A common fixed point theorem of Greguš type

By M. L. DIVICCARO (Naples), B. FISHER (Leicester), and S. SESSA (Naples)

Abstract. Let T and I be two selfmaps of a closed convex subset C of a Banach space X satisfying the inequality

$$\|Tx-Ty\|^p \leqq a\|Ix-Iy\|^p+(1-a)\max\{\|Tx-Ix\|^p, \|Tx-Iy\|^p\}$$

for all x, y in C, where $0<a<1/2^{p-1}$ and $p\geqq 1$. Further, let I weakly commute with T, i.e. $\|TIx-ITx\|\leqq\|Tx-Ix\|$ for any x in C and let I be linear and nonexpansive in C. It is proved that if $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

Let T and I be two mappings of a metric space (X, d) into itself. Sessa [6] defined T and I to be *weakly commuting* if $d(TIx, ITx)\leqq d(Ix, Tx)$ for any x in X. Clearly two commuting mappings weakly commute but two weakly commuting mappings in general do not commute. See Example 1 below. It is well known that [5] a mapping I of (X, d) into itself is called *nonexpansive* if $d(Ix, Iy)\leqq d(x, y)$ for all x, y in X. This implies that I continuous on X.

From now on, C denotes a closed convex subset of a Banach space X. Following the basic ideas of a paper of FISHER and SESSA [3] and drawing inspiration from a recent work of DELBOSCO, FERRERO and ROSSATI [1], we now establish the following result:

Theorem. *Let T and I two weakly commuting mappings of C into itself satisfying the inequality*

$$(1) \qquad \|Tx-Ty\|^p \leqq a\cdot\|Ix-Iy\|^p+(1-a)\cdot\max\{\|Tx-Ix\|^p, \|Ty-Iy\|^p\}$$

for all x, y in C, where $0<a<1/2^{p-1}$ and $p\geqq 1$. If I is linear, nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed at which T is continuous.

PROOF. Let $x=x_0$ be an arbitrary point of C and, since $I(C)$ contains $T(C)$, let x_1, x_2, x_3 be points such that

$$Ix_1 = Tx, \quad Ix_2 = Tx_1, \quad Ix_3 = Tx_2,$$

so that $Tx_{r-1}=Ix_r$ for $r=1, 2, 3$. Using inequality (1) we have

$$\|Tx_r-Ix_r\|^p = \|Tx_r-Tx_{r-1}\|^p \leqq a\|Tx_{r-1}-Ix_{r-1}\|^p+$$
$$+(1-a)\cdot\max\{\|Tx_r-Ix_r\|^p, \|Tx_{r-1}-Ix_{r-1}\|^p\}$$

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which implies

$$\|Tx_r - Ix_r\|^p \leqq \|Tx_{r-1} - Ix_{r-1}\|^p$$

for $r=1, 2, 3$. It follows that

$$\|Tx_r - Ix_r\| \leqq \|Tx - Ix\| \tag{2}$$

for $r=1, 2, 3$. Using (1) again, we have from (2):

$$\|Tx_2 - Ix_1\|^p = \|Tx_2 - Tx\|^p \leqq$$

$$\leqq a \cdot \|Ix_2 - Ix\|^p + (1-a) \cdot \max\{\|Tx_2 - Ix_2\|^p, \|Tx - Ix\|^p\} \leqq \tag{3}$$

$$\leqq a \cdot [\|Tx_1 - Ix_1\| + \|Tx - Ix\|]^p + (1-a) \cdot \|Tx - Ix\|^p \quad (2^p a + 1 - a) \cdot \|Tx - Ix\|^p.$$

Now we define a point z by

$$z = \frac{1}{2} x_2 + \frac{1}{2} x_3$$

Since C is convex and z is in C, we have

$$Iz = \frac{1}{2} Ix_2 + \frac{1}{2} Ix_3 = \frac{1}{2} Tx_1 + \frac{1}{2} Tx_2$$

because I is linear. Then we have from (3):

$$\|Iz - Ix_1\|^p = \left\| \frac{1}{2} \cdot (Ix_2 - Ix_1) + \frac{1}{2} \cdot (Ix_3 - Ix_1) \right\|^p \leqq$$

$$\leqq \frac{1}{2} \|Tx_1 - Ix_1\|^p + \frac{1}{2} \|Tx_2 - Ix_1\|^p \leqq \frac{(2^p a + 2 - a)}{2} \cdot \|Tx - Ix\|^p \tag{4}$$

and from (2):

$$\|Iz - Ix_2\|^p = \frac{1}{2^p} \|Ix_3 - Ix_2\|^p = \frac{1}{2^p} \cdot \|Tx_2 - Ix_2\|^p \leqq \frac{1}{2^p} \cdot \|Tx - Ix\|^p. \tag{5}$$

Using (1) and (2), we achieve

$$\|Tz - Iz\|^p = \left\| \frac{1}{2} \cdot (Tz - Tx_1) + \frac{1}{2} \cdot (Tz - Tx_2) \right\|^p \leqq$$

$$\leqq \frac{1}{2} \|Tz - Tx_1\|^p + \frac{1}{2} \|Tz - Tx_2\|^p \leqq$$

$$\leqq \frac{1}{2} [a \cdot \|Iz - Ix_1\|^p + (1-a) \cdot \max\{\|Tz - Iz\|^p, \|Tx_1 - Ix_1\|^p\}] +$$

$$+ \frac{1}{2} [a \cdot \|Iz - Ix_2\|^p + (1-a) \cdot \max\{\|Tz - Iz\|^p, \|Tx_2 - Ix_2\|^p\}] \leqq$$

$$\leqq \frac{1}{2} a \cdot [\|Iz - Ix_1\|^p + \|Iz - Ix_2\|^p] + (1-a) \cdot \max\{\|Tz - Iz\|^p, \|Tx - Ix\|^p\}.$$

After some computations, using (4) and (5), one deduces from the foregoing inequality:

$$\|Tz-Iz\|^p \leqq \lambda \cdot \max\{\|Tz-Iz\|^p, \|Tx-Ix\|^p\}$$

where

$$\lambda = \frac{1}{2}\left[2+\frac{2^{p-1}a^2(2^p-1)}{2^p}-\frac{a(2^p-1)}{2^p}\right].$$

Since $0<a<1/2^{p-1}$, it is easily seen that $0<\lambda<1$. Thus we have

(6) $$\|Tz-Iz\| \leqq \lambda^{1/p}\cdot\|Tx-Ix\|$$

with $0<\lambda^{1/p}<1$. Now (6) implies

$$\inf\left\{\|Tz-Iz\|: z=\frac{1}{2}x_2+\frac{1}{2}x_3\right\} \leqq \lambda^{1/p}\cdot\inf\{\|Tx-Ix\|: x\in C\}$$

and on the other hand, since

$$\inf\{\|Tx-Ix\|: x\in C\} \leqq \inf\left\{\|Tz-Iz\|: z=\frac{1}{2}x_2+\frac{1}{2}x_3\right\},$$

we can claim

$$\inf\{\|Tx-Ix\|: x\in C\} = 0.$$

Then the sets defined by

$$K_n = \{x\in C: \|Tx-Ix\| \leqq 1/n\},\ H_n = \{x\in C: \|Tx-Ix\| \leqq 2/\{[1-(1-a)^{1/p}]\cdot n\}$$

for $n=1, 2, \ldots$ must be nonempty and obviously

$$K_1 \supseteqq K_2 \supseteqq \ldots \supseteqq K_n \supseteqq \ldots .$$

Consequently $\overline{TK_n}$, which denotes the closure of TK_n, is nonempty for $n=1, 2, \ldots$ and

$$\overline{TK_1} \supseteqq \overline{TK_2} \supseteqq \ldots \supseteqq \overline{TK_n} \supseteqq \ldots .$$

Now we observe that (1) implies

(7) $$\|Tx-Ty\| \leqq a^{1/p}\cdot\|Ix-Iy\|+(1-a)^{1/p}\cdot\max\{\|Tx-Ix\|, \|Ty-Iy\|\}$$

for all x, y in C. Then we have for all x, y in K_n:

$$\|Tx-Ty\| \leqq a^{1/p}\cdot[\|Tx-Ix\|+\|Tx-Ty\|+\|Ty-Iy\|]+(1-a)^{1/p} \leqq$$
$$\leqq 2a^{1/p}/n+a^{1/p}\cdot\|Tx-Ty\|+(1-a)^{1/p}/n \leqq 2/n+a^{1/p}\cdot\|Tx-Ty\|+1/n$$

which gives

$$\|Tx-Ty\| \leqq 3/[(1-a^{1/p})]\cdot n.$$

Thus

$$\lim_{n\to\infty}\operatorname{diam} TK_n = \lim_{n\to\infty}\operatorname{diam}\overline{TK_n} = 0,$$

i.e. $\overline{TK_n}$ is a decreasing sequence of nonempty closed subsets of C whose sequence $\{\operatorname{diam}\overline{TK_n}\}$ of the diameters converges to zero and, by a well known Theorem of Cantor the intersection $\bigcap_{n=1}^{\infty}\overline{TK_n}$ consists of a single point w.

Let y be an arbitrary point of $\overline{TK_n}$. Then for arbitrary $\varepsilon>0$, there exists a point y' in K_n such that

$$\|Ty'-y\| < \varepsilon. \tag{8}$$

Since T weakly commutes with I and I is nonexpansive, we obtain from (7) and (8):

$$\begin{aligned}\|Ty-Iy\| &\leqq \|Ty-TIy'\|+\|TIy'-ITy'\|+\|ITy'-Iy\| \leqq \\ &\leqq a^{1/p}\cdot\|Iy-I^2y\|+(1-a)^{1/p}\cdot\max\{\|Ty-Iy\|, \|TIy'-I^2y'\|\}+ \\ &\quad +\|Iy'-Ty'\|+\|Ty'-y\| \leqq \\ &\leqq a^{1/p}\cdot\|y-Iy'\|+(1-a)^{1/p}\cdot\max\{\|Ty-Iy\|, \|TIy'-ITy'\|+\|Ty'-Iy'\|\}+ \\ &\quad +1/n+\varepsilon \leqq a^{1/p}\cdot[\|y-Ty'\|+\|Ty'-Iy'\|]+ \\ &\quad +(1-a)^{1/p}\cdot\max\{\|Ty-Iy\|, 2\|Ty'-Iy'\|\}+1/n+\varepsilon \leqq \\ &\leqq (1+a^{1/p})\cdot\varepsilon+(1+a^{1/p})/n+(1-a)^{1/p}\cdot\max\{\|Ty-Iy\|, 2/n\}.\end{aligned}$$

Since ε is arbitrary, we get

$$\|Ty-Iy\| \leqq (1+a^{1/p})/n+(1-a)^{1/p}\cdot\max\{\|Ty-Iy\|, 2/n\}. \tag{9}$$

If

$$\|Ty-Iy\| \leqq 2/n,$$

then we also have

$$\|Ty-Iy\| \leqq 2/n \leqq 2/\{[1-(1-a)^{1/p}]\cdot n\}.$$

If

$$\|Ty-Iy\| > 2/n,$$

(9) implies

$$\|Ty-Iy\| \leqq 2/n+(1-a)^{1/p}\cdot\|Ty-Iy\|$$

because $1+a^{1/p}<2$. Then we have, in both cases, that y belongs to H_n for $n=1, 2, \ldots$. It follows that

$$\|Tw-Iw\| \leqq 2/\{[1-(1-a)^{1/p}]\cdot n\}$$

for $n=1, 2, \ldots$ and therefore $Tw=Iw$. Since I weakly commutes with T, we have

$$\|TIw-ITw\| \leqq \|Iw-Tw\| = 0,$$

i. e.

$$T^2w = TIw = ITw. \tag{10}$$

Using (1), we achieve

$$\begin{aligned}\|T^2w-Tw\|^p &\leqq a\cdot\|ITw-Iw\|^p+(1-a)\cdot\max\{\|T^2w-ITw\|^p, \|Tw-Iw\|^p\} = \\ &= a\cdot\|T^2w-Tw\|^p\end{aligned}$$

and so $T^2w=Tw$, i.e. $Tw=w'$ is a fixed point of T. (10) implies also $w'=Tw'=Iw'$ and this means that w' is a fixed point of I. Let w'' another common fixed point of T and I. Then using (1),

$$\begin{aligned}\|w'-w''\| = \|Tw'-Tw''\|^p &\leqq a\cdot\|Iw'-Iw''\|^p+ \\ +(1-a)\cdot\max\{\|Tw'-Iw'\|^p, \|Tw''-Iw''\|^p\} &= a\cdot\|w'-w''\|^p\end{aligned}$$

and this guarantees the unicity of the common fixed point of T and I. Now let $\{y_n\}$ be a sequence of points of C with limit w'. Using (7), we have

$$\|Ty_n - Tw'\| \leqq a^{1/p} \cdot \|Iy_n - Iw'\| + (1-a)^{1/p} \cdot \max\{\|Ty_n - Iy_n\|, \|Tw' - Iw'\|\}.$$

Remembering that I is continuous in C, the foregoing inequality implies

$$\limsup_{n\to\infty} \|Ty_n - Tw''\| \leqq (1-a)^{1/p} \cdot \limsup_{n\to\infty} \|Ty_n - Tw'\|,$$

i. e.

$$\lim_{n\to\infty} \|Ty_n - Tw'\| = 0$$

and this means that T is continuous at w'. *This concludes the proof.*

Example 1. Let X be the reals and $C=[0, 1]$ with euclidean norm. Define two mappings T and I of C into itself by

$$Tx = \frac{1}{4}x - \frac{1}{8}x^2, \quad Ix = \frac{1}{2}x$$

for all x in C. We have

$$\|TIx - ITx\| = \frac{1}{16}x^2 - \frac{1}{32}x^2 = \frac{1}{32}x^2 \leqq \frac{1}{4}x + \frac{1}{8}x^2 = \|Tx - Ix\|$$

for all x in C. So T weakly commutes with I but they do not commute since

$$Ix = \frac{1}{8}x - \frac{1}{32}x^2 \neq \frac{1}{8}x - \frac{1}{16}x^2 = ITx$$

for any non-zero x in C. Clearly T is linear, nonexpansive in C and

$$T(C) = [0, 1/8] \subseteq [0, 1/2] = I(C).$$

Further we have

$$\|Tx - Ty\|^p = \left\|\frac{1}{4}\cdot(x-y) - \frac{1}{8}\cdot(x+y)\cdot(x-y)\right\|^p =$$

$$= \frac{1}{4^p}\cdot\|x-x\|^p\cdot\left\|1 - \frac{1}{2}\cdot(x+y)\right\|^p \leqq \frac{1}{2^p}\cdot\left(\frac{1}{2}\|x-y\|\right)^p = \frac{1}{2^p}\cdot\|Ix - Iy\|^p$$

for all x, y in C and for any $p \geqq 1$. Assuming $a=1/2^p$, then all the assumptions of our Theorem are satisfied and 0 is the unique common fixed point of T and I.

Remark 1. If $p=1$, we obtain a result of Fisher and Sessa [3].
Assuming $I=$identity of X, we have the following

Corollary 1. *Let T be a mapping of C into itself satisfying the inequality*

(11) $$\|Tx - Ty\|^p \leqq a\cdot\|x-y\|^p + (1-a)\cdot\max\{\|Tx-x\|^p, \|Ty-y\|^p\}$$

for all x, y in C, where $0<a<1/2^{p-1}$ and $p \geqq 1$. Then T has a unique fixed point in C.

The result of this corollary was also given in [3], but it was deduced as a consequence of a common fixed point theorem involving a pair of mappings under a different contractive condition.

Remark 2. Delbosco, Ferrero and Rossati [1], generalizing the result of Gregus [4], considered the inequality

$$(12) \qquad \|Tx-Ty\|^p \leqq a\cdot\|x=y\|^p+b\cdot\|Tx-x\|^p+c\cdot\|Ty-y\|^p$$

for all x, y in C, where $0<a<1/2^{p-1}$, $p\geqq 1$, $b\geqq 0$, $c\geqq 0$ and $a+b+c=1$. Due to the the symmetry, one may suppose $b=c$ and clearly (11) is more general than (12).

Remark 3. For $p=1$, the result of the Corollary 1 was established by Fisher in [2].

The condition "$I(C)$ contains $T(C)$" is necessary in our Theorem as shown in the following.

Example 2. Let X the reals with euclidean norm and $C=[0, 1]$. Define two mappings T and I of C into itself by $Tx=1$ and $Ix=0$ for all x in C. We have

$$\|ITx-TIx\| = 1 = \|Ix-Tx\|$$

for all x in C. Thus T and I weakly commute in C. Then all the assumptions of our theorem are trivially satisfied except that $I(C)$ contains $T(C)$ since $I(C)=\{0\}$ and $T(C)=\{1\}$, but T and I do not have common fixed points.

The condition that I weakly commutes with T is also necessary in our Theorem as it was proved in the example 3 of [3] for $p=1$. However, here we give another example using a general $p\geqq 1$:

Example 3. Let X the reals with euclidean norm and $C=[0, +\infty)$. Define two mappings T and I of C into itself by

$$Tx = \frac{1}{4}x+1 \quad \text{and} \quad Ix = \frac{1}{2}x$$

for all x in C. We have

$$T(C) = [1, \infty) = [0,+\infty) = I(C)$$

and

$$\|Tx-Ty\|^p = \left(\frac{1}{4}\right)^p\cdot\|x-y\|^p = \frac{1}{2^p}\cdot\frac{1}{2^p}\cdot\|x-y\|^p = \frac{1}{2^p}\cdot\|Ix-Iy\|^p$$

for all x, y in C. Since I is linear and nonexpansive, all the assumptions of our Theorem with $a=1/2^p$ for any $p\geqq 1$ hold except the weak commutativity of T and I since for $4\leqq x<6$:

$$\|ITx-TIx\| = \frac{1}{2} > \frac{1}{4}x-1 = \frac{1}{2}x-\frac{1}{4}x-1 = \|Ix-Tx\|.$$

On the other hand, T and I I do not have common fixed points.

It is well known that the class of the continuous functions contains the class of the nonexpansive mappings. We do not know if our Theorem holds assuming I continuous instead of nonexpansive. Moreover, it is not yet known if the hypothesis of the linearity of I is necessary in our result.

Remark 4. See example 4 of [3] where it is proved for $p=1$ that T and I, although have a unique common fixed point, possess infinite fixed points.

We conclude exhibiting the following

Corollary 2. *Let T and I two weakly commuting mappings of C into itself satisfying the inequality*

$$(13)\qquad \|Tx-Ty\| \leqq a\cdot\|Ix-Iy\| + \frac{1}{2}(1-a)\cdot\max\{\|Tx-Iy\|, \|Ty-Ix\|\}$$

for all x, y in C, where $0<a<1$. If I is linear and nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

PROOF. The inequality (13) implies

$$\|Tx-Ty\| \leqq a\cdot\|Ix-Iy\| +$$
$$\frac{1}{2}(1-a)\cdot\max\{\|Tx-Ix\|+\|Ix-Iy\|, \|Ty-Iy\|+\|Ix-Iy\|\} \leqq$$
$$\leqq \frac{1}{2}(a+1)\cdot\|Ix-Iy\| + \frac{1}{2}(1-a)\cdot\max\{\|Tx-Ix\|, \|Ty-Iy\|\}$$

for all x, y in C. Since $(1-a)/2=1-(a+1)/2$, the thesis follows immediately from our Theorem for $p=1$.

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M. L. DIVICCARO AND S. SESSA
UNIVERSITA' DI NAPOLI
FACOLTA' DI ARCHITETTURA
ISTITUTO MATEMATICO
VIA MONTEOLIVETO 3
80 134 NAPOLI
ITALY

B. FISHER
UNIVERSITY OF LEICESTER
DEPARTMENT OF MATHEMATICS
LEICESTER
LE1 7RH
ENGLAND

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