On the different definitions of Finsler-connection

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In the case of tangent bundles, a very elegant Koszul-type definition of a Finsler-connection was given and discussed by Z. I. Szabó [8]. A further important development is Miron's recent work [6] in the theory of Finsler-connections, which extends the notion of a Finsler-connection to an arbitrary vector bundle. In this note we study a relation between these two theories, comparing them also with Matsumoto's well-known theory ([5]).

1. Introduction

Let $\xi=(E, \pi, B, F)$ be a real vector bundle of finite rank over the base manifold B. (For notations and terminology, see the monographs [3], [4].) The tangent bundle of a manifold M is denoted by τ_M and its total space by TM; $\operatorname{Sec}\xi$ is the $C^\infty(B)$-module of the sections of ξ and — in particular — $\mathfrak{X}(M):=\operatorname{Sec}\tau_M$ is the module of the vector fields on M. $\pi^*(\tau_B)$ is the pull-back of τ_B over π.

Our basic starting objects in this paper are the exact sequences

(SEQ 1) $$0 \to \pi^*(\tau_B) \xrightarrow{\lambda} \tau_{TB} \xrightarrow{\mu} \pi^*(\tau_B) \to 0$$

and

(SEQ 2) $$0 \to v\xi \xrightarrow{i} \tau_E \xrightarrow{j} \pi^*(\tau_B) \to 0.$$

We fix a right splitting of (SEQ 1) and one of (SEQ 2) and — for the sake of simplicity — we denote both of them by $\mathscr{H}$. We also need the left splitting $\mathscr{V}$ complementary to $\mathscr{H}$ (for which $\operatorname{Im}\mathscr{H}=\operatorname{Ker}\mathscr{V}$), the horizontal and vertical projections h and v belonging to $\mathscr{H}$ and finally the almost tangent structure

$$J = \lambda\circ\mu\colon \tau_{TB} \to \tau_{TB}.$$

In conclusion, we recall the notion of pseudoconnection in ξ with respect to $\tilde{\xi}$, this will be called simply a pseudoconnection.

Definition. Let the vector bundles $\xi=(E, \pi, B, F)$ and $\tilde{\xi}=(\tilde{E}, \tilde{\pi}, B, \tilde{F})$ be given. A pseudoconnection in ξ with respect to $\tilde{\xi}$ is a pair (∇, A), where $A\colon \tilde{\xi}\to\tau_B$ is a B-morphism and $\nabla\colon \operatorname{Sec}\tilde{\xi}\times\operatorname{Sec}\xi\to\operatorname{Sec}\xi$ is a mapping satisfying the following conditions:

(I) $$\nabla_{X_1+X_2}Y = \nabla_{X_1}Y+\nabla_{X_2}Y,$$

(II) $$\nabla_X(Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2,$$

(III) $$\nabla_{fX} Y = f\nabla_X Y,$$

(IV) $$\nabla_X fY = (A\circ X) fY + f\nabla_X Y$$

$(X, X_1, X_2 \in \mathrm{Sec}\,\tilde{\xi};\; Y, Y_1, Y_2 \in \mathrm{Sec}\,\xi;\; f \in C^\infty(B))$.

This concept is a generalization of I. Candela's pseudoconnection. (Cf. [2].)

2. Finsler-connections in τ_B

In Szabó's theory [8] the module of so-called Finsler-vector fields plays a central role. For our purposes it will be convenient to replace this module by $\mathrm{Sec}\,\pi^*(\tau_B)$. After this modification Szabó's definition of a Finsler-connection gets the following form:

Definition. A linear connection $\nabla\colon \mathfrak{X}(TB)\times\mathfrak{X}(TB)\to\mathfrak{X}(TB)$ is called a Finsler-connection if $\forall X, Y \in \mathfrak{X}(TB)$

(F) $$\nabla_X Y = \mathscr{H}\circ\nabla^h_{\mu\circ X}\mu\circ Y + \lambda\circ\nabla^h_{\mu\circ X}\mathscr{V}\circ Y + \mathscr{H}\circ\nabla^v_{\mathscr{V}\circ X}\mu\circ Y + \lambda\circ\nabla^v_{\mathscr{V}\circ X}\mathscr{V}\circ Y$$

where

$$(\nabla^h, \mathscr{H}),\ \nabla^h\colon \mathrm{Sec}\,\pi^*(\tau_B)\times\mathrm{Sec}\,\pi^*(\tau_B)\to\mathrm{Sec}\,\pi^*(\tau_B) \quad\text{and}$$

$$(\nabla^v, \lambda),\ \nabla^v\colon \mathrm{Sec}\,\pi^*(\tau_B)\times\mathrm{Sec}\,\pi^*(\tau_B)\to\mathrm{Sec}\,\pi^*(\tau_B)$$

are pseudoconnections in the vector bundle $\pi^*(\tau_B)$. In this case we use the notation $\nabla = \mathrm{Lift}\,(\nabla^h, \nabla^v)$.

We note that the representation (F) of a Finsler-connection is unique; in particular we have:

(1) $$\nabla^h_X Y = \mu\circ\nabla_{\mathscr{H}\circ X}\mathscr{H}\circ Y = \mathscr{V}\circ\nabla_{\mathscr{H}\circ X}\lambda\circ Y,$$

(2) $$\nabla^v_X Y = \mu\circ\nabla_{\lambda\circ X}\mathscr{H}\circ Y = \mathscr{V}\circ\nabla_{\lambda\circ X}\lambda\circ Y$$

$(X, Y \in \mathrm{Sec}\,\pi^*(\tau_B))$.

Now one can make the following simple but useful observation:

Proposition 1. *A linear connection in τ_{TB} is a Finsler-connection if and only if the conditions*

(3) $$\forall X, Y \in \mathfrak{X}(TB)\colon\ v\nabla_X hY = 0,$$
$$h\nabla_X vY = 0;$$

(4) $$\forall X, Y \in \mathfrak{X}(TB)\colon\ \nabla_{hX} J\circ Y = J\circ\nabla_{hX} hY,$$
$$\nabla_{vX} J\circ Y = J\circ\nabla_{vX} hY$$

are fulfilled.

Proof. If ∇ is a linear connection in τ_{TB} satisfying (3) and (4) then $(\nabla^h, \mathscr{H})$ and (∇^v, λ) can be defined by (1) and (2) owing to (3) and then (4) implies that $\nabla = \mathrm{Lift}\,(\nabla^h, \nabla^v)$. The converse is a simple calculation. □

Combining this Proposition 1 with the following result of M. ANASTASIEI, we get a relation between Szabó's concept and Matsumoto's theory.

Proposition 2. ([1], [7].) *Let ∇ be a linear connection in τ_{TB}. A triad $(\mathscr{H}, J, \nabla)$ determines a Finsler-connection in Matsumoto's sense if and only if conditions* (3) *and* (4) *are valid.* □

3. Finsler-connection in a vector bundle

In the case of general vector bundles condition (4) has no meaning (because we have no canonical almost tangent structure) but retaining property (3) one gets the following generalization ([6]):

Definition. A linear connection

$$\nabla\colon \mathfrak{X}(E)\times\mathfrak{X}(E) \to \mathfrak{X}(E)$$

is called a Finsler-connection in Miron's sense if

$$\forall X, Y\in\mathfrak{X}(E)\colon\ v\nabla_x hY = 0, \qquad h\nabla_X vY = 0. \tag{5}$$

Thus from Prop. 1 we immediately get the following

Corollary. *If a linear connection in τ_{TB} is a Finsler-connection, then it is a Finsler-connection in Miron's sense.* □

Now we consider the inverse problem.

Theorem. (i) *Suppose that $\nabla\colon \mathfrak{X}(TB)\times\mathfrak{X}(TB)\to\mathfrak{X}(TB)$ is a Finsler-connection in Miron's sense. Then there are unique Finsler-connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, such that*

$$\forall X, Y\in\mathfrak{X}(TB)$$

$$\nabla_X Y = v\overset{1}{\nabla}_X Y + h\overset{2}{\nabla}_X Y. \tag{M}$$

(ii) *If $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are Finsler-connections, then (M) defines a Finsler-connection in Miron's sense.*

PROOF. We concern ourselves only with the assertion (i) because (ii) can be verified by a similar reasoning and calculation. Consider the pseudoconnections

$$(\overset{1}{\nabla}{}^h, \mathscr{H}),\ \overset{1}{\nabla}{}^h_X Y := \mathscr{V}\circ\nabla_{\mathscr{H}\circ X}\lambda\circ Y,$$

$$(\overset{1}{\nabla}{}^v, \lambda),\ \overset{1}{\nabla}{}^v_X Y := \mathscr{V}\circ\nabla_{\lambda\circ X}\lambda\circ Y,$$

$$(\overset{2}{\nabla}{}^h, \mathscr{H}),\ \overset{2}{\nabla}{}^h_X Y := \mu\circ\nabla_{\mathscr{H}\circ X}\mathscr{H}\circ Y,$$

$$(\overset{2}{\nabla}{}^v, \lambda),\ \overset{2}{\nabla}{}^v_X Y := \mu\circ\nabla_{\lambda\circ X}\mathscr{H}\circ Y \quad (X, Y\in\operatorname{Sec}\pi^*(\tau_B)).$$

Let $\overset{1}{\nabla}:=\mathrm{Lift}(\overset{1}{\nabla}{}^h, \overset{1}{\nabla}{}^v)$, $\overset{2}{\nabla}:=\mathrm{Lift}(\overset{2}{\nabla}{}^h, \overset{2}{\nabla}{}^v)$ then by virtue of (F) we get the relations

$$\text{(6)}\qquad \overset{1}{\nabla}_X vY = \lambda\circ\mathscr{V}\circ\nabla_{\mathscr{H}\circ\mu\circ X}\lambda\circ\mathscr{V}\circ Y+\lambda\circ\mathscr{V}\circ\nabla_{\lambda\circ\mathscr{V}\circ X}\lambda\circ\mathscr{V}\circ Y=$$

$$= v\nabla_{hX}vY+v\nabla_{vX}vY = v\nabla_X vY = \nabla_X vY \quad (X, Y\in\mathfrak{X}(TB))$$

and in the same way

$$\text{(7)}\qquad \overset{2}{\nabla}_X hY = \nabla_X hY \quad (X, Y\in\mathfrak{X}(TB)).$$

By Miron's condition (5) implies that

$$v\overset{1}{\nabla}_X Y+h\overset{2}{\nabla}_X Y \overset{(3)}{=} v\overset{1}{\nabla}_X vY+h\overset{2}{\nabla}_X hY \overset{(6),(7)}{=\!=} v\nabla_X vY+h\nabla_X hY =$$

$$\overset{(5)}{=} \nabla_X vY+\nabla_X hY = \nabla_X Y,$$

proving (M). The unicity of the pair $(\overset{1}{\nabla}, \overset{2}{\nabla})$ is clear since (M) implies the relations (6), (7) which uniquely determine $(\overset{1}{\nabla}, \overset{2}{\nabla})$. (See (1) and (2).)

Finally we return from the tangent bundle τ_B to the general vector bundle ξ.

One can put the question whether or not a linear connection ∇ in τ_E is a Finsler-connection in Miron's sense. We note that the answer depends again on a representation (F*) of ∇ very similar to (F). Namely we have the following

Theorem. ([9]) *Let ∇ be a linear connection in τ_E. The following conditions are equivalent:* (i) *∇ is a Finsler-connection in Miron's sense.* (ii) *There are pseudo-connections*

$$(\nabla^h, \mathscr{H}),\ \nabla^h\colon \mathrm{Sec}\,\pi^*(\tau_B)\times\mathrm{Sec}\,\pi^*(\tau_B) \to \mathrm{Sec}\,\pi^*(\tau_B),$$

$$(\tilde{\nabla}^h, \mathscr{H}),\ \tilde{\nabla}^h\colon \mathrm{Sec}\,\pi^*(\tau_B)\times\mathrm{Sec}\,V\xi \to \mathrm{Sec}\,V\xi,$$

$$(\nabla^v, i),\ \nabla^v\colon \mathrm{Sec}\,V\xi\times\mathrm{Sec}\,V\xi \to \mathrm{Sec}\,V\xi,$$

$$(\tilde{\nabla}^v, i),\ \tilde{\nabla}^v\colon \mathrm{Sec}\,V\xi\times\mathrm{Sec}\,\pi^*(\tau_B) \to \mathrm{Sec}\,\pi^*(\tau_B)$$

such that

$$\text{(F*)}\qquad \nabla_X Y = \mathscr{H}\circ\nabla^h_{j\circ X}j\circ Y+\nabla^v_{vX}vY+\mathscr{H}\circ\tilde{\nabla}^v_{vX}j\circ Y+\tilde{\nabla}^h_{j\circ X}vY \quad (X, Y\in\mathfrak{X}(TE).$$

In this case the representation (F*) is unique. □

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