On additive arithmetical functions with values in topological groups II

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1. Let G be an additively written Abelian topological group with the translation invariant metric ϱ. A mapping $\varphi\colon \mathbf{N}\to G$ is called to be a completely additive function, if

$$\varphi(mn) = \varphi(m)+\varphi(n) \quad \forall m, n\in \mathbf{N} \tag{1.1}$$

holds.

Continuing our work [1] we shall consider completely additive functions under the condition

$$\sum_{n=1}^{\infty} \frac{\varrho\big(\varphi(n), \varphi(n+1)\big)}{n} < \infty. \tag{1.2}$$

A complete characterization of completely additive functions subject to (1.2) for $G=\mathbf{R}/Z$ was given by the second named author in a series of papers [2].

Let Q_x, resp. R_x be the multiplicative groups of positive rationals and the positive reals. We can extend the domain of φ to Q_x by the relation

$$\varphi\left(\frac{m}{n}\right) := \varphi(m)-\varphi(n),$$

uniquely. Then φ satisfies the relation

$$\varphi(rs) = \varphi(r)+\varphi(s) \quad \forall r, s\in Q_x,$$

so $\varphi\colon Q_x\to G$ is a homomorphism. We shall say that φ is continuous at the point 1, if $r_\nu\in Q_x$, $r_\nu\to 1$ implies that $\varphi(r_\nu)\to 0$. In [1] we proved the next

Lemma 1. *Let G be an additively written closed Abelian topological group, $\varphi\colon Q_x\to G$ be a homomorphism that is continuous at the point 1. Then its domain can be extended onto $\mathbf{R}_x$ by the relation*

$$\varphi(\alpha) := \lim_{\substack{r_\nu\to\alpha \\ r_\nu\in Q_x}} \varphi(r_\nu) \quad (\alpha\in\mathbf{R}_x)$$

uniquely. The so obtained mapping $\varphi\colon \mathbf{R}_x\to G$ is a continuous homomorphism, consequently

$$\varphi(\alpha\beta) = \varphi(\alpha)+\varphi(\beta) \quad (\forall \alpha, \beta\in\mathbf{R}_x). \tag{1.3}$$

Theorem. *If φ satisfies* (1.1), (1.2) *then it is a continuous* $\mathbf{R}_x \to G$ *homomorphism.*

2. PROOF OF THEOREM. The proof is based on the following

Lemma 2. *Let $p, q \in \mathbf{N}$ be coprime integers, $q < p^2$; $M, N \in \mathbf{N}$. Let (k_t, l_t) be such a sequence in $\mathbf{N}^2$ for which*

$$r_t := \frac{p^{k_t}}{q^{l_t}} \to \frac{M}{N}$$

holds. Then

$$\varphi(r_t) \to \varphi(M/N). \quad \blacksquare$$

First we deduce our theorem from this lemma. Let $r_j = N_j \cdot M_j^{-1} \in Q_x$, $r_j \to 1$. Let p, q be fixed integers satisfying the conditions stated in Lemma 2. Then for every r_j there exists a suitable sequence $(a_\nu, b_\nu) \in \mathbf{N}^2$ such that

$$p^{a_\nu} \cdot p^{-b_\nu} \to r_j \quad (\nu \to \infty),$$

and so by Lemma 2

$$\varphi(p^{a_\nu} q^{-b_\nu}) \to \varphi(r_j) \quad (\nu \to \infty).$$

Consequently there exists such a sequence $(k_j, l_j) \in \mathbf{N}^2$ for which

$$p^{k_j} q^{-l_j} - r_j \to 0, \quad \varphi(p^{k_j} \cdot q^{-l_j}) - \varphi(r_j) \to 0.$$

Observing now that $p^{k_j} q^{-l_j} \to 1$, from Lemma 2 $(M = N = 1)$ we get that

$$\varphi(p^{k_j} q^{-l_j}) \to \varphi(1) = 0.$$

Consequently $\varphi(r_j) \to 0$, the conditions in Lemma 1 are satisfied, and so the Theorem in true.

Let us prove finally Lemma 2.

Let K be a large constant,

$$f(n) = \max_{a \in [-K, K]} \varrho(\varphi(n), \varphi(n+a)).$$

From (1.2) we have

$$\sum_{n \geqq 1} \frac{f(n)}{n} < \infty.$$

Let

$$t(y) = \sum_{n \geqq y} \frac{f(n)}{n},$$

furthermore

$$A_m := [Np^m, Np^{m+1}); \; B_m := [Mq^m, Mq^{m+1}).$$

Let $H_1 \in \mathbf{N}$ be a large constant, $(k, l) \in \mathbf{N}^2$ be such that

$$\left| \frac{Np^k}{Mq^l} - 1 \right| < \frac{1}{p^{H_1}}, \tag{2.1}$$

and $H_2 \in \mathbf{N}$ be defined so that $B_{l+H_2} \subseteq A_{k+H_1}$. For an $n \in B_{l+H_2}$ let

$$n_j = n_j(n) = \left[\frac{n}{p^j}\right], \quad m_j = m_j(n) = \left[\frac{n}{q^j}\right] \quad (j = 0, 1, 2, \ldots).$$

It is clear that $n_j \in A_{k+H_1-j}$, $m_j \in A_{l+H_2-j}$, furthermore each n_j occurs for at most p^j of n, and each m_j occurs for at most q^j of n. Furthermore

$$n_j = pn_{j+1} + b_j, \quad m_j = qm_{j+1} + c_j, \quad 0 \leqq b_j < p, \quad 0 \leqq c_j < q.$$

Hence we get that

$$\varrho(\varphi(n_j), \varphi(pn_{j+1})) \leqq f(n_j), \quad \varrho(\varphi(m_j), \varphi(qm_{j+1})) \leqq f(m_j),$$

assuming that $K > p (> q)$.

We have

$$\varrho(\varphi(n), \varphi(p^k n_k(n))) \leqq \sum_{s=0}^{k-1} \varrho(\varphi(p^s n_s(n)), \varphi(p^{s+1} n_{s+1}(n))) \leqq \sum_{s=0}^{k-1} f(n_s(n)),$$

and similarily that

$$\varrho(\varphi(n), \varphi(q^l m_l(n))) \leqq \sum_{s=0}^{l-1} f(m_s(n)).$$

Consequently

$$\varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) \leqq \sum_{s=0}^{k-1} f(n_s(n)) + \sum_{s=0}^{l-1} f(m_s(n)),$$

and so

$$\sum_{n \in B_{l+H_2}} \frac{1}{n} \varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) \leqq t(p^{H_1-1}) + t(q^{H_2-1}). \tag{2.2}$$

We have, by (2.1)

$$\begin{aligned} |Mn_k(n) - Nm_l(n)| &\leqq n \left| \frac{M}{p^k} - \frac{N}{q^l} \right| + (M+N) \leqq \frac{nM}{p^k} \left| 1 - \frac{Np^k}{Mq^l} \right| + M + N \leqq \\ &\leqq \frac{p^{k+H_1+k} \cdot M}{p^k} \cdot \frac{1}{p^{H_1}} + (M+N) \leqq 2M + N. \end{aligned} \tag{2.3}$$

By using the translation invariant property of ϱ, we get that

$$\begin{aligned} \varrho\left(\varphi\left(\frac{p^k}{M}\right), \varphi\left(\frac{q^l}{N}\right)\right) &= \varrho\left(\varphi(p^k n_k(n)), \varphi\left(\frac{q^l}{N} Mn_k(n)\right)\right) \leqq \\ &\leqq \varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) + \varrho(\varphi(Mn_k(n)), \varphi(Nm_l(n))). \end{aligned}$$

Hence, by (2.2) we deduce that

$$\begin{aligned} \varrho\left(\varphi\left(\frac{p^k}{M}\right), \varphi\left(\frac{q^l}{N}\right)\right) \cdot \sum_{n \in B_{l+H_2}} \frac{1}{n} &\leqq t(p^{H_1-1}) + t(q^{H_2-1}) + \\ &+ \sum_{n \in B_{l+H_2}} \varrho(\varphi(Mn_k(n)), \varphi(Nm_l(n))) \frac{1}{n}. \end{aligned} \tag{2.4}$$

The sum on the right hand side is majorated by $\ll t(p^{H_1})$. Since

$$1 \leqq \sum_{n \in B_{l+H_2}} \frac{1}{n} < q,$$

we get that

$$\varrho\left(\varphi\left(\frac{p^k N}{q^l M}\right), 0\right) \ll t(p^{H_1-1}) + t(q^{H_2-1}).$$

Let now $(k, l)=(k_t, l_t)$. Then (2.1) is satisfied by a suitable sequence $H_1= =H_1(t)\to\infty$. This implies that

$$\varphi\left(\frac{p^{k_t} N}{q^{l_t} M}\right) \to 0,$$

and so the proof of Lemma 2 is finished.

References

[1] Z. DARÓCZY and I. KÁTAI, On additive arithmetical functions with values in topological groups I *Publ. Math. (Debrecen)* **32** (1985),

[2] I. KÁTAI, Multiplicative functions with regularity properties I., II., III. *Acta Math. Hung.* **42** (1983), 295—308, **43** (1984), 105—130, **43** (1984), 259—272.

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