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A polar-coordinate model of the hyperbolic space

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Abstract. This article presents a d-dimensional polar-coordinate model, a new model of the d-dimensional hyperbolic space. Since we can establish a bijection between this model and the Weierstrass model with a proper projection, we can give a simple discussion of the geometric configurations, which is the even simpler in case $d = 2$ than in [Wiegand, 1992].

1. Introduction

The different models of the d-dimensional ($d > 2$) hyperbolic space (the Cayley-Klein model, the Klein-Poincaré model, Poincaré's hemisphere model, the Weierstrass model, etc.) can be found in several books ([Bolyai, 1987], [Faber, 1983], [Liebmann, 1992], [Rozenfeld, 1969], [Szász, 1973]), but a complete survey is difficult today. We can find a polar-coordinate model of the hyperbolic plane in the article of [Wiegand, 1992]. As a generalisation we give a polar-coordinate ($\mathcal{P}^d$) model of the d-dimensional hyperbolic space. The examination of the geometric configurations becomes more simple even in the case $d = 2$ when we establish a bijection between the Weierstrass model ($\mathcal{W}^d$) and our polar-coordinate model by a projection [Németh]. We deal with the reflection in a hyperplane, as the generating element of the congruence transformations.

2. The Weierstrass and the polar-coordinate models of the hyperbolic space

Let us consider the sheet of a hyperboloid H^d given by the equation $x_1^2+x_2^2+\ldots+x_d^2-x_{d+1}^2 = -1$, $x_{d+1} > 0$ in the $(d+1)$-dimensional Euclidean

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space. The Weierstrass model ($\mathcal{W}^d$) of the d-dimensional hyperbolic space can be built on this surface. Let the points of the hyperbolic space be the points of the surface H^d, and each k-dimensional ($k < d$) plane corresponds to a $(k+1)$-dimensional Euclidean space section of the surface, which contains the point $\boldsymbol{O}(0,0,\dots,0)$. Thus the lines are intersections of H^d with 2-dimensional planes containing the point $\boldsymbol{O}$, namely branches of hyperbolas. Incidence and order are Euclidean. The definitions of distance and angle are the same as in the case $d = 2$ ([Faber, 1983], [Rozenfeld, 1969]). Let A^d be the half asymptotic cone of the surface H^d. Two planes with dimensions s and r are called parallel if their intersecting $(s+1)$ and $(r+1)$-dimensional Euclidean planes have one and only one common half generator of A^d.

We can define a polar-coordinate model ($\mathcal{P}^d$) of the d-dimensional hyperbolic space on the base of a polar-coordinate model of the hyperbolic plane ([Wiegand, 1992]) the following way. We denote by $\mathbb{H}^d$ the hyperbolic space and by $\mathbb{E}^d$ the Euclidean space. Let us consider a rectangular coordinate system $(\boldsymbol{O}^h, x_1^h, x_2^h, \dots, x_d^h)$ in $\mathbb{H}^d$ and assign the rectangular coordinate system $(\boldsymbol{O}^e, x_1^e, x_2^e, \dots, x_d^e)$ in $\mathbb{E}^d$ to the previous coordinate system. Let us assign the point $\boldsymbol{P}^e \in \mathbb{E}^d$ with polar coordinates $(\sinh p, \varphi_1, \dots, \varphi_{d-1})$ to the point $\boldsymbol{P}^h \in \mathbb{H}^d$ ($\boldsymbol{P}^h \neq \boldsymbol{O}^h$) with polar coordinates $(p, \varphi_1, \dots, \varphi_{d-1})$. The point $\boldsymbol{O}^e$ corresponds to the point $\boldsymbol{O}^h$. This assignment is a bijection between the spaces $\mathbb{H}^d$ and $\mathbb{E}^d$ because of the invertibility of the function hyperbolic sine. Hence we can model the d-dimensional hyperbolic space in the space $\mathbb{E}^d$.

3. The discussion of the d-dimensional polar-coordinate model

By the help of the Weierstrass model and of an orthogonal projection we are able to describe our d-dimensional polar-coordinate model in a simple way (in the case $d = 2$ see [Németh]).

3.1. Points, lines, planes and hyperplanes

Lemma 1. *Let the hyperbolic distance between the points $\boldsymbol{K}(0,\dots,0,1) \in \mathbb{E}^{d+1}$ and $\boldsymbol{P} \in \mathcal{W}^d$ be ρ and the orthogonal projections of the points $\boldsymbol{K}$ and $\boldsymbol{P}$ onto the $x_1x_2\dots x_d$-hyperplane be $\boldsymbol{O}(0,0,\dots,0)$ and $\boldsymbol{P}'$, respectively. Then the Euclidean distance between $\boldsymbol{O}$ and $\boldsymbol{P}'$ is equal to* $\sinh\rho$.

Proof. As the surface $\mathbb{H}^d$ has rotational symmetry (in the Euclidean sense) about the x_{d+1}-axis, we can rotate it so, that the points $\boldsymbol{P}$ and $\boldsymbol{P}'$

be in the coordinate plane $x_d x_{d+1}$ (Fig. 1). Then the orthogonal projection of the point $\boldsymbol{P}\left(0,\dots,0,p_d,\sqrt{p_d^2+1}\right)$ is the point $\boldsymbol{P}'(0,\dots,0,p_d,0)$.

Figure 1

The Euclidean distance between $\boldsymbol{O}$ and $\boldsymbol{P}'$ is $|p_d|$, the hyperbolic distance ρ between the points $\boldsymbol{K}$ and $\boldsymbol{P}$ is given by the expression $\cosh\rho = -\langle \boldsymbol{K},\boldsymbol{P}\rangle = \sqrt{p_d^2+1}$ [Faber, 1983], from which $\rho = \operatorname{arcosh}\sqrt{p_d^2+1}$.
Using $\cosh^2 x - \sinh^2 x = 1$, we obtain that

$$\sinh\rho = \sinh\left(\operatorname{arcosh}\sqrt{p_d^2+1}\right) = \sqrt{\cosh^2\left(\operatorname{arcosh}\sqrt{p_d^2+1}\right)-1} = |p_d|.$$

$\square$

Theorem 1. *The orthogonal projection of the model $\mathcal{W}^d$ onto the coordinate plane $x_1x_2\dots x_d$ is the model $\mathcal{P}^d$ of the d-dimensional hyperbolic space.*

Proof. Take the rectangular coordinate system in the model $\mathcal{W}^d$, whose orthogonal projection is the rectangular coordinate system $(\boldsymbol{O},x_1,x_2,\dots,x_d)$ in the $x_1x_2\dots x_d$-hyperplane. The centres ($\boldsymbol{K}$ and $\boldsymbol{O}$) and the axes of the coordinate systems correspond to each other. The angle of the lines passing through the point $\boldsymbol{K}$ in the model $\mathcal{W}^d$ is equal to the Euclidean angle of the tangent vectors at the point, and is equal to the Euclidean angle of the projections of the tangent vectors, since the tangent vectors are parallel to the $x_1x_2\dots x_d$-hyperplane. Then the hyperbolic angle of the lines passing through the point $\boldsymbol{K}$ is equal to the

Euclidean angle of their projections (these are lines passing through the point $\boldsymbol{O}$ in the $x_1x_2\ldots x_d$-hyperplane). Using Lemma 1 a point $\boldsymbol{P}'$ with Euclidean polar coordinates $(\sinh p, \varphi_1, \ldots, \varphi_{d-1})$ is assigned to a point $\boldsymbol{P} \in \mathcal{W}^d$ with polar coordinates $(p, \varphi_1, \ldots, \varphi_{d-1})$. This assignment is a bijection. $\square$

On the base of Theorem 1, we define our polar-coordinate model as the orthogonal projection of the Weierstrass model. This definition and the previous definition (see 1.) are equivalent. So the points of $\mathcal{P}^d$ are the points of the $x_1x_2\ldots x_d$-hyperplane.

Definition. We call the imaginary throat-sphere of a two-sheeted hyperboloid of revolution D^d the sphere, which is the intersection of D^d and the hyperplane that is orthogonal to the axis of revolution of D^d and contains the center of D^d.

Theorem 2. *The k-dimensional planes in our polar-coordinate model are the Euclidean k-dimensional planes passing through the point $\boldsymbol{O}$, and the sheets of those two-sheeted hyperboloids of revolution whose throat-sphere's radius is the imaginary unit and the vertices of their asymptotic cones fall into the point $\boldsymbol{O}$.*

Proof. The k-dimensional planes of the model $\mathcal{W}^d$ are the plane sections of the surface H^d with $(k+1)$-dimensional planes containing the point $\boldsymbol{O}$. Then the vertices of the asymptotic cones of the plane sections are the point $\boldsymbol{O}$. If the $(k+1)$-dimensional (Euclidean) subspace belonging to the k-dimensional (hyperbolic) plane $L \in \mathcal{W}^d$ is perpendicular to the $x_1x_2\ldots x_d$-hyperplane, then the orthogonal projection L' is a k-dimensional plane containing the point $\boldsymbol{O}$ in the $x_1x_2\ldots x_d$-hyperplane. If the subspace belonging to L is not perpendicular to the $x_1x_2\ldots x_d$-hyperplane, then the orthogonal projection L' is a k-dimensional sheet of a hyperboloid. Since the orthogonal projection preserves incidence, the orthogonal projection of the half asymptotic cone of L is the half asymptotic cone of L', and its vertex is the point $\boldsymbol{O}$. The common points of H^d and the $x_1x_2\ldots x_d$-plane satisfy the equation $x_1^2 + x_2^2 + \ldots + x_d^2 = -1$, it is the throat-sphere of H^d. The subspace belonging to L intersects this throat-sphere in an imaginary-sphere (throat-sphere of the surface L). This sphere lies in the $x_1x_2\ldots x_d$-hyperplane, therefore its orthogonal projection is itself and because of the preservation of incidence it is the throat-sphere of

L' as well. This throat-sphere is really a sphere, then L' is a hyperboloid of revolution. The imaginary radius of the throat-sphere of L' is equal to 1. If L' is a branch of a hyperbola in the Euclidean sense, then its conjugate axis corresponds to its throat-sphere. $\square$

If the hyperbolic distance between a hyperplane perpendicular to the x_1-axis and the point $\boldsymbol{O}$ is equal to t ($t \neq 0$) and the hyperplane is in the $x_1 > 0$ halfspace, then the equation of the hyperplane in the model (from Lemma 1 and Theorem 2) is:

$$-\frac{x_1^2}{\sinh^2 t} + x_2^2 + \ldots + x_d^2 = -1, \quad x_1 > 0.$$

Corollary. *In the Euclidean sense the lines in our polar-coordinate model are the lines passing through the point $\boldsymbol{O}$ and the branches of hyperbolas with unit semiconjugate axes whose asymptotes pass through the point $\boldsymbol{O}$.*

3.2. Incidence, order and parallelism

Since incidence and order are Euclidean in the model $\mathcal{W}^d$ and the orthogonal projection preserves incidence, furthermore, it preserves order for the corresponding branches of hyperbolas, the incidence and order are Euclidean in the model $\mathcal{P}^d$ too.

In the model $\mathcal{W}^d$ two elements of the space are parallel if and only if there exists a common half-generator of their asymptotic cones. Then parallelism is the following in the model $\mathcal{P}^d$:

Definition. Two planes (they may have different dimensions) are called parallel in the model $\mathcal{P}^d$ if their half-asymptotic cones have one and only one common half-generator. In this case we consider the half-asymptotes of a line (as a branch of a hyperbola) to be the half-asymptotic cone of the line, and the plane (or line) containing the point $\boldsymbol{O}$ itself is considered to be the asymptotic cone of the plane (or line).

3.3. Reflection in a hyperplane

Definition. A reflection in a hyperplane L in the model $\mathcal{W}^d$ is the affinity whose fixed plane is the d-dimensional Euclidean hyperplane that intersects the surface H^d in the hyperbolic hyperplane L. The direction of the affinity is the conjugate direction of this fixed plane with respect to the surface H^d and its ratio is equal to -1.

We can prove that this affinity assigns the surface H^d to itself and the above definition satisfies the properties of the reflection in a hyperplane.

Definition. The reflection in a hyperplane in the model $\mathcal{P}^d$ is the orthogonal projection of the reflection in a hyperplane in the model $\mathcal{W}^d$.

3.4. Sphere, parasurface and equidistant surface

The $(d-1)$-dimensional spheres of radius r in the model $\mathcal{W}^d$ are the hyperplane sections of the surface H^d with hyperplanes not containing the point $\boldsymbol{O}$ and intersecting each element of A^d (in the case $d=2$ see [Faber, 1983]). The centre of a sphere is the common point of H^d and of the tangent plane of H^d that is parallel to the sphere's intersecting hyperplane. These surfaces in the Euclidean sense are $(d-1)$-dimensional spheres or ellipsoids. The orthogonal projections of these surfaces in the $x_1x_2\dots x_d$-hyperplane are the $(d-1)$-dimensional spheres of $\mathcal{P}^d$, as the assignment between the two models is a bijection, preserves incidence and the centres of the corresponding spheres correspond to each other. The spheres of $\mathcal{P}^d$ in the Euclidean sense are the spheres with the centre $\boldsymbol{O}$, or ellipsoids of revolution whose line of major axis passes through the point $\boldsymbol{O}$. These surfaces are surfaces of revolution because the two-dimensional planes in the model $\mathcal{P}^d$ containing the point $\boldsymbol{O}$ (that are two-dimensional Euclidean planes as well) and the centre of the sphere intersects this ellipsoid in congruent ellipses. Their line of major axis passes through the point $\boldsymbol{O}$ because the circles are determined by the same center and radius.

If the coordinates of the centre of a d-dimensional sphere of radius r are $\boldsymbol{C}(\rho, 0, \dots, 0)$ in the hyperbolic space, then its equation is (based on the case $d=2$, when we got the equations of the circles in $\mathcal{P}^2$ from the equations of the circles in $\mathcal{W}^2$ [Németh]):

$$\frac{(x_1 - \sinh\rho\cdot\cosh r)^2}{\cosh^2\rho\cdot\sinh^2 r} + \frac{x_2^2}{\sinh^2 r} + \dots + \frac{x_d^2}{\sinh^2 r} = 1,$$

if $\rho = 0$, then:

$$x_1^2 + x_2^2 + \dots + x_d^2 = \sinh^2 r.$$

The $(d-1)$-dimensional parasurfaces in the model $\mathcal{W}^d$ are the hyperplane sections of H^d which are $(d-1)$-dimensional paraboloids (in the case $d=2$ see [Faber, 1983]). Their orthogonal projections in the $x_1x_2\dots x_d$-hyperplane are also paraboloids, that are the $(d-1)$-dimensional parasurfaces in the model $\mathcal{P}^d$ because of the properties of the assignment between the two models. Since the axes of the surfaces contain the point $\boldsymbol{O}$ and each hyperplane section of these surfaces containing their axes is congruent, therefore these surfaces are paraboloids of revolution (that are the parasurfaces), whose axes contain the point $\boldsymbol{O}$. If the axis is the x_1-axis, its equation has the form (based on the case $d = 2$):

$$\frac{1}{2(\cosh t - \sinh t)} \cdot (x_2^2 + \ldots + x_d^2) + \sinh t = x_1.$$

The equidistant surfaces of a hyperplane L in the model $\mathcal{W}^d$ are the hyperplane sections of H^d which are sheets of hyperboloids, and its hyperplanes are parallel to the hyperplane that intersects H^d in the given plane L (in case $d = 2$ see [Faber, 1983]). Their orthogonal projections in the $x_1 x_2 \ldots x_d$-hyperplane are $(d-1)$-dimensional Euclidean planes (if L' contains the point $\boldsymbol{O}$), or sheets of hyperboloids of revolution, which will be the equidistant surfaces in the model $\mathcal{P}^d$ too, because of the properties of the assignment between the two models. If the plane L' contains the point $\boldsymbol{O}$, its equidistant surfaces are planes parallel to L' in the Euclidean sense. If L' does not contain the point $\boldsymbol{O}$, then the axes of revolution of the equidistant surfaces coincide with the axis of revolution of the hyperboloid L' and their asymptotic cones will be parallel to each other.

If the axis of the equidistant surface is the x_1-axis, the parameter of the plane is t, and the distance is ρ, then its equation has the form (based on the case $d = 2$):

$$-\frac{(x_1 \pm \sinh t \cdot \sinh \rho)^2}{\sinh^2 t \cdot \cosh^2 \rho} + \frac{x_2^2}{\cosh^2 \rho} + \ldots + \frac{x_d^2}{\cosh^2 \rho} = -1.$$

The intersections of $(d-1)$-dimensional cycles and $(k+1)$-dimensional (hyperbolic) planes in the model are k-dimensional cycles.

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