On the spectrum of a liminal C^*-algebra

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1. Introduction

In this paper we prove that:

(i) If E is a unital separable η-homogeneous C^*-algebra, its spectrum is metrisable.

(ii) If E is a separable (unital) C^*-algebra the pure states set is a polish set. Also, if E has a Souslin pure states set (i.e. the underlying topological space is Souslin) the state space is metrisable.

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2. Preliminaries

For general results on C^*-algebras notation and terminology about derivations, $\mathscr{K}$-Souslin sets and on compact convex sets we refer to ([3], [8], [9], [7], [6]) respectively. In particular we have the following definitions.

A *derivation* on an algebra E is a linear map $\delta\colon E\to E$ such that

$$\delta(xy)=(\delta x)y+x(\delta y)\quad (x,y\in E).$$

An element a of some possible larger algebra is said to implement δ if

$$\delta x = ax-xa\quad (x\in E)$$

and δ is said to be *inner* if such an a can be found in E. Otherwise δ is said to be *outer* ([8]).

A C^*-algebra is said to be *n-homogeneous* if all its irreducible $*$-representations are of the same finite dimension n.

A C^*-algebra is said to be *liminal* if, for every irreducible $*$-representation Π of E and each $x\in E$, $\pi(x)$ is compact. The C^*-algebra E is said to be *postliminal* if every non-zero quotient C^*-algebra of E possesses a non-zero liminal closed two-sided ideal.

All the algebras that we are concerned here are of type I.

A topological space E is said to be *polish*, if it is separable and if there exists a metric on E for which the topology is τ and $E[\tau]$ is complete.

A Hausdorff space $E[\tau]$ is said to be *Lusin* (resp. *Souslin*) if it is the injective continuous (resp. continuous) image of a polish space.

A point x of a convex set K is an *extreme point* of K iff x is not an interior point of any line segment whose endpoints belong to K.

3. On the Spectrum

We state and prove the following

Theorem 3.1. *If E is a unital separable η-homogeneous C^*-algebra then, $\hat{E}$ is metrisable.*

PROOF. Since E is a unital η-homogeneous C^*-algebra the set of pure states $P(E)$ is w^*-compact ([10]). Since E is separable, E'_s is a Lusin space ([7, II, p. 115]). Now the state space $S(E)\subseteq E'_s$ is a compact Lusin space and thus metrisable ([ibid, p. 106]). Also, it is well known, that the canonical map $P(E)\to\hat{E}$ is continuous (open) and onto, and thus $\hat{E}$ is w^*-compact and metrisable since the continuous image of a compact metrisable space in a Hausdorff space ([4, Th. 4.2]), is compact and metrisable.

Corollary 3.2. *If E is a unital separable (post) liminal C^*-algebra, with all derivations inner, then the spectrum of E is metrisable.*

PROOF. If every derivation on E is inner, E is the direct sum of finitely many unital homogeneous C^*-algebras ([1.5.5]), but the spectrum of an η-homogeneous C^*-algebra with identity is a compact Hausdorff space ([4, Th. 4.2]), and thus $\hat{E}$ is a compact Hausdorff space. Now, we continue as at the previous theorem.

Now, we state some examples in relation with Theorem 3.1.

(α) Let E be a compact Hausdorff space and we suppose that is not second countable. The $C(E)$ is a unital postliminal non-separable C^*-algebra, with all derivations inner. Then, its spectrum E is not metrisable.

(β) Let E be the C^*-algebra of all $m=\{m_\eta\}$, of 2×2 complex matrices for which $\sup\limits_{\eta=1,2,\dots}\|m_\eta\|$ is finite, with coordinatewise operations and $\|m\|=\sup\limits_{\eta=1,2,\dots}\|m_\eta\|$ such that m_η converges to a matrix of the form

$$\begin{pmatrix}\lambda(m) & 0\\ 0 & \mu(m)\end{pmatrix}$$

as $\eta\to\infty$.

E is a unital, separable, liminal C^*-algebra with outer derivations and the spectrum is not Hausdorff (see: [8, p. 534]).

4. Note on the extreme points

Let K be a compact convex set and $\partial_e K$ the set of the extreme points. We state and prove the following:

Proposition 4.1. *Let E be a unital C^*-algebra.*

(a) *If E is separable, the set of pure states is a polish set.*

(b) *If the set of pure states of E is Souslin, the state space of E is metrisable.*

For the proof of this proposition we need the following.

Lemma 4.2. *Let K be a compact convex metrisable space. Then, $\partial_e K$ is a polish set.*

PROOF. Obvious by ([6, 1.3], [7, II]).

PROOF OF PROPOSITION 4.1. (a) Let $S(E) \subseteqq E_s'$ the state space of E, $S(E)$ is w^*-compact Souslin set and thus metrisable. Now, by the above Lemma, the set of pure states is a polish subset of $S(E)$. (See also: [5, p. 101].)

(b) It is proved by ([2]), that a compact convex set is metrisable, if the set of the extreme points is the continuous image of a complete separable metric space. By the above and according well known definitions ([7, II]), the state space is metrisable.

References

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