On the convergence of linear martingales

By ISTVÁN FAZEKAS (Debrecen)

§ 1. Introduction

The following generalization of the notion of martingale has been studied by MACQUEEN [7]. Let $(\xi_t, F_t, t=1, 2, \ldots)$ be an adapted sequence of random variables and suppose that

$$E\{\xi_s | F_{s-1}\} = a_1\xi_{s-1} + \ldots + a_m\xi_{s-m} \tag{1}$$

for $s>m$, where m is a fixed positive integer and $a_1, \ldots, a_m$ are non-random coefficients. From equation (1) we get that

$$\xi_s = a_1\xi_{s-1} + \ldots + a_m\xi_{s-m} + \delta_s, \tag{2}$$

where δ_s is a martingale difference. This autoregressive scheme is widely studied in the literature. Deep investigations have been devoted in particular to the stationarity of the process ξ_s in the Gaussian case (see [1], p. 108). However, it has been pointed out in [7], that the process ξ_s is more closely related to a martingale than a stationary process, if the coefficients a_k are positive and $\sum_{k=1}^{m} a_k = 1$. In this special case classical martingale convergence theorems of Doob are true for ξ_s (see [7], Section 3).

The aim of this paper is to give a new proof for the results of MacQueen (part (a) and (b) of our Theorem 3). Part (c) of Theorem 3 is new.

Our method is the following: we reduce our problem with the help of the transformation described in (6) and (7) to the study of an m-dimensional process X_t for which

$$E\{X_t | F_{t-1}\} = \Lambda X_{t-1}. \tag{3}$$

We prove some general results for the vector-valued process X_t (Theorems 1 and 2) which imply the convergence of ξ_t.

§ 2. Definitions and preliminary remarks

Definition 1. Let (Ω, F, P) be a probability space, F_s $(s=1, 2, \ldots)$ an increasing sequence of σ-subalgebras of F and ξ_s $(s=1, 2, \ldots)$ real random variables (r.v.'s) defined on (Ω, F, P).

We call the process $(\xi_s, F_s, s=1, 2, \ldots)$ a linear martingale if ξ_s is F_s-measurable, $E|\xi_s|<\infty$ for every s and

$$E\{\xi_s|F_{s-1}\} = a_1(s)\xi_{s-1}+\ldots+a_m(s)\xi_{s-m} \tag{4}$$

for all $s>m$, where m is a fixed positive integer, and $a_k(s)$ $(k=1, \ldots, m;\ s>m)$ are nonnegative non-random coefficients for which $\sum_{k=1}^{m} a_k(s)=1$ $(s>m)$. (We suppose that there exists an s for which $a_m(s)\neq 0$.)

From equation (4) it follows that

$$\xi_s = a_1(s)\xi_{s-1}+\ldots+a_m(s)\xi_{s-m}+\delta_s, \tag{5}$$

where $\delta_s=\xi_s-E\{\xi_s|F_{s-1}\}$ is a martingale difference.

If the initial r.v.'s $\xi_1, \ldots, \xi_m$, the coefficients

$$a_k(s)\quad (k = 1, \ldots, m;\ s = m+1, m+2, \ldots)$$

and the martingale difference $(\delta_s,\ F_s,\ s=m+1, m+2, \ldots)$ are given, then there exists a process $(\xi_s,\ F_s,\ s=1, 2, \ldots)$ satisfying (5) and thus also (4).

In order to study the convergence properties of the process ξ_t we introduce the m-dimensional vectors

$$X_t = \begin{pmatrix}\xi_t\\ \vdots\\ \xi_{t-m+1}\end{pmatrix}, \quad \Delta_t = \begin{pmatrix}\delta_t\\ 0\\ \vdots\\ 0\end{pmatrix}, \tag{6}$$

and a matrix of type $m\times m$:

$$\Lambda(t) = \begin{pmatrix} a_1(t) & \ldots & a_m(t)\\ 1 & & 0 \\ & \ddots & \\ 0 & 1 & 0\end{pmatrix}. \tag{7}$$

With these notations we have

$$E\{X_t|F_{t-1}\} = \Lambda(t)X_{t-1} \tag{8}$$

and

$$X_t = \Lambda(t)X_{t-1}+\Delta_t \tag{9}$$

for $t>m$.

Remark 1. This transformation is widely used to study discrete time autoregressive processes ([1], p. 109).

For the sake of brevity a process $(X_t,\ F_t,\ t=1, 2, \ldots)$ with property (8) will be called a Λ-martingale:

Definition 2. An adapted m-dimensional stochastic process $(X_t,\ F_t,\ t=1, 2, \ldots)$ is called a Λ-martingale if X_t is integrable and $E\{X_{t+1}|F_t\}=\Lambda(t+1)X_t$ for $t=1, 2, \ldots$, where $\Lambda(t)$ is a given non-random matrix $(t=1, 2, \ldots)$. (Equivalently, (X_t, F_t) is a Λ-martingale if (9) is satisfied, where (Δ_t, F_t) is a martingale difference.)

From equation (9) one easily deduces that X_t has the following representation

$$X_t = A(t,s)X_s + \sum_{u=s+1}^{t} A(t,u)\Delta_u \quad (t > s), \tag{10}$$

where the matrices $A(t,u)$ are the solutions of the equations

$$\begin{aligned} A(t,u) &= \Lambda(t)A(t-1,u) \quad \text{for} \quad t > u, \\ A(u,u) &= I \quad \text{(the identity matrix).} \end{aligned} \tag{11}$$

Equations (10) and (11) can be considered as the discrete analogues of the solution of a stochastic differential equation (see Theorem 4.2.4 of [4]).

In the sequel $\|\cdot\|$ denotes the norm of a vector or a matrix.

§ 3. Convergence of Λ-martingales

We shall prove that under certain conditions the classical martingale convergence theorems of Doob are true for Λ-martingales. We assume that the limit

$$\lim_{t\to\infty} A(t,u) = A(u) \tag{12}$$

exists for every $u=1, 2, \ldots$.

Let us introduce the accompanying martingale of X_t:

$$Y_t = \sum_{u=1}^{t} A(u)\Delta_u,$$

where $\Delta_1 = X_1$.

Lemma 1. *If the Λ-martingale (X_t, F_t) is bounded in L_α $(\alpha \geqq 1)$, i.e.*

$$\sup_t E\|X_t\|^\alpha \leqq c < \infty,$$

then its accompanying martingale (Y_t, F_t) is also bounded in L_α:

$$\sup_t E\|Y_t\|^\alpha \leqq c.$$

PROOF. Let us consider the following martingale:

$$Y_{t,s} = \sum_{u=1}^{s} A(t,u)\Delta_u$$

for $1 \leqq s \leqq t$, where t is fixed. For the submartingale $\|Y_{t,s}\|^\alpha$ $(1 \leqq s \leqq t)$ we have

$$E\|Y_{t,s}\|^\alpha \leqq E\|Y_{t,t}\|^\alpha = E\|X_t\|^\alpha \leqq c$$

for every $s \leqq t$. Since

$$\lim_{t\to\infty} Y_{t,s} = \sum_{u=1}^{s} A(u)\Delta_u = Y_s$$

for every fixed s, we get by Fatou's lemma $E\|Y_s\|^\alpha \leqq c$.

In addition to assumption (12), we need the following stability condition for the solution of equation (11):

$$\|A(t,u)-A(u)\| \leqq c_{t-u} \quad (t \geqq u), \tag{13}$$

where $\sum_{s=0}^{\infty} c_s < \infty$.

In the sequel, we suppose that there exists a positive function $C_1(\omega)$ for which

$$C_1(\omega)\,\|\Delta_u(\omega)\| \leqq \|A(u)\,\Delta_u(\omega)\| \tag{14}$$

for every $u \geqq 1$ and $\omega \in \Omega$.

Theorem 1. *Let* $(X_t,\ F_t,\ t=1,2,\dots)$ *be a* A*-martingale satisfying conditions* (13) *and* (14).

(a) *If* $\sup_t E\|X_t\| < c < \infty$, *then* $\lim_{t\to\infty} X_t = X_\infty$ *almost surely (a.s.) and* $E\|X_\infty\| < < \infty$.

(b) X_t *converges in* L_1 *as* $t \to \infty$ *if and only if the family* $\{X_t;\ t=1,2,\dots\}$ *is uniformly integrable.*

Proof. (a) Let Y_t be the accompanying martingale of X_t. By Lemma 1 $\sup_t E\|Y_t\| < c$. It follows from Doob's theorem (see [5], p. 319) that $\lim_{t\to\infty} Y_t = Y_\infty$ a.s. and $E\|Y_\infty\| < \infty$. Condition (13) implies that

$$\|X_t - Y_t\| = \Big\|\sum_{u=1}^{t} [A(t,u)-A(u)]\Delta_u\Big\| \leqq \sum_{u=1}^{t} c_{t-u}\|\Delta_u\| = \tag{15}$$

$$= \sum_{s=0}^{t-1} \|\Delta_{t-s}\| c_s = \sum_{s=0}^{n} \|\Delta_{t-s}\| c_s + \sum_{s=n+1}^{t-1} \|\Delta_{t-s}\| c_s.$$

It has been proved that the series $\sum_{u=1}^{t} A(u)\Delta_u = Y_t$ is convergent a.s. Therefore by (14) $\lim_{u\to\infty} \Delta_u(\omega) = 0$ and $\sup_u \|\Delta_u(\omega)\| \leqq c(\omega) < \infty$ for almost all $\omega \in \Omega$. Now inequality (15) and condition $\sum_{s=0}^{\infty} c_s < \infty$ together imply that $\lim_{t\to\infty} \|X_t - Y_t\| = 0$ a.s. So, in view of $\lim_{t\to\infty} Y_t = Y_\infty$ a.s. we have $\lim_{t\to\infty} X_t = X_\infty = Y_\infty$ a.s.

(b) Uniform integrability implies that $\sup_t E\|X_t\| < c < \infty$. By part (a) of this theorem $\lim_{t\to\infty} X_t = X_\infty$ a.s. hence X_t converges also in L_1 because of the uniform integrability.

Conversely, convergence in L_1 always implies uniform integrability.

Remark 2. If $A(t,u) \to A(u)$ as $t \to \infty$, then the accompanying martingale Y_t has the form $Y_t = A(t)X_t$, $t \geqq 1$.

Indeed, by (10) and (11)

$$A(s,t)X_t = \sum_{u=1}^{t} A(s,t)A(t,u)\Delta_u = \sum_{u=1}^{t} A(s,u)\Delta_u$$

for $s > t$, where $\Delta_1 = X_1$. If s tends to infinity we get $Y_t = A(t)X_t$.

Remark 3. (a) Under conditions (13) and (14) uniform integrability of X_t implies uniform integrability of Y_t. Indeed, from equation

$$E\{X_{u+t}|F_u\} = A(t, u)X_u$$

for $t\to\infty$ we get

$$E\{X_\infty|F_u\} = A(u)X_u = Y_u.$$

(b) If in (14) $C_1(\omega)=C_1=\text{const.}$, then the uniform integrability of Y_t implies the uniform integrability of X_t. Indeed, by the proof of Theorem 1 $\lim_{t\to\infty}\|X_t-Y_t\|=0$ in L_1.

(c) The decomposition $X_t=Y_t+Z_t$ is the (unique) Riesz decomposition of the uniformly integrable Λ-martingale X_t.

Remark 4. Let B be a Banach space with the Radon—Nikodym property (cf. [2]). Theorem 1 is valid for B-valued Λ-martingales too.

Theorem 2. *Let (X_t, F_t) be a Λ-martingale. Suppose that $\|A(t,u)\|\leqq K$ for $t\geqq u$. Assume that conditions* (12) *and* (14) *hold and in* (14) $C_1(\omega)=C_1=const.$ *If $\sup_t E\|X_t\|^\alpha<\infty$, where $\alpha>1$, then $\lim_{t\to\infty} X_t=X_\infty$ in L_α (and a.s.).*

Proof. It follows from Lemma 1 that $\sup_t E\|Y_t\|^\alpha<\infty$ for the accompanying martingale Y_t. By the theorem of Doob $\lim_{t\to\infty} Y_t=Y_\infty$ in L_α. From this and from the inequality of Burkholder (cf. [3], p. 384) we infer that

$$E\Big\|\sum_{u=s+1}^{t} A(t,u)\Delta_u\Big\|^\alpha \leqq B_1E\Big(\sum_{u=s+1}^{t}\|A(t,u)\Delta_u\|^2\Big)^{\alpha/2}\leqq$$

$$\leqq B_1K^\alpha E\Big(\sum_{u=s+1}^{t}\|\Delta_u\|^2\Big)^{\alpha/2}\leqq B_1K^\alpha C_1^{-\alpha}E\Big(\sum_{u=s+1}^{t}\|A(u)\Delta_u\|^2\Big)^{\alpha/2}\leqq$$

$$\leqq B_1K^\alpha C_1^{-\alpha}B_2^{-1}E\Big\|\sum_{u=s+1}^{t}A(u)\Delta_u\Big\|^\alpha = B_1K^\alpha C_1^{-\alpha}B_2^{-1}E\|Y_t-Y_s\|^\alpha<\varepsilon$$

if $s>s_\varepsilon$ for every $t>s$.

Finally, from equality

$$X_t-Y_\infty = (Y_s-Y_\infty)+\Big(\sum_{u=1}^{s}A(t,u)\Delta_u-Y_s\Big)+\sum_{u=s+1}^{t}A(t,u)\Delta_u$$

we get the desired convergence.

§ 4. Convergence of linear martingales

Let $(\xi_t, F_t, t=1,2,\ldots)$ be a linear martingale. Assume that the coefficients $a_k(s)$ do not depend on s: $a_k(s)=a_k$, $s>m$, $k=1,2,\ldots,m$. Let d be the greatest common divisor of those integers k for which $a_k>0$.

Theorem 3. *Let us suppose that $d=1$.*

(a) *If $\sup_t E|\xi_t|<\infty$, then ξ_t converges almost surely as $t\to\infty$.*

(b) ξ_t *converges in* L_1 *as* $t\to\infty$ *iff the family* $\{\xi_t\colon t=1,2,\dots\}$ *is uniformly integrable.*

(c) *Let* $\alpha>1$. *If* $\sup_t E|\xi_t|^\alpha<\infty$, *then* ξ_t *converges in* L_α *as* $t\to\infty$.

PROOF. In § 2 we have constructed the Λ-martingale $X(t)$, the matrix $\Lambda(t)$ and the martingale difference Δ_t corresponding to ξ_t. By the conditions of our theorem $\Lambda=\Lambda(t)$ is the transition matrix of a non-decomposable acyclic Markov chain with m states. From the theory of Markov chains it is well known that the elements of the matrices $A(t,u)=\Lambda^{t-u}$ converge exponentially fast to the elements of the matrix $A(u)=A=(a_{ij})$ as $t\to\infty$, where $a_{ij}=p_j$ $(i,j=1,\dots,m)$ and $p_j=\sum_{i=j}^m a_i/\sum_{i=1}^m ia_i$ $(j=1,\dots,m)$ is the unique stationary distribution of the Markov chain.

Therefore conditions (13) and (14) are true. Thus Theorems 1 and 2 imply the present result.

Remark 5. The first component of the martingale Y_t:

$$\eta_t=\Big(\sum_{k=1}^m\Big(\sum_{i=k}^m a_i\Big)\xi_{t+1-k}\Big)\Big/\sum_{i=1}^m ia_i$$

can be regarded as an accompanying martingale of ξ_t.

Remark 6. (a) In the case of $d>1$ Theorem 3 implies the convergence of the subsequences $\xi_i,\xi_{i+d},\xi_{i+2d},\dots$ for every $1\le i<d$.

(b) Let (ξ_t,F_t) and (η_t,G_t) be independent martingales and suppose that $\lim_{t\to\infty}\xi_t\ne\lim_{t\to\infty}\eta_t$. The linear martingale $\xi_1,\eta_1,\xi_2,\eta_2,\dots$ is not convergent (case $d>1$).

Acknowledgement. The author wishes to thank Professor YU. A. ROZANOV for his support during the preparation of this paper and Professor C. C. HEYDE for references [6] and [7].

References

[1] M. ARATÓ, Linear Stochastic Systems with Constant Coefficients, Lecture Notes in Control and Information Sciences, *Springer-Verlag (Berlin—Heidelberg—New York, 1982)*.

[2] S. D. CHATTERJI, Martingale convergence and the Radon—Nikodym theorem in Banach spaces, *Math. Scand.* **22** (1968), 21—41.

[3] Y. S. CHOW and H. TEICHER, Probability Theory, *Springer-Verlag (New York—Heidelberg—Berlin, 1978)*.

[4] M. H. A. DAVIS, Linear Estimation and Stochastic Control, *Chapman and Hall Ltd. (London, 1977)*.

[5] J. L. DOOB, Stochastic Processes, *John Wiley & Sons (New York, 1953)*.

[6] C. C. HEYDE, On a probabilistic analogue of the Fibonacci sequence, *J. Appl. Prob.* **17** (1980), 1079—1082.

[7] J. B. MACQUEEN, A linear extension of the martingale convergence theorem, *Ann. Prob.* **1** (1973), 263—271.

(Received March 10, 1985)