On torsion-free modules over valuation domains

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This note is devoted to several results on torsion-free modules of infinite rank over arbitrary (commutative) valuation domains. The results are related to the projective dimensions (p.d.) of these modules and serve as a prelude to a study of pure submodules of free modules.

A well-known lemma by AUSLANDER [1] states that if R is any ring and M is any (left) R-module, and if

$$0 = M_0 < M_1 < \dots < M_\alpha < \dots < M_\lambda = M$$

is a well-ordered ascending chain of submodules which is continuous (in the sense that $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for limit ordinals β), then p.d. $M_{\alpha+1}/M_\alpha \leqq n$ for all $\alpha < \lambda$ implies p.d. $M \leqq n$. In another version (which is equivalent for $n \geqq 1$ and limit λ), p.d. $M_\alpha \leqq n-1$ for all $\alpha < \lambda$ implies p.d. $M \leqq n$. Concentrating on torsion-free modules M over valuation domains R, we show (Theorem 1) that if the M_α are pure in M and if $\operatorname{cof} \lambda \leqq \omega_n$, then p.d. $M_\alpha \leqq n$ implies p.d. $M \leqq n$. (Here ω_n stands for the first ordinal of cardinality $\aleph_n$.) The case $\operatorname{cof} \lambda = \omega_{n+1}$ requires additional hypothesis (Theorem 5).

We also study briefly pure submodules of free modules over valuation domains R, and give a necessary and sufficient criterion for R to have all these pure submodules again free (Theorem 9).

1. Preliminaries

R will denote throughout a valuation domain; Q will stand for its field of quotients. We consider only torsion-free R-modules M. The *rank* of M is the dimension of the Q-vector space $Q \otimes M$. A submodule N of M is *pure* if M/N is torsion-free. For any subset X of M, $\langle X \rangle$ will denote the submodule of M generated by X.

For the proof of the following lemma we refer to [4]; $\varkappa$ stands for a cardinal.

Lemma A. *If a torsion-free R-module M is $\varkappa$-generated (i.e. can be generated by $\varkappa$ elements), then the same holds for all pure submodules of M.*

We shall need the following results from [4]. $\aleph_{-1}$ denotes "finite".

Lemma B. *A finite rank torsion-free R-module M which can be generated by $\aleph_n$ but not by $\aleph_{n-1}$ elements has projective dimension $n+1$.*

Lemma C. *A torsion-free R-module M of rank* $\aleph_n$ *satisfies* p.d. $M \leqq n$ *if and only if its pure submodules of rank* $<\aleph_n$ *have projective dimensions* $\leqq n$.

By a *tight* submodule of M is meant a submodule N such that p.d. N and p.d. $M/N \leqq$ p.d. M. A *tight system* for M (with p.d. $M=n$) is a family T of submodules M_i $(i \in I)$ of M such that

(i) $0,\ M \in T$;

(ii) T is closed under unions of chains;

(iii) if $M_i \leqq M_j$, both in T, then p.d. $M_j/M_i \leqq n$;

(iv) if $M_i \in T$ and X is a subset of M of cardinality $\leqq \aleph_{n-1}$, then there is an $M_j \in T$ satisfying $\langle M_i, X\rangle \leqq M_j$ such that M_j/M_i is $\aleph_{n-1}$-generated.

In [2] it is proved:

Lemma D. *Every torsion-free R-module M has a tight system consisting of pure submodules. If* p.d. $M=n$, *then a pure submodule of rank* $\leqq \aleph_{n-1}$ *in M is* $\aleph_{n-1}$-*generated.*

A torsion-free R-module M is called *separable* if every finite set of its elements is contained in a summand of M which is the direct sum of rank one submodules. The following holds (see [5]):

Lemma E. *Pure submodules of separable modules are separable. Separable modules of countable rank are direct sums of rank one modules.*

2. Chains of pure submodules

We start with the proof of the following theorem.

Theorem 1. *Let M be a torsion-free R-module and n a non-negative integer. Assume*

$$0 = M_0 \leqq M_1 \leqq \ldots \leqq M_\alpha \leqq \ldots \leqq M_{\omega_n} = M \tag{1}$$

is a continuous chain of submodules such that

(a) *each* M_α *is pure in* M;

(b) p.d. $M_\alpha \leqq n$ *for each* $\alpha < \omega_n$.

Then p.d. $M \leqq n$.

PROOF. Because of Lemma D, there is a tight system T_α in each $M_\alpha (\alpha < \omega_n)$ such that the members are pure submodules of M_α. Using these T_α, we first verify:

Lemma 2. *Under the hypotheses of Theorem* 1, *given any* $\aleph_n$-*generated submodule H of M, there is a submodule* $\bar{H}$ *of M satisfying*

(α) *H is contained in* $\bar{H}$;

(β) *the rank of* $\bar{H}$ *is* $\leqq \aleph_n$;

(γ) $\bar{H} \cap M_\alpha \in T_\alpha$ *for each* $\alpha < \omega_n$;

(δ) $\bar{H} + M_\alpha$ *is pure in M for each* α.

PROOF OF LEMMA 2. We describe two processes whose combination will yield a submodule with the required properties.

First, for each $\alpha<\omega_n$ choose an $\aleph_n$-generated submodule $T_{1\alpha}\in T_\alpha$ such that $H\cap M_\alpha\leqq T_{1\alpha}$. Clearly, the submodule $H_2=\langle H, T_{1\alpha}\,(\alpha<\omega_n)\rangle$ is $\aleph_n$-generated. Repeat this process with H_2 in place of H to obtain a submodule $H_3=\langle H_2, T_{2\alpha}\,(\alpha<\omega_n)\rangle$ with $\aleph_n$-generated $T_{2\alpha}\in T_\alpha$ such that $H_2\cap M_\alpha\leqq T_{2\alpha}$, etc. The chain $H\leqq H_2\leqq \leqq\ldots\leqq H_m\leqq\ldots$ of these $\aleph_n$-generated submodules has a union H_* which evidently satisfies

$$H_*\cap M_\alpha\in T_\alpha \quad \text{for all} \quad \alpha<\omega_n.$$

Next, consider the submodule $(H+M_\alpha)/M_\alpha$ of M/M_α. View M/M_α as the union of its submodules M_β/M_α $(\alpha<\beta<\omega_n)$. Since p.d. $M_{\beta+1}/M_\beta\leqq n+1$ because of (b), from Auslander's Lemma we infer that p.d. $M/M_\alpha\leqq n+1$ as well. From Lemma D it follows that all pure submodules of rank $\leqq\aleph_n$ in M/M_α are $\aleph_n$-generated. Consequently, there exists an $\aleph_n$-generated pure submodule $(H_\alpha^2+M_\alpha)/M_\alpha$ of M/M_α that contains $(H+M_\alpha)/M_\alpha$; here $H_\alpha^2(\geqq H)$ can be chosen so as to be $\aleph_n$-generated. The submodule $H^2=\langle H, H_\alpha^2\,(\alpha<\omega_n)\rangle$ is again of rank $\leqq\aleph_n$, so the same process can be repeated with H^2 playing the role of H to obtain a larger $\aleph_n$-generated submodule H^3 such that $(H^3+M_\alpha)/M_\alpha$ contains the purification of $(H^2+M_\alpha)/M_\alpha$ in M/M_α, etc. The union H^* of the chain $H\leqq H^2\leqq\ldots\leqq H^m\leqq\ldots$ will be of rank $\leqq\aleph_n$ and for each $\alpha<\omega_n$ it will satisfy:

$$H^*+M_\alpha \quad \text{is pure in } M.$$

To conclude the proof of Lemma 2, we alternate the two processes and define $\bar{H}$ as the union of the ascending chain $H\leqq H_*\leqq(H_*)^*\leqq((H_*)^*)_*\leqq\ldots$. Obviously, $\bar{H}$ will satisfy $(\alpha)-(\delta)$. □

Resuming the proof of Theorem 1, we proceed to establish the existence of a continuous chain

$$(2) \qquad 0=H_0\leqq H_1\leqq\ldots\leqq H_\nu\leqq\ldots\leqq H_\lambda=M$$

of submodules in M with the following properties:

(i) $H_{\nu+1}/H_\nu$ is $\aleph_n$-generated for $\nu<\lambda$;

(ii) $H_\nu\cap M_\alpha\in T_\alpha$ for $\alpha<\omega_n$ and $\nu<\lambda$;

(iii) $H_\nu+M_\alpha$ is pure in M for $\alpha<\omega_n$, $\nu<\lambda$.

Here λ denotes a suitable ordinal.

Define H_ν by transfinite induction. Setting $H_0=0$, assume that the H_μ $(\mu<\nu)$ have been selected so as to have properties (i)—(iii).

If ν is a limit ordinal, then $H_\nu=\bigcup_{\mu<\nu} H_\mu$. (ii) and (iii) will obviously hold for H_ν.

If ν is a successor ordinal, say $\nu=\mu+1$, and if $H_\mu<M$, then in (1) we pass mod H_μ and consider the chain

$$(3) \qquad 0=H_\mu/H_\mu\leqq\ldots\leqq(H_\mu+M_\alpha)/H_\mu\leqq\ldots \quad (\alpha<\omega_n).$$

By $(H_\mu+M_\alpha)/H_\mu\cong M_\alpha/(H_\mu\cap M_\alpha)$ and $H_\mu\cap M_\alpha\in T_\alpha$, we see that in (3) all modules have projective dimensions $\leqq n$. Thus (3) is a chain like (1), so that Lemma 2 can be applied to a non-zero cyclic submodule of M/H_μ to obtain a submodule $\bar{H}/H_\mu$ of M/H_μ satisfying (β)—(δ). It only remains to put $H_\nu=\bar{H}$ and to check that (i)—(iii) hold (which is routine), completing the proof of (2).

In order to verify Theorem 1, by Auslander's Lemma it will be enough to show that in (2) p.d. $H_{\nu+1}/H_\nu \leqq n$ for each $\nu<\lambda$. Note that $H_{\nu+1}/H_\nu$ is the union of the following continuous well-ordered ascending chain:

$$(4) \qquad 0 \leqq [(H_{\nu+1}\cap M_1)+H_\nu]/H_\nu \leqq \dots \leqq [(H_{\nu+1}\cap M_\alpha)+H_\nu]/H_\nu \leqq \dots$$

with $\alpha<\omega_n$. Because of (iii), here $(H_{\nu+1}\cap M_\alpha)+H_\nu=H_{\nu+1}\cap(H_\nu+M_\alpha)$ is pure in M; thus the chain (4) consists of pure submodules of $H_{\nu+1}/H_\nu$. Therefore, (i) implies that the modules in (4) are $\aleph_n$-generated (cf. Lemma A). Furthermore, in view of (ii),

$$[(H_{\nu+1}\cap M_\alpha)+H_\nu]/H_\nu \cong (H_{\nu+1}\cap M_\alpha)/(H_\nu\cap M_\alpha)$$

has projective dimension $\leqq n$. From Lemma C it is easy to derive that p.d. $H_{\nu+1}/H_\nu\leqq n$. □

The special case $n=0$ is most interesting:

Corollary 3. *The union of a countable ascending chain* $0=F_0\leqq F_1\leqq\dots\leqq F_m\leqq\dots$ *of free R-modules* F_m *is again free whenever each* F_m *is pure in* F_{m+1}. □

Another corollary is the following result generalizing Theorem 1.

Corollary 4. *Let M be a torsion-free R-module and n, k non-negative integers. If*

$$0 = M_0 < M_1 < \dots < M_\alpha < \dots < M_{\omega_n} = M$$

is a well-ordered continuous chain of submodules of M such that

(a) *each* M_α *is pure in M,*

(b) *for each* $\alpha<\omega_n$, p.d. $M_\alpha\leqq n+k$,

then p.d. $M\leqq n+k$.

Proof. We induct on k, the case $k=0$ being covered by Theorem 1. Suppose $k\geqq 1$. For each α, consider the canonical projective resolution of M_α:

$$(5) \qquad 0 \to H_\alpha \to F_\alpha = \bigoplus_{a\in M_\alpha} Rx_a \xrightarrow{\Phi_\alpha} M_\alpha \to 0$$

where F_α is free and $\Phi_\alpha(x_a)=a$. Here H_α is pure in F_α and p.d. $H_\alpha\leqq n+k-1$. The obvious embeddings $M_\alpha\to M_\beta$ $(\alpha<\beta)$ give rise to a direct system of exact sequences (5) whose direct limit is the resolution (5) for $\alpha=\omega_n$. As $H_{\omega_n}=\bigcup_{\alpha<\omega_n} H_\alpha$, the induction hypothesis can be applied to H_{ω_n} to conclude that p.d. $H_{\omega_n}\leqq n+k-1$. Hence p.d. $M\leqq n+k$, indeed. □

If we wish to consider longer chains in (1), and to retain the same conclusion on p.d. M, then we need a cardinality restriction on the M_α's as well as an additional hypothesis on the $M_{\alpha+1}/M_\alpha$'s.

Theorem 5. *Assume that n is a non-negative integer and*

$$(6) \qquad 0 = M_0 \leqq M_1 \leqq \dots \leqq M_\alpha \leqq \dots \leqq M_{\omega_{n+1}} = M$$

is a continuous well-ordered chain of submodules of the torsion-free R-module M, satisfying the following conditions:

(i) *each* M_α *is pure in* M;
(ii) *each* M_α *is* $\aleph_{n+1}$*-generated;*
(iii) p.d. $M_\alpha \leqq n$ *for each* $\alpha < \omega_{n+1}$;
(iv) *the pure submodules of rank* $\leqq \aleph_{n-1}$ *in* $M_{\alpha+1}/M_\alpha$ *are* $\aleph_{n-1}$*-generated for each* $\alpha < \omega_{n+1}$.

Then p.d. $M \leqq n$.

Proof. We construct another chain (7) replacing the given (6). For each $\alpha < \omega_{n+1}$, fix a tight system T_α in M_α, consisting of pure submodules. We want a continuous chain

$$0 = A_0 \leqq A_1 \leqq \dots \leqq A_\alpha \leqq \dots \leqq A_{\omega_{n+1}} = M \tag{7}$$

of submodules, subject to the conditions:

1) each A_α is $\aleph_n$-generated $(\alpha < \omega_{n+1})$;
2) $A_\alpha \in T_\alpha$ whenever $\alpha < \omega_{n+1}$ is a non-limit ordinal;
3) $A_\alpha \cap M_\beta \in T_\beta$ for all $\beta < \alpha < \omega_{n+1}$;
4) $A_\alpha + M_\beta$ is pure in M_α for $\beta < \alpha < \omega_{n+1}$.

Observe that $\alpha < \omega_{n+1}$ implies $\operatorname{cof} \alpha \leqq \omega_n$, thus Lemma 2 can be applied in the same way as in the proof of (2) to establish a chain (7) with the desired properties. In order to ascertain that $A_{\omega_{n+1}} = M$, well-order a generating set of M: $\{a_\alpha | \alpha < \omega_{n+1}\}$ with the proviso that $a_\beta \in A_\alpha$ for all $\beta < \alpha$. This will hold whenever $A_{\alpha+1}$ is constructed so as to include a_α $(\alpha < \omega_{n+1})$.

Once (7) has been established, it is sufficient to verify that

$$\text{p.d. } A_{\alpha+1}/A_\alpha \leqq n \quad \text{for} \quad \alpha < \omega_{n+1}.$$

In the exact sequence

$$0 \to (A_{\alpha+1} \cap M_\alpha)/A_\alpha \to A_{\alpha+1}/A_\alpha \to A_{\alpha+1}/(A_{\alpha+1} \cap M_\alpha) \to 0$$

the last non-zero module is $\cong (A_{\alpha+1} + M_\alpha)/M_\alpha$. This is $\aleph_n$-generated and 4) implies that it has property (iv). A simple reference to Lemma C shows that its projective dimension is at most n. Hence it remains only to show that

$$\text{p.d. } (A_{\alpha+1} \cap M_\alpha)/A_\alpha \leqq n. \tag{8}$$

We distinguish two cases according as α is a successor or a limit ordinal.

In the first alternative, 2) ensures $A_\alpha \in T_\alpha$. Furthermore, by 3), $A_{\alpha+1} \cap M_\alpha \in T_\alpha$ likewise, whence (8) follows at once.

If α is a limit ordinal, then we view $(A_{\alpha+1} \cap M_\alpha)/A_\alpha$ as the union of its submodules $[(A_{\alpha+1} \cap M_\beta) + A_\alpha]/A_\alpha$ for $\beta < \alpha$. These are isomorphic to $(A_{\alpha+1} \cap M_\beta)/(A_\alpha \cap M_\beta)$; here both intersections belong to T_β, so the projective dimension of their quotient is $\leqq n$. Furthermore, $(A_{\alpha+1} \cap M_\beta) + A_\alpha = A_{\alpha+1} \cap (M_\beta + A_\alpha)$ are pure submodules. Since $\operatorname{cof} \alpha \leqq \omega_n$, Theorem 1 can be applied to conclude that (8) holds true. □

Again the case $n=0$ deserves special attention:

Corollary 6. *If*

$$0 = F_0 \leqq F_1 \leqq \dots \leqq F_\alpha \leqq \dots \quad (\alpha < \omega_1)$$

is a continuous chain of $\aleph_1$*-generated free R-modules* F_α *such that* F_α *is pure in* $F_{\alpha+1}$ *for each* α *and in* $F_{\alpha+1}/F_\alpha$ *the finite rank pure submodules are finitely generated, then the union* $\cup F_\alpha$ *is again a free R-module.* □

3. Pure submodules of free modules

Submodules of free R-modules need not be free, not even when they are pure, so the problem of studying pure submodules of free R-modules arises. So far they have not been investigated systematically and here we can only establish a few relevant properties.

An easy, but important observation is as follows.

Proposition 7. *Pure submodules of free R-modules are separable, and their $\aleph_0$-generated pure submodules are free.*

PROOF. This is an immediate consequence of Lemma E. □

The second part of the assertion generalizes easily:

Proposition 8. *Pure submodules of a free R-module of rank $\aleph_n$ have projective dimension $\leqq n$.*

PROOF. A pure submodule of a free R-module of rank $\aleph_n$ can have projective dimension $d \geqq n+1$ only if it contains a finite rank pure submodule of projective dimension d; this is an easy consequence of Lemma C; see [4]. However, this would contradict Proposition 7. □

In view of the definition of purity, the equivalence of conditions (i) and (ii) in the following theorem is obvious [gl.d. means global dimension].

Theorem 9. *For a valuation domain R, the following are equivalent:*

(i) *pure submodules of free R-modules are free;*

(ii) p.d. $M \leqq 1$ *for all torsion-free R-modules M;*

(iii) gl.d. $R \leqq 2$ *and* p.d. $Q=1$.

PROOF. It remains to verify the equivalence of (ii) and (iii). Observe that (ii) implies p.d. $I \leqq 1$ for all the ideals I of R. Therefore gl.d. $R = \sup$ p.d. $I+1 \leqq 2$, and (iii) follows.

Assume now that (iii) holds. Then p.d. $I \leqq 1$ for all the ideals I of R. Given any torsion-free R-module M, we can find a well-ordered ascending continuous chain of submodules of M whose factors are rank one torsion-free modules. By hypothesis, these factors have projective dimension $\leqq 1$, so by Auslander's Lemma, p.d. $M \leqq 1$. □

With the aid of the preceding result one can show that if R admits non-free pure submodules in free R-modules, then this already occurs at the cardinality $\aleph_1$.

Lemma 10. *If R is such that all pure submodules of the free R-module of rank $\aleph_1$ are free, then all pure submodules of free R-modules are free.*

PROOF. Otherwise, either gl.d. $R \geqq 3$ or p.d. $Q \geqq 2$. Thus either an ideal I of R or Q has projective dimension $\geqq 2$. By Lemma B, there is a rank one torsion-free module M with $\aleph_1$ generators which cannot be countably generated. If $0 \to H \to \to F \to M \to 0$ is a free resolution of this M with F free of rank $\aleph_1$, then p.d. $M=2$ implies p.d. $H=1$. This H is a non-free pure submodule of F. □

As far as the converse of Proposition 7 is concerned, we offer a counterexample.

Example. There is a valuation domain R which has a separable torsion-free R-module M of rank $\aleph_1$ whose $\aleph_0$-generated pure submodules are free, but M is not embeddable as a pure submodule in a free R-module. Let R be such that gl.d. $R \leqq 2$ and p.d. $Q=1$. Choose a free resolution $0 \to H \to F \to Q \to 0$ with H, F countably generated free. Using a fixed isomorphism $\varphi \colon F \to H$, it is easy to construct a chain

$$0 = F_0 < F_1 = H < F_2 = F < F_3 < \dots$$

of countably generated free R-modules such that $F_{n+1}/F_n \cong Q$ for $n \geqq 0$. By Corollary 3, $\bigcup_{n<\omega} F_n$ is likewise free, so we can proceed transfinitely and get a well-ordered continuous ascending chain of countably generated free R-modules F_α for every $\alpha < \omega_1$ such that $F_{\alpha+1}/F_\alpha \cong Q$ for each $\alpha < \omega_1$. Let M be $\bigcup F_\alpha$ for $\alpha < \omega_1$. Then Eklof's Theorem [3] shows p.d. $M=1$. By Theorem 9, M is not isomorphic to any pure submodule of a free R-module. As every countable rank submodule of M is contained in some F_α, it follows that M is separable and its $\aleph_0$-generated pure submodules are free.

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