On boundary value problems for nonlinear elliptic equations on unbounded domains

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Introduction

In [1] it has been proved the existence of variational solutions of boundary value problems for the equation

$$\sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u, \dots, D^\beta u, \dots) + g(x, u) = F, \quad x \in \Omega$$

where Ω is a possibly unbounded domain in R^n, $|\beta| \leqq m$. The terms $f_\alpha(x, \xi)$ are required to have polynomial growth in ξ, in the term $g(x, u)$, however, no growth restriction is imposed but it is supposed that g (essentially) satisfies the sign condition $g(x, u)u \geqq 0$ and for all $t > 0$, $x \mapsto \sup_{|u| \leqq t} |g(x, u)| \in L^1(\Omega)$.

In the present paper it will be proved the existence of variational solutions of boundary value problems for the elliptic equation

$$(0.1) \qquad \sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u \dots, D^\beta u, \dots) + \sum_{|\alpha|\leqq l} (-1)^{|\alpha|} D^\alpha g_\alpha(x, u, \dots, D^\beta u, \dots) = F$$

where $|\beta| \leqq m$, l is an integer with the property $l < m - \frac{n}{p}(1 - p + \varrho)$, p and ϱ are real numbers such that $1 < p < \infty$, $p - 1 < \varrho \leqq p$. Functions f_α satisfy the same conditions as in [1] and in [2], g_α are supposed to satisfy (essentially)

$$(0.2) \qquad g_\alpha(x, \xi)\, \xi_\alpha \geqq 0,$$

$$(0.3) \qquad |g_\alpha(x, \xi)| \leqq C(\xi')\big(K(x) + |\xi''|^\varrho\big)$$

where $\xi = (\xi', \xi'')$ and ξ' contains those coordinates ξ_β of ξ for which $|\beta| < m - \frac{n}{p}$; $K \in L^{p/\varrho}(\Omega)$.

In [3] and in [4] it is shown that there exist variational solutions of problems for (0.1) but g_α are supposed to satisfy other conditions instead of (0.3). In [5] the Dirichlet problem in bounded Ω for second order equations is considered with $l = 0$ if $g_\alpha = g$ satisfies a condition of type (0.3).

§ 1. Preliminaries

Let $\Omega \subset R^n$ be a (possibly) unbounded domain, $p>1$, m a positive integer. Assume that Ω has the weak cone property (see [6]) and for all sufficiently large μ, there exists a bounded $\Omega_\mu \subset \Omega$ with the weak cone property such that $\Omega_\mu \supset \{x\in\Omega: |x|<\mu\}$. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u whose distributional derivatives of order $\leqq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is

$$\|u\|_{W_p^m(\Omega)} = \Big\{\sum_{|\alpha|\leqq m} \int_\Omega |D^\alpha u|^p\,dx\Big\}^{1/p},$$

where $\alpha=(\alpha_1, \dots, \alpha_n)$, $|\alpha|=\sum_{j=1}^n \alpha_j$, $D^\alpha = D_n^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j=\dfrac{\partial}{\partial x_j}$. By $W_{p,0}^m(\Omega)$ will be denoted the closure in $\|\cdot\|_{W_p^m(\Omega)}$ of $C_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support contained in Ω.

Let N be the number of multiindices α satisfying $|\alpha|\leqq m$. The vectors $\xi=(\xi_0, \dots, \xi_\beta, \dots)\in R^N$ will be written in the form $\xi=(\eta, \zeta)$ where η consists of those ξ_β for which $|\beta|\leqq m-1$. Assume that

I. Functions f_α: $\Omega\times R^N \to R$ satisfy the Carathéodory conditions, i.e. they are measurable with respect to x for each fixed $\xi\in R^N$ and continuous with respect to ξ for almost all $x\in\Omega$.

II. There exist a constant $c_1>0$ and a function $K_1\in L^q(\Omega)$ (where $1/p+1/q=1$) such that

$$|f_\alpha(x, \xi)| \leqq c_1|\xi|^{p-1}+K_1(x)$$

for all $|\alpha|\leqq m$, a.e. $x\in\Omega$ and all $\xi\in R^N$.

III. For all $(\eta, \zeta), (\eta, \zeta')\in R^N$ with $\zeta\neq\zeta'$ and a.e. $x\in\Omega$

$$\sum_{|\alpha|=m} [f_\alpha(x, \eta, \zeta)-f_\alpha(x, \eta, \zeta')](\xi_\alpha-\xi'_\alpha) > 0.$$

IV. There exist a constant c_2 and a function $K_2\in L^1(\Omega)$ such that for a.e. $x\in\Omega$ and all $\xi\in R^N$

$$\sum_{|\alpha|\leqq m} f_\alpha(x, \xi)\,\xi_\alpha \geqq c_2|\xi|^p - K_2(x).$$

V. Functions p_α, r_α: $\Omega\times R^N\to R$ satisfy the Carathéodory conditions and

$$g_\alpha = p_\alpha + r_\alpha.$$

VI. $p_\alpha(x, \xi)\xi_\alpha \geqq 0$ and $|r_\alpha(x, \xi)|\leqq h_\alpha(x)$ for all $\xi\in R^N$ and a.e. $x\in\Omega$ where $h_\alpha\in L^{p/\varrho}(\Omega)$.

VII. There exist a continuous function C and a function $K\in L^{p/\varrho}(\Omega)$ such that

$$|p_\alpha(x, \xi)| \leqq C(\xi')\big(K(x)+|\xi''|^\varrho\big)$$

for all $|\alpha|\leqq l$, $\xi=(\xi', \xi'')\in R^N$ and a.e. $x\in\Omega$.

VIII. V is a closed subspace of $W_p^m(\Omega)$ with the property: $v\in V$, $\varphi\in C_0^\infty(R^n)$ imply that $\varphi v\in V$. (V may be e.g. $W_p^m(\Omega)$ or $W_{p,0}^m(\Omega)$.)

Set

$$p_{\alpha,\mu}(x,\xi)=\begin{cases} p_\alpha(x,\xi) & \text{if } |x|\leqq\mu,\ |p_\alpha(x,\xi)|\leqq\mu,\\ \mu\dfrac{p_\alpha(x,\xi)}{|p_\alpha(x,\xi)|} & \text{if } |x|\leqq\mu,\ |p_\alpha(x,\xi)|>\mu,\\ 0 & \text{if } |x|>\mu,\end{cases} \tag{1.1}$$

$$r_{\alpha,\mu}(x,\xi)=\begin{cases} r_\alpha(x,\xi) & \text{if } |x|\leqq\mu,\ |r_\alpha(x,\xi)|\leqq\mu,\\ \mu\dfrac{r_\alpha(x,\xi)}{|r_\alpha(x,\xi)|} & \text{if } |x|\leqq\mu,\ |r_\alpha(x,\xi)|>\mu,\\ 0 & \text{if } |x|>\mu,\end{cases} \tag{1.2}$$

$$g_{\alpha,\mu}(x,\xi)=p_{\alpha,\mu}(x,\xi)+r_{\alpha,\mu}(x,\xi). \tag{1.3}$$

Assumptions I., II., V., VI. and (1.1)—(1.3) imply that formulas

$$\langle T(u),v\rangle=\sum_{|\alpha|\leqq m}\int_\Omega f_\alpha(x,u,\ldots,D^\beta u,\ldots)D^\alpha v\,dx,$$

$$\langle S_\mu(u),v\rangle=\sum_{|\alpha|\leqq l}\int_\Omega g_{\alpha,\mu}(x,u,\ldots,D^\beta u,\ldots)\,D^\alpha v\,dx$$

define linear continuous functionals $T(u)$ resp. $S_\mu(u)$ on V for any fixed (sufficiently large) μ.

By assumptions I.—V. and (1.1)—(1.3) operator $T+S_\mu$ satisfies the conditions of [2] for (sufficiently large) fixed μ and thus we have

Lemma 1. *For any $F\in V^*$ there exists $u_\mu\in V$ such that*

$$(T+S_\mu)(u_\mu)=F.$$

Lemma 2. *Assume that $(u_j)\to u$ weakly in V and for any bounded domain $\omega\subset\Omega$*

$$\lim_{j\to\infty}\int_\omega h_j\,dx=0 \tag{1.4}$$

where

$$h_j(x)=\sum_{|\alpha|=m}[f_\alpha(x,u_j,\ldots,D^\gamma u_j,\ldots,D^\beta u_j,\ldots)-$$

$$-f_\alpha(x,u_j,\ldots,D^\gamma u_j,\ldots,D^\beta u,\ldots)](D^\alpha u_j-D^\alpha u) \tag{1.5}$$

$|\gamma|<m$, $|\beta|=m$. Then there is a subsequence (u_j') of (u_j) such that $D^\beta u_j'\to D^\beta u$ a.e. in Ω for all β with $|\beta|\leqq m$.

Proof. Since $(u_j)\to u$ weakly in V thus there is a subsequence $(\tilde u_j)$ of (u_j) such that

$$D^\gamma\tilde u_j\to D^\gamma u\quad\text{a.e. in}\quad\Omega\quad\text{for}\quad|\gamma|<m$$

(see e.g. [7]). Further, by assumption III. $h_j\geqq 0$ and so (1.4) and Fatou's lemma imply that

$$h_j\to 0\quad\text{a.e. in } \omega.$$

Thus there exists $\omega_0 \subset \omega$ of measure 0 such that for $x \in \omega \setminus \omega_0$

$$|D^\beta u(x)| < \infty,\ |K_1(x)| < \infty,\ |K_2(x)| < \infty \tag{1.6}$$

and

$$D^\gamma \tilde{u}_j(x) \to D^\gamma u(x) (|\gamma| < m),\ \tilde{h}_j(x) \to 0. \tag{1.7}$$

Set

$$\xi^{(j)}(x) = (\ldots, D^\beta \tilde{u}_j(x), \ldots)$$

where $|\beta| = m$. By assumptions I., II., IV. and by (1.6), (1.7)

$$\begin{aligned}\tilde{h}_j(x) \geqq & \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \ldots, D^\gamma \tilde{u}_j, \ldots, D^\beta \tilde{u}_j, \ldots) D^\alpha \tilde{u}_j - \\ & - \sum_{|\alpha|=m} |f_\alpha(x, \tilde{u}_j, \ldots, D^\gamma \tilde{u}_j, \ldots, D^\beta \tilde{u}_j, \ldots) D^\alpha u| - \\ & - \sum_{|\alpha|=m} |f_\alpha(x, \tilde{u}_j, \ldots, D^\gamma \tilde{u}_j, \ldots, D^\beta u, \ldots) (D^\alpha \tilde{u}_j - D^\alpha u) \geqq \\ & \geqq c_2 |\xi^{(j)}(x)|^p - c_3(x) [1 + |\xi^{(j)}(x)|^{p-1} + |\xi^{(j)}(x)|] \quad \text{if} \quad x \in \omega \setminus \omega_0.\end{aligned}$$

$(D^\gamma \tilde{u}_j(x)$ is bounded for a fixed $x \in \omega \setminus \omega_0$.$)$ Since by (1.7) $\tilde{h}_j(x)$ is bounded for a fixed $x \in \omega \setminus \omega_0$ thus $\xi^{(j)}(x)$ is bounded, too. Consequently, $(\xi^{(j)}(x))$ contains a subsequence which converges to a vector $\xi^*(x)$.

Now we show that

$$\xi^*(x) = \xi(x) = (\ldots, D^\beta u(x), \ldots). \tag{1.8}$$

Indeed, applying (1.5) to the subsequence of $(\tilde{h}_j(x))$ and letting $j \to \infty$ (by (1.7)) we obtain

$$\begin{aligned}0 = & \sum_{|\alpha|=m} [f_\alpha(x, u(x), \ldots, D^\gamma u(x), \ldots, \xi^*(x)) - \\ & - f_\alpha(x, u(x), \ldots, D^\gamma u(x), \ldots, \xi(x)) [\xi^*_\alpha(x) - \xi_\alpha(x)]\end{aligned}$$

which implies (1.8) in virtue of assumption III.

So we have shown that all convergent subsequences of $(\xi^{(j)}(x))$ tend to $\xi(x)$. Therefore $\lim_{j \to \infty} \xi^{(j)}(x) = \xi(x)$ and thus by (1.7) $D^\beta \tilde{u}_j \to D^\beta u$ a.e. in ω for all β with $|\beta| \leqq m$. Hence (by a "diagonal process") easily follows Lemma 2 since ω is an arbitrary bounded subset of Ω.

§ 2. The existence theorem

Theorem. *Suppose that conditions* I.—VIII. *are fulfilled. Then for any* $F \in V^*$ *there exists* $u \in V$ *such that*

$$\begin{aligned}& \sum_{|\alpha| \leqq m} \int_\Omega f_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha v \, dx + \\ & + \sum_{|\alpha| \leqq l} \int_\Omega g_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha v \, dx = \langle F, v \rangle\end{aligned} \tag{2.1}$$

for all $v \in V$.

PROOF. By Lemma 1 for any $j=j_0, j_0+1, j_0+2, \ldots$ there is $u_j \in V$ such that

$$\langle (T+S_j)(u_j), v\rangle = \langle F, v\rangle \quad \text{for all} \quad v\in V. \tag{2.2}$$

From assumptions IV., VI. and (1.1)—(1.3) it follows that (u_j) is bounded in V. Thus there exist a subsequence (u_{j_k}) of (u_j) and $u\in V$ such that

$$(u_{j_k}) \to u \quad \text{weakly in} \quad V, \tag{2.3}$$

$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{a.e. in} \quad \Omega \quad \text{for} \quad |\gamma| \leqq m-1 \tag{2.4}$$

(see [7]).

Consider an arbitrary bounded domain $\omega\subset\Omega$ and take a function $\Theta\in C_0^\infty(R^n)$ such that $\Theta\geqq 0$ and $\Theta(x)=1$ for $x\in\omega$. By Sobolev's imbedding theorems (see e.g. [6]) it may be supposed that

$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{in} \quad L^p(\Omega\cap \operatorname{supp}\Theta) \quad \text{for} \quad |\gamma| \leqq m-1 \tag{2.5}$$

and

$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{in} \quad L^{q_1}(\Omega\cap \operatorname{supp}\Theta) \quad \text{for} \quad |\gamma| \leqq l < m-\frac{n}{p}(1-p+\varrho) \tag{2.6}$$

where q_1 is defined by $\dfrac{1}{p/\varrho}+\dfrac{1}{q_1}=1$.

By assumption VIII. $\Theta(u_{j_k}-u)\in V$ and so by (2.2)

$$\sum_{|\alpha|\leqq m} \int_\Omega f_\alpha(x, u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)\, D^\alpha[\Theta(u_{j_k}-u)]\, dx + \tag{2.7}$$

$$+\sum_{|\alpha|\leqq l} \int_\Omega g_{\alpha, j_k}(x, u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)\, D^\alpha[\Theta(u_{j_k}-u)]\, dx = \langle F, \Theta(u_{j_k}-u)\rangle.$$

Since $(u_{j_k}-u)\to 0$ weakly in V thus

$$\Theta(u_{j_k}-u) \to 0 \quad \text{weakly in} \quad V. \tag{2.8}$$

In virtue of (2.7) we have

$$\sum_{|\alpha|=m} \int_\Omega [f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)- \tag{2.9}$$

$$-f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)]\, \Theta D^\alpha(u_{j_k}-u)\, dx =$$

$$= \sum_{|\alpha|=m} \int_\Omega f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)\, \Theta D^\alpha(u-u_{j_k})\, dx +$$

$$+\sum_{|\alpha|=m} \int_\Omega f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots) \sum_{|\gamma|\leqq m-1} c_\gamma D^\gamma(u-u_{j_k})\, D^{\alpha-\gamma}\Theta\, dx +$$

$$+\sum_{|\alpha|\leqq m-1} \int_\Omega f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)\, D^\alpha[\Theta(u-u_{j_k})]\, dx +$$

$$+\sum_{|\alpha|\leqq l} \int_\Omega g_{\alpha, j_k}(x, u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)\, D^\alpha[\Theta(u-u_{j_k})]\, dx + \langle F, \Theta(u_{j_k}-u)\rangle$$

where $|\gamma|<m$, $|\beta|=m$.

Now we show that all the terms in the right of (2.9) converge to 0 as $k\to\infty$. By (2.3) $D^\alpha(u_{j_k}-u)\to 0$ weakly in $L^p(\Omega)$, further by (2.4), assumption I.

$$\Theta f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)\to\Theta f_\alpha(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots) \tag{2.10}$$

a.e. in Ω and so by assumption II., (2.5) and Vitali's theorem (2.10) is valid in $L^q(\Omega)$ norm, too. Thus the first term in the right of (2.9) converges to 0.

Since by assumptions I., II.

$$f_\alpha(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)$$

is bounded in $L^q(\Omega)$ thus (2.5) implies that the second and third terms in the right of (2.9) converge to 0 as $k\to\infty$.

From assumptions V.—VII. it follows that

$$g_{\alpha, j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)$$

is bounded in $L^{p/\varrho}(\Omega\cap\operatorname{supp}\Theta)$ thus (2.6) implies that the fourth term in the right of (2.9) converges to 0 as $k\to\infty$. Finally, from (2.8) it follows that the last term in the right of (2.9) converges to 0 as $k\to\infty$.

Thus we have shown that the term in the left of (2.9) converges to 0 as $k\to\infty$ and so by assumption III. and by $\Theta\geqq 0$ we find that (1.4) is valid for any bounded $\omega\subset\Omega$. Consequently, from Lemma 2 we obtain that (u_{j_k}) contains a subsequence $(u_{j'_k})$ such that

$$D^\beta u_{j'_k}\to D^\beta u \quad \text{a.e. in} \quad \Omega \quad \text{if} \quad |\beta|\leqq m. \tag{2.11}$$

Thus assumption I. implies that

$$f_\alpha(x, u_{j'_k}, \ldots, D^\beta u_{j'_k}, \ldots)\to f_\alpha(x, u, \ldots, D^\beta u, \ldots)$$

a.e. in Ω and so by assumption II., Hölder's inequality Vitali's theorem shows that for any $v\in V$

$$\lim_{k\to\infty}\langle T(u_{j'_k}), v\rangle = \langle T(u), v\rangle. \tag{2.12}$$

By using (1.1)—(1.3), assumption V. and (2.11) we obtain

$$g_{\alpha, j'_k}(x, u_{j'_k}, \ldots, D^\beta u_{j'_k}, \ldots)\to g_\alpha(x, u, \ldots, D^\beta u, \ldots)$$

a.e. in Ω and so by assumptions VI., VII., Hölder's inequality Vitali's theorem shows that for any $v\in V$

$$\lim_{k\to\infty}\langle s_{j'_k}(u_{j'_k}), v\rangle = \sum_{|\alpha|\leqq l}\int_\Omega g_\alpha(x, u, \ldots, D^\beta u, \ldots)\, D^\alpha v\, dx. \tag{2.13}$$

Thus from (2.2), (2.12), (2.13) one obtains (2.1) and the proof of the theorem is complete.

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