem 2.5)

$$(2.43)\qquad \begin{cases} D(L_k) = C_\pi^\infty, \\ L_k\varphi = L\varphi \quad (= L(x, D)\varphi) \quad \text{for} \quad \varphi\in D(L_k). \end{cases}$$

Then L_k is densely defined and because of the existence of the formal transpose L', one sees that L_k is closable in H_k^π. This is based on the fact that the topology of H_k^π is stronger than the topology of D'_π. Let $L_k^\sim$ be the smallest closed extension of L_k (cf. [8], pp. 77—79).

Furthermore, let $L_k'^{\#}$ be a linear operator $H_k^\pi \to H_k^\pi$ defined by

$$(2.44)\qquad \begin{cases} D(L_k'^{\#}) = \{u \in H_k^\pi \mid \text{there exists an element } f \in H_k^\pi \text{ such that} \\ \qquad\qquad u(L_k'\varphi) = f(\varphi) \text{ for all } \varphi \in C_\pi^\infty\}, \\ L_k'^{\#} u = f. \end{cases}$$

The operator $L_k'^{\#}$ is closed and $L_k^\sim \subset L_k'^{\#}$ (that is, $L_k'^{\#}$ is an extension of $L_k^\sim$).

3. The construction of a one-sided inverse of $L_k'^{\#}$

3.1. We at first establish some semi-Fredholm properties of the minimal operator $L_k^\sim$. Afterwards we give the existence results for the solutions of the maximal equation $L_k'^{\#} u=f$ (by employing the duality between H_k^π and $H_{1/k^\vee}^\pi$). Let $L_k^*: H_k^{\pi *} \to H_k^{\pi *}$ be the dual operator of L_k. The kernel (the range) is denoted by $N(L_k^\sim)$ (and $R(L_k^\sim)$, resp.). We show

Theorem 3.1. *Suppose that a linear operator $L: C_\pi^\infty \to C_\pi^\infty$ obeys condition* (ii) *of Theorem* 2.5 *and that there exist constants $C_1>0$ and $C_2 \geqq 0$ such that*

$$(3.1)\qquad \|L\varphi\|_k \geqq C_1 \|\varphi\|_{kk^\sim} - C_2 \|\varphi\|_k \quad \textit{for all} \quad \varphi \in C_\pi^\infty,$$

where $k^\sim \in K_\pi$ is chosen so that

$$(3.2)\qquad k^\sim(l) \to \infty, \quad \textit{with} \quad |l| \to \infty.$$

Then the operator $L_k^\sim$ is a semi-Fredholm operator with

$$(3.3)\qquad \dim N(L_k^\sim) < \infty.$$

Proof. In virtue of (3.1) for all $u \in D(L_k^\sim)$

$$(3.4)\qquad C_1 \|u\|_{kk^\sim} \leqq \|L_k^\sim u\|_k + C_2 \|u\|_k.$$

Hence every bounded sequence of $N(L_k^\sim)$ possesses a convergent subsequence (since the imbedding $\lambda: H_{kk^\sim}^\pi \to H_k^\pi$ is compact). This shows the validity of (3.3).

Furthermore, by taking a sequence $\{L_k^\sim u_n\}$ in H_k^π, where $\{u_n\}$ is bounded in $D(L_k^\sim)$, one sees due to (3.4) that $\{u_n\}$ has a convergent subsequence $\{u_{n_j}\}$. Since $\|L_k^\sim u_{n_j} - f\|_k \to 0$ with some $f \in H_k$ with $j \to \infty$, the limit u of $\{u_{n_j}\}$ lies in $D(L_k^\sim)$ and $L_k^\sim u = f$. Hence $L_k^\sim(B)$ is closed when B is closed and bounded in $D(L_k^\sim)$, which implies that $R(L_k^\sim)$ is closed (cf. [5], pp. 99—100). This completes the proof. ∎

Corollary 3.2. *Let L be such as in Theorem* 3.1. *Then the relations*

$$R(L_k^{\sim}) = N(L_k^*)^{\perp} := \{f \in H_k^{\pi} | Tf = 0 \text{ for all } T \in N(L_k^*)\} \tag{3.5}$$

and

$$R(L_k^*) = {}^{\perp}N(L_k^{\sim}) := \{T \in H_k^{\pi *} | Tu = 0 \text{ for all } u \in N(L_k^{\sim})\} \tag{3.6}$$

hold.

This is a standard consequence of the properties of semi-Fredholm operators by taking into account that $L_k^{\sim *} = L_k^*$.

3.2. Suppose that the inclusion

$$D(L'^{\sim}_{1/(kk^{\sim})^{\nu}}) \subset H^{\pi}_{1/k^{\nu}} \tag{3.7}$$

holds, where $k^{\sim} \in K_{\pi}$ satisfies

$$k^{\sim}(l) \geqq 1 \quad \text{for all} \quad l \in \mathbf{Z}^n. \tag{3.8}$$

Then $L'^{\sim}_{1/(kk^{\sim})^{\nu}}$ is a closed operator $H^{\pi}_{1/k^{\nu}} \to H^{\pi}_{1/(kk^{\sim})^{\nu}}$ so that the dual operator $L'^{*}_{1/(kk^{\sim})^{\nu}}$ is a closed operator $H^{\pi *}_{1/(kk^{\sim})^{\nu}} \to H^{\pi *}_{1/k^{\nu}}$. Let J_k be the isometrical isomorphism $H_k^{\pi} \to H^{\pi *}_{1/k^{\nu}}$ for $k \in K_{\pi}$ (established in Theorem 2.2). In the sequel we exhibit the connection between the operators $L_k'^{\#}$ and $L'^{*}_{1/(kk^{\sim})^{\nu}}$.

Theorem 3.3. *Suppose that a linear operator* $L\colon C_{\pi}^{\infty} \to C_{\pi}^{\infty}$ *obeys the condition* (ii) *of Theorem* 2.5 *and that the inclusion* (3.7) *holds with* $k^{\sim} \geqq 1$. *Then the relation*

$$L_k'^{\#}|_{H^{\pi}_{kk^{\sim}}} = J_k^{-1} \circ (L'^{*}_{1/(kk^{\sim})^{\nu}}) \circ J_{kk^{\sim}} \tag{3.9}$$

is valid.

Proof. *A.* At first we show that $L_k'^{\#}|_{H^{\pi}_{kk^{\sim}}} \subset J_k^{-1} \circ (L'^{*}_{1/(kk^{\sim})^{\nu}}) \circ J_{kk^{\sim}}$. Suppose that u lies in $H^{\pi}_{kk^{\sim}} \cap D(L_k'^{\#})$ and that $L_k'^{\#} u = f$. Let $F = J_k f$ (and $U = J_{kk^{\sim}} u$) belong to $H^{\pi *}_{1/k^{\nu}}$ (and to $H^{\pi *}_{1/(kk^{\sim})^{\nu}}$, resp.) such that

$$F\varphi = f(\varphi) \quad \text{for all} \quad \varphi \in C_{\pi}^{\infty} \tag{3.10}$$

and

$$U\varphi = u(\varphi) \quad \text{for all} \quad \varphi \in C_{\pi}^{\infty}. \tag{3.11}$$

The relation $L_k'^{\#} u = f$ implies that

$$F\varphi = f(\varphi) = u(L_k'\varphi) = U(L'_{1/(kk^{\sim})^{\nu}}\varphi) \tag{3.12}$$

for all $\varphi \in C_{\pi}^{\infty} = D(L'_{1/(kk^{\sim})^{\nu}})$. Hence U lies in $D(L'^{*}_{1/(kk^{\sim})^{\nu}})$ and

$$L'^{*}_{1/(kk^{\sim})^{\nu}}(J_{kk^{\sim}} u) = L'^{*}_{1/(kk^{\sim})^{\nu}} U = F = J_k f, \tag{3.13}$$

which proves the first part of the assertion.

B. On the other hand we assume that u lies in $D\big(J_k^{-1}\circ(L'^{*}_{1/(kk^{\sim})^{\nu}})\circ J_{kk^{\sim}}\big)$. Then u lies in $H^{\pi}_{kk^{\sim}}$ and for all $\varphi\in C^{\infty}_{\pi}$

$$u(L'_k\varphi)=(J_{kk^{\sim}}u)(L'_k\varphi)=(J_{kk^{\sim}}u)(L'_{1/kk^{\sim})^{\nu}}\varphi)= \tag{3.14}$$
$$=\big(L'^{*}_{1/(kk^{\sim})^{\nu}}(J_{kk}u)\big)(\varphi)=J_k^{-1}\big(L'^{*}_{1/(kk^{\sim})^{\nu}}(J_{kk^{\sim}}u)\big)(\varphi),$$

since $J_{kk^{\sim}}u$ belongs to $D(L'^{*}_{1/(kk^{\sim})^{\nu}})$. Hence the proof is complete. ∎

Corollary 3.4. *Suppose that a linear operator* L: $C^{\infty}_{\pi}\to C^{\infty}_{\pi}$ *obeys condition* (ii) *of Theorem* 2.5. *Then the range* $R(L'^{\#}_k)$ *is closed in* H^{π}_k *if and only if the range* $R(L'^{\sim}_{1/k^{\nu}})$ *is closed in* $H^{\pi}_{1/k^{\nu}}$.

Proof. The assertion follows immediately from Theorem 3.3 with $k^{\sim}\equiv 1$ and from the fact that $R(L'^{\sim}_{1/k^{\nu}})$ is closed if and only if $R(L'^{*}_{1/k^{\nu}})$ is closed. ∎

3.3. In the sequel we construct a one-sided continuous inverse for $L'^{\#}_k$ on $R(L'^{\#}_k)$. More precisely, we show that (under certain conditions) there exists a continuous operator K: $R(L'^{\#}_k)\to H^{\pi}_k$ satisfying

$$L'^{\#}_k(Kf)=f \quad \text{for all} \quad f\in R(L'^{\#}_k). \tag{3.15}$$

Theorem 3.5. *Suppose that a linear operator* L: $C^{\infty}_{\pi}\to C^{\infty}_{\pi}$ *obeys the condition* (ii) *of Theorem* 2.5 *and that there exist constants* $C_1>0$ *and* $C_2\geqq 0$ *such that*

$$\|L'\varphi\|_{1/k^{\nu}}\geqq C_1\|\varphi\|_{k^{\sim}/k^{\nu}}-C_2\|\varphi\|_{1/k^{\nu}} \tag{3.16}$$

for all $\varphi\in C^{\infty}_{\pi}$, *where* $k^{\sim}$ *satisfies* (3.2). *Then there exists a continuous linear operator* K: $R(L'^{\#}_k)\to H^{\pi}_k$ *with the property*

$$L'^{\#}_k(Kf)=f \quad \textit{for all} \quad f\in R(L'^{\#}_k). \tag{3.17}$$

Proof. In virtue of Theorem 3.1, the assumption (3.16) implies that $R(L'^{\sim}_{1/k^{\nu}})$ is closed. Furthermore, one sees by Theorem 3.3 that

$$L'^{\#}_k=J_k^{-1}\circ(L'^{*}_{1/k^{\nu}})\circ J_k. \tag{3.18}$$

As a Hilbert space the space H^{π}_k can be expressed as the orthogonal sum

$$H^{\pi}_k=N(L'^{\#}_k)\oplus N, \tag{3.19}$$

where N is closed in H^{π}_k. Define now a linear operator $\mathscr{L}$: $H^{\pi}_k\to H^{\pi}_k$ as the restriction $L'^{\#}_k|_N$. Then the kernel $N(\mathscr{L})$ is $\{0\}$.

Since the range $R(L'^{\sim}_{1/k^{\nu}})$ is closed it follows that the range $R(L'^{*}_{1/k^{\nu}})$ is closed in $H^{\pi *}_{1/k^{\nu}}$. Hence due to (3.18) the range $R(\mathscr{L})=R(L'^{\#}_k)$ is closed in H^{π}_k. Furthermore the closedness of N implies that the operator $\mathscr{L}$: $H^{\pi}_k\to R(L'^{\#}_k)$ is closed. In view of the Closed Graph Theorem the operator $K:=\mathscr{L}^{-1}$: $R(L'^{\#}_k)\to H^{\pi}_k$ is continuous. Thus the proof is complete. ∎

Corollary 3.6. *Suppose that a linear operator* $L\colon C_\pi^\infty \to C_\pi^\infty$ *obeys the condition* (ii) *of Theorem* 2.5 *and that there exists* $\varkappa > 0$ *such that*

$$\|L'\varphi\|_{1/(kk^\sim)^\nu} \geqq \varkappa \|\varphi\|_{1/k^\nu} \quad \text{for all} \quad \varphi \in C_\pi^\infty, \tag{3.20}$$

where $k^\sim \geqq 1$. *Then there exists a continuous linear operator* $K\colon H_k^\pi \to H_{kk^\sim}^\pi$ *such that*

$$L_k'^{\#}(Kf) = f \quad \text{for all} \quad f \in H_k^\pi. \tag{3.21}$$

Proof. The inequality (3.20) yields that $R(L_{1/(kk^\sim)^\nu}'^\sim)$ is closed in $H_{1/(kk^\sim)^\nu}^\pi$ and that $N(L_{1/(kk^\sim)^\nu}'^\sim) = \{0\}$. The space $H_{kk^\sim}^\pi$ can be expressed as the sum

$$H_{kk^\sim}^\pi = N(L_{kk^\sim}'^{\#}) \oplus N$$

where N is closed in $H_{kk^\sim}^\pi$. One sees (as in the proof of Theorem 3.5) that the inverse of $L_k'^{\#}|_N\colon N \to H_k^\pi$ satisfies the required conditions. ∎

Corollary 3.7. *Suppose that a linear operator* $L\colon C_\pi^\infty \to C_\pi^\infty$ *obeys the condition* (ii) *of Theorem* 2.5 *and that there exists a constant* $\gamma > 0$ *such that for all* $\varphi \in C_\pi^\infty$

$$\|L\varphi\|_k \geqq \gamma \|\varphi\|_{kk^\sim} \tag{3.22}$$

and

$$\|L'\varphi\|_{1/(kk^\sim)^\nu} \geqq \gamma \|\varphi\|_{1/k^\nu}, \tag{3.23}$$

where $k^\sim \geqq 1$. *Then there exists a continuous linear operator* $E\colon H_k^\pi \to H_{kk^\sim}^\pi$ *with the properties*

$$L_k'^{\#}(Ef) = f \quad \text{for all} \quad f \in H_k^\pi \tag{3.24}$$

and

$$E(L_k^\sim u) = u \quad \text{for all} \quad u \in D(L_k^\sim). \tag{3.25}$$

Proof. In virtue of the inequalities (3.22) and (3.23) the ranges $R(L_k^\sim)$ and $R(L_{1/(kk^\sim)^\nu}'^\sim)$ are closed and

$$N(L_k^\sim) = N(L_{1/(kk^\sim)^\nu}'^\sim) = \{0\}. \tag{3.26}$$

As a Hilbert space the space H_k^π can be expressed as the orthogonal sum

$$H_k^\pi \in R(L_k^\sim) \oplus R, \tag{3.27}$$

and the projection $p\colon H_k^\pi \to R(L_k^\sim)$ is continuous.

Define a linear operator $E\colon H_k^\pi \to H_{kk^\sim}^\pi$ via the formula

$$Ef = L_k^{\sim\,-1}(pf) + K((I-p)f), \tag{3.28}$$

where K is the operator $H_k^\pi \to H_{kk^\sim}^\pi$ constructed in Corollary 3.6. Then E is continuous and since

$$L_k'^{\#}(L_k^{\sim\,-1}(pf)) = pf$$

and

$$(I-p)f = 0 \quad \text{for all} \quad f \in R(L_k^\sim),$$

a direct computation shows the relations (3.24)—(3.25). This finishes the proof. ∎

4. On properties of the one-sided inverse when k is "large"

4.1. Let L be (as in the previous chapter) a linear operator $C_\pi^\infty \to C_\pi^\infty$ possessing the formal transpose L': $C_\pi^\infty \to C_\pi^\infty$. As we have shown in Theorem **3.5**, the inequality

$$\|L'\varphi\|_{1/k^\vee} \geqq C_1 \|\varphi\|_{k^\sim/k^\vee} - C_2 \|\varphi\|_{1/k^\vee} \quad \text{for all} \quad \varphi \in C_\pi^\infty, \tag{4.1}$$

where $k^\sim \in K_\pi$ satisfies (3.2), is sufficient to guarantee the existence of a linear continuous operator K: $R(L_k'^{\#}) \to H_k^\pi$ such that

$$L_k'^{\#}(Kf) = f \quad \text{for all} \quad f \in R(L_k'^{\#}).$$

Furthermore, in Corollary 3.6 we showed that the inequality

$$\|L'\varphi\|_{1/(kk^\sim)^\vee} \geqq \varkappa \|\varphi\|_{1/k^\vee} \quad \text{for all} \quad \varphi \in C_\pi^\infty, \tag{4.2}$$

where $k^\sim \geqq 1$, implies the existence of a linear continuous operator K: $H_k^\pi \to H_{kk^\sim}^\pi$ such that

$$L_k'^{\#}(Kf) = f \quad \text{for all} \quad f \in H_k^\pi.$$

In this chapter we turn to the problem of seeking sufficient conditions under which the operator K can be expressed as a Fourier series operator

$$(K\varphi)(x) = \lambda_n \sum_{l \in \mathbf{Z}^n} K(x, l)\, \varphi_l e^{i(l,x)} \quad \text{for} \quad \varphi \in C_\pi^\infty \cap R(L_k'^{\#}), \tag{4.3}$$

where the mapping $K(\cdot, \cdot)$ obeys certain regularity properties. The essential tool will be the boundedness of K on $R(L_k'^{\#})$.

Let μ_n be a positive number such that

$$\mu_n := \inf\{\lambda > 0 \,|\, \sum_{l \in \mathbf{Z}^n} 1/(1+|l|^2)^\lambda < \infty\}.$$

The linear space of all m-times continuously differentiable periodic functions $\mathbf{R}^n \to \mathbf{C}$ will be denoted by C_π^m. First we show

Lemma 4.1. *Suppose that S is a continuous linear operator $H_k^\pi \to H_k^\pi$, where $k \in K_\pi$ satisfies the inequality*

$$k_{\gamma+m}(l) := (1+|l|^2)^{(\gamma+m)/2} \leqq Ck(l) \quad \textit{for all} \quad l \in \mathbf{Z}^n \tag{4.4}$$

with constants $C > 0$, $\gamma > \mu_n$ and $m \in \mathbf{N}_0$. Then there exists a mapping g: $W \to H_k^{\pi *}$ such that

1^0 *the function $x \to g(x)(v)$ lies in C_π^m for each $v \in H_k^\pi$,*

2^0 *for all $v \in H_k^\pi$ and $\varphi \in C_\pi^\infty$ one has*

$$(Sv)(\varphi) = \int_W g(x)(v)\, \varphi(x)\, dx, \tag{4.5}$$

3^0 *for all $x, y \in W$ one has*

$$\|g(x) - g(y)\| \leqq M \|S\| \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |e^{i(l,x)} - e^{i(l,y)}|^2 / k_{(\gamma+m)}^2(l)\big)^{1/2} \tag{4.6}$$

4^0 *the mapping g: $W \to H_k^{\pi *}$ is continuous and bounded on W.*

PROOF. *A*. For every $\varphi \in C_\pi^\infty$ we obtain by (4.4)

$$(4.7) \qquad |(D_x^\alpha \varphi)(x)| = \lambda_n \Big| \sum_{l \in \mathbf{Z}^n} l^\alpha \varphi_l e^{i(l,x)} \Big| \leqq$$

$$\leqq \lambda_n \sum_{l \in \mathbf{Z}^n} |\varphi_l| \, k_{\gamma+|\alpha|}(l) \big(|l|^{|\alpha|}/k_{\gamma+|\alpha|}(l)\big) \leqq$$

$$\leqq \big(\lambda_n \sum_{l \in \mathbf{Z}^n} (|l|^{|\alpha|}/k_{\gamma+|\alpha|}(l))^2\big)^{1/2} \|\varphi\|_{k_{\gamma+|\alpha|}} \leqq C' \|\varphi\|_k$$

for each $|\alpha| \leqq m$, where we applied Hölder's inequality. Hence for every $u \in H_k^\pi$ there exists a function f_u in C_π^m such that

$$(4.8) \qquad u(\varphi) = \int_W f_u(x) \varphi(x) \, dx \quad \text{for all} \quad \varphi \in C_\pi^\infty$$

(since C_π^m is complete equipped with the usual norm topology).

B. Define now a relation g through

$$(4.9) \qquad g(x)(v) := f_{Sv}(x) \quad \text{for} \quad x \in W.$$

Then for all $x \in W$ and $v \in H_k^\pi$ one has by (4.7) (with $\alpha = 0$)

$$(4.10) \qquad |g(x)(v)| = |f_{Sv}(x)| \leqq C' \|Sv\|_k \leqq C' \|S\| \, \|v\|_k.$$

In virtue of (4.7) the function $g(x)\colon v \to g(x)(v)$ is well-defined for each $x \in W$ and by (4.10) $g(x)$ lies in $H_k^{\pi *}$. Similarly it is clear (because of (4.9)) that the functions $x \to g(x)(v)$ (for a fixed $v \in H_k^\pi$) and $x \to g(x)$ are well-defined. Furthermore the function $x \to g(x)(v) = f_{Sv}(x)$ lies in C_π^m and by (4.8) one has

$$(4.11) \qquad (Sv)(\varphi) = \int_W f_{Sv}(x) \varphi(x) \, dx = \int_W g(x)(v) \varphi(x) \, dx.$$

C. For each pair $(x, y) \in W \times W$ we obtain

$$(4.12) \qquad \big|\big(g(x) - g(y)\big)(\varphi)\big| = |f_{S\varphi}(x) - f_{S\varphi}(y)| =$$

$$= \lambda_n \Big| \sum_{l \in \mathbf{Z}^n} (f_{S\varphi})_l (e^{i(l,x)} - e^{i(l,y)}) \Big| \leqq$$

$$\leqq \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |e^{i(l,x)} - e^{i(l,y)}|^2 / k_{\gamma+m}^2(l)\big)^{1/2} \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |(S\varphi)_l \, k_{\gamma+m}(l)|^2\big)^{1/2} \leqq$$

$$\leqq \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |e^{i(l,x)} - e^{i(l,y)}|^2 / k_{\gamma+m}^2(l)\big)^{1/2} C \|S\| \, \|\varphi\|_k$$

for all $\varphi \in C_\pi^\infty$. This inequality yields us

$$(4.13) \qquad \|g(x) - g(y)\| \leqq C \|S\| \big(\lambda_n \sum_{l \in \mathbf{Z}^n} |e^{i(l,x)} - e^{i(l,y)}|^2 / k_{\gamma+m}^2(l)\big)^{1/2}.$$

Since the series $\sum_{l \in \mathbf{Z}^n} |e^{i(l,x)} - e^{i(l,y)}|^2 / k_{\gamma+m}^2(l)$ is uniformly convergent (in $W \times W$) we see that the right hand side of (4.13) is tending to zero with $x \to y$. The boundedness of g follows immediately from (4.13). Hence the proof is complete. ∎

4.2. For each $L := g(x) \in H_k^{\pi *}$ there exists $V := g^\sim(x) \in H_{1/k^\nu}^\pi$ such that

$$(4.14) \qquad g(x)(\varphi) = g^\sim(x)(\varphi) \quad \text{for all} \quad \varphi \in C_\pi^\infty$$

and $\|g(x)\|=\|g^{\sim}(x)\|_{1/k^{\nu}}$ for all $x\in W$ (cf. Lemma 2.2). In the sequel we denote the function $f_{S\varphi}$ by $S\varphi$.

Theorem 4.2. *Suppose that S is a continuous linear operator $H_k^\pi\to H_k^\pi$, where $k\in K_\pi$ satisfies the inequality* (4.4). *Then the operator S can be expressed as the extension of a Fourier series operator*

$$(S\varphi)(x)=\lambda_n\sum_{l\in\mathbf{Z}^n}S(x,l)\,\varphi_l e^{i(l,x)}\quad\text{for}\quad \varphi\in C_\pi^\infty, \tag{4.15}$$

where the mapping $S(\cdot,\cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ satisfies

1^0 the mapping $x\to S(x,l)$ lies in C_π^m for each $l\in\mathbf{Z}^n$,

2^0 for each $\alpha\in\mathbf{N}_0^n$, $|\alpha|\leqq m$ there exists $C_\alpha>0$ such that for all $x\in W$ and $l\in\mathbf{Z}^n$

$$|D_x^\alpha S(x,l)|\leqq C_\alpha k_{\gamma+|\alpha|}(l). \tag{4.16}$$

Proof. In virtue of Lemma 4.1 one has

$$\begin{aligned}(S\varphi)(x)&=g(x)(\varphi)=g^{\sim}(x)(\varphi)=\\ &=\sum_{l\in\mathbf{Z}^n}(g^{\sim}(x))_{-l}\,\varphi_l=\lambda_n\sum_{l\in\mathbf{Z}^n}\lambda_n^{-1}(g^{\sim}(x))_{-l}\,e^{-i(l,x)}\,\varphi_l\,e^{i(l,x)}\end{aligned} \tag{4.17}$$

for all $x\in W$ and $\varphi\in C_\pi^\infty$, where we used the fact that

$$T\varphi=\sum_{l\in\mathbf{Z}^n}T_{-l}\,\varphi_l\quad\text{for all}\quad T\in D'_\pi\quad\text{and}\quad\varphi\in C_\pi^\infty. \tag{4.18}$$

Define now a function $S(\cdot,\cdot)\colon\mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ by

$$S(x,l)=\lambda_n^{-1}(g^{\sim}(x))_{-l}\,e^{-i(l,x)}. \tag{4.19}$$

Then for all $x\in W$ one obtains

$$S(x,l)=\lambda_n^{-1}g^{\sim}(x)(e^{i(l,\cdot)})\,e^{-i(l,x)}=\lambda_n^{-1}S(e^{i(l,\cdot)})(x)\,e^{-i(l,x)}. \tag{4.20}$$

Hence the function $x\to S(x,l)$ lies in C_π^m and

$$\begin{aligned}|D_x^\alpha S(x,l)|&\leqq\lambda_n^{-1}\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}\left|D_x^\beta\big(S(e^{i(l,\cdot)})\big)(x)\right||l^{\alpha-\beta}|\leqq\\ &\leqq\lambda_n^{-1}\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}C_\beta\,\|S(e^{i(l,\cdot)})\|_{k_{\gamma+|\beta|}}|l^{\alpha-\beta}|\leqq\\ &\leqq\lambda_n^{-1}\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}C_\beta\,\|S\|\,\|e^{i(l,\cdot)}\|_{k_{\gamma+|\beta|}}|l^{\alpha-\beta}|\leqq C_\alpha k_{\gamma+|\alpha|}(l)\end{aligned} \tag{4.21}$$

where we applied the inequality

$$|(D_x^\alpha v)(x)|\leqq C_\alpha\|v\|_{k_{\gamma+|\alpha|}} \tag{4.22}$$

for all $v\in H_k^\pi\subset H_{k_{\gamma+m}}^\pi\,H_{k_{\gamma+|\alpha|}}^\pi$ (cf. (4.7)). This completes the proof. ∎

Combining Theorems 3.5 and 4.2 we obtain

Corollary 4.3. *Suppose that a linear operator* $L\colon C_\pi^\infty \to C_\pi^\infty$ *possesses a formal transpose* $L'\colon C_\pi^\infty \to C_\pi^\infty$ *and that there exist constants* $C_1>0$ *and* $C_2\geqq 0$ *such that for all* $\varphi\in C_\pi^\infty$

$$\|L'\varphi\|_{1/k^\nu} \geqq C_1\|\varphi\|_{k^\sim/k^\nu} - C_2\|\varphi\|_{1/k^\nu}, \tag{3.17}$$

where $k\in K_\pi$ *satisfies* (4.4) *and* $k^\sim\in K_\pi$ *satisfies* (3.2). *Then there exists a continuous linear operator* $K\colon R(L_k'^{\#})\to H_k^\pi$ *with the properties*

1^0 *for all* $f\in R(L_k'^{\#})$

$$L_k'^{\#}(Kf)=f, \tag{3.18}$$

2^0 *for all* $\varphi\in C_\pi^\infty\cap R(L_k'^{\#})$

$$(K\varphi)(x)=\lambda_n\sum_{l\in\mathbf{Z}^n}K(x,l)\,\varphi_l e^{i(l,x)}, \tag{4.23}$$

where the mapping $K(\cdot,\cdot)$ *satisfies*

3^0 *the function* $x\to K(x,l)$ *lies in* C_π^m,

4^0 *for each* $\alpha\in\mathbf{N}_0^n$, $|\alpha|\leqq m$

$$|D_x^\alpha K(x,l)|\leqq C_\alpha k_{\gamma+|\alpha|}(l)\quad\text{for}\quad x\in W\quad\text{and}\quad l\in\mathbf{Z}^n. \tag{4.24}$$

Proof. In virtue of Theorem 3.5 there exists a continuous operator $K\colon R(L_k'^{\#})\to H_k^\pi$ satisfying (3.18). Let $\overline{K}=K\circ q$, where $q\colon H_k^\pi\to R(L_k'^{\#})$ is the continous projection (of Corollary 3.4). Since k obeys the inequality (4.4) we obtain by Theorem 4.2 that

$$(K\varphi)(x)=\lambda_n\sum_{l\in\mathbf{Z}^n}K(x,l)\,\varphi_l e^{i(l,x)}\quad\text{for all}\quad\varphi\in C_\pi^\infty,$$

where $K(\cdot,\cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ satisfies 3^0 and 4^0. Hence we have proved the assertion. ∎

Similarly the combination of Corollary 3.6 and Theorem 4.2 yields

Corollary 4.4. *Suppose that a linear operator* $L\colon C_\pi^\infty\to C_\pi^\infty$ *possesses a formal transpose* $L'\colon C_\pi^\infty\to C_\pi^\infty$ *and that there exists a constant* $\varkappa>0$ *such that*

$$\|L'\varphi\|_{1/(kk^\sim)^\nu}\geqq\varkappa\|\varphi\|_{1/k^\nu}\quad\text{for all}\quad\varphi\in C_\pi^\infty, \tag{4.25}$$

where $k\in K_\pi$ *satisfies* (4.4) *and where* $k^\sim\geqq 1$. *Then there exists a continuous linear operator* $K\colon H_k^\gamma\to H_{kk^\sim}^\pi$ *with the properties*

1^0 *for all* $f\in H_k^\pi$

$$L_k'^{\#}(Kf)=f,$$

2^0 *for all* $\varphi\in C_\pi^\infty$

$$(K\varphi)(x)=\lambda_n\sum_{l\in\mathbf{Z}^n}K(x,l)\,\varphi_l e^{i(l,x)},$$

where the mapping $K(\cdot,\cdot)$ *satisfies* 3^0 *and* 4^0 *of Corollary* 4.3.

5. On properties of the one-sided inverse when L is (globally) hypoelliptic

Suppose that a linear operator $L\colon C_\pi^\infty \to C_\pi^\infty$ owns a formal transpose $L'\colon C_\pi^\infty \to C_\pi^\infty$. Define a linear operator $L'^{\#}_{-\infty}\colon D'_\pi \to D'_\pi$ such that

$$\begin{cases} D(L'^{\#}_{-\infty}) = \{u \in D'_\pi \mid \text{there exists an element } f \in D'_\pi \text{ such that} \\ \qquad\qquad u(L'\varphi) = f(\varphi) \text{ for all } \varphi \in C_\pi^\infty\}, \\ L'^{\#}_{-\infty} u = f. \end{cases}$$

Then one sees easily that the domain $D(L'^{\#}_{-\infty})$ is the whole space D'_π.

We say that the linear operator $L\colon C_\pi^\infty \to C_\pi^\infty$ possessing a formal transpose $L'\colon C_\pi^\infty \to C_\pi^\infty$ is (D'_π, C_π^∞)-hypoelliptic when the solutions of the equation $L'^{\#}_{-\infty} u = f$ with $f \in C_\pi^\infty$ lie in C_π^∞.

It is well-known that the operator $L(D)$ defined by

$$(5.1) \qquad \big(L(D)\varphi\big)(x) = \lambda_n \sum_{l \in \mathbf{Z}^n} L(l)\,\varphi_l\, e^{i(l,x)},$$

where $L(\cdot)\colon \mathbf{R}^n \to \mathbf{C}$ is polynomial is (D'_π, C_π^∞)-hypoelliptic if and only if there exist constants $C > 0$, $R \geqq 0$ and $\varkappa \in \mathbf{R}$ such that

$$(5.2) \qquad |L(l)| \geqq C k_\varkappa(l) \quad \text{for all} \quad l \in \mathbf{Z}^n \quad \text{with} \quad |l| \geqq R$$

(cf. [6]). We show

Theorem 5.1. *Suppose that L is a linear operator $C_\pi^\infty \to C_\pi^\infty$ possessing a formal transpose $L'\colon C_\pi^\infty \to C_\pi^\infty$. Furthermore we assume that L is (D'_π, C_π^∞)-hypoelliptic and that there exists a constant $\varkappa > 0$ such that for all $\varphi \in C_\pi^\infty$*

$$(3.16) \qquad \|L'\varphi\|_{1/(kk^{\sim})^{\nu}} \geqq \varkappa \|\varphi\|_{1/k^{\nu}},$$

where $k^{\sim} \geqq 1$. Then there exists a continuous linear operator $K\colon H_k^\pi \to H_{kk^{\sim}}^\pi$ with the properties

1^0 *for all $f \in H_k^\pi$*

$$(3.21) \qquad L_k'^{\#}(Kf) = f,$$

2^0 *the restriction of K on C_π^∞ can be expressed as a Fourier series operator*

$$(5.3) \qquad (K\varphi)(x) = \lambda_n \sum_{l \in \mathbf{Z}^n} K(x,l)\,\varphi_l\, e^{i(l,x)} =: \big(K(x,D)\varphi\big)(x),$$

where

3^0 *the mapping $x \to K(x,l)$ lies in C_π^∞ for each $l \in \mathbf{Z}^n$*

and

4^0 *for each $\alpha \in \mathbf{N}_0^n$ there exist $C_\alpha > 0$ and $\mu_\alpha \in \mathbf{R}$ such that*

$$(5.4) \qquad |D_x^\alpha K(x,l)| \leqq C_\alpha k_{\mu_\alpha}(l) \quad \text{for all} \quad x \in W \quad \text{and} \quad l \in \mathbf{Z}^n.$$

Proof. In virtue of Corollary 3.6 there exists a continuous linear operator $K\colon H_k^\pi \to H_{kk^{\sim}}^\pi$ satisfying (3.21). Because of (3.21) and (D'_π, C_π^∞)-hypoellipticity of L one sees that K maps C_π^∞ into itself (with $D(K) = C_\pi^\infty$).

Let $\{\psi_n\}$ be a sequence in C_π^∞ such that $\psi_n \to \psi$ and $K\psi_n \to f$ (in C_π^∞). Because the inclusion $C_\pi^\infty \subset H_k^\pi$ holds (also topologically) we get that $K\psi_n \to K\psi$ in $H_{kk^\sim}^\pi$. Hence $K\psi = f$, which shows that $K\colon C_\pi^\infty \to C_\pi^\infty$ is closed. Due to the Closed Graph Theorem K is continuous. Thus Lemma 2.4 completes the proof. ∎

We remark that the relations (3.21) and (5.3) yield the validity of the equation

$$L(x, D)\big(K(x, D)\varphi\big) = \varphi \quad \text{for all} \quad \varphi \in C_\pi^\infty. \tag{5.5}$$

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