Operational calculus on a subset of R^n

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Abstract

The space of distributions on $[0, a_1)\times\ldots\times[0, a_n)$ is characterized as the algebra of operators (on a class of test functions) which commute with convolution. With convolution identified with multiplication, some operational formulas are developed.

If X is in R^n, then x_i will be understood to denote the i-th component of X. Thus, $X=(x_1, x_2, \ldots, x_n)$. For $0<a_i\leqq\infty$, define L to be the set of locally integrable complex-valued functions of n real variables on the subset $J=\prod_{i=1}^{n}[0, a_i)$ of R^n. These functions are extended to be zero for any $x_i<0$.

Definition. For f and g in L we define the convolution

$$f*g(X)=\int_0^{x_n}\ldots\int_0^{x_1}f(X-U)g(U)\,du_1\ldots du_n \quad (X \text{ in } J). \tag{1}$$

The space L is closed under this operation, which is commutative and associative.

Let Q be the subset of L consisting of those functions which are infinitely differentiable and which, along with all partial derivatives, vanish on $\{X\in J\mid \text{some } x_i=0\}$. A mapping A from Q into Q is said to be *perfect* if $A(p*q)=Ap*q$ for all p and q in Q. If we define the product of two operators to be the composition of the operators, then the set P of all perfect operators is an algebra.

For $i=1, \ldots, n$, we denote by f_i the partial derivative of f with respect to the i-th variable and by D_i the perfect operator which maps q into q_i.

Denote by $\mathscr{D}'_+$ the space of all distributions on $\prod_{i=1}^{n}(-\infty, a_i)$ having support in J. The space $\mathscr{D}'_+$ can be viewed as the dual of the space $\mathscr{D}_-$ of infinitely differentiable functions having support in some subset $\prod_{i=1}^{n}(-\infty, c_i]$ of $\prod_{i=1}^{n}(-\infty, a_i)$.

Definition. For F and G in $\mathscr{D}'_+$ we define

$$\langle F*G, \varphi\rangle=\langle F(X), \langle G(U), \varphi(U+X)\rangle\rangle \quad (\varphi \text{ in } \mathscr{D}_-). \tag{2}$$

The space $\mathscr{D}'_+$ is closed under this operation. For f and g in L, the definitions of $f*g$ given in (1) and in (2) agree. For each F in $\mathscr{D}'_+$ and each q in Q the distribution $F*q$ belongs to Q and is given by the equation

$$F*q(X)=\langle F(U), q(X-U)\rangle$$

(cf. [4, Theorem 27.5]).

Definition. For F in $\mathscr{D}'_+$ we define $\{F\}q(X)=F*q(X)$ for all q in Q.

By the associativity of convolution, $\{F\}$ is a perfect operator and $\{F*G\}=\{F\}\{G\}$ for all F and G in $\mathscr{D}'_+$.

For each X in R^n we denote by X^i the vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For each i we denote by K_i the subset $\{X_i|X\in J\}$ of R^{n-1} and by L_i the set of locally integrable functions on K_i. These functions are extended to be zero for any $x_i<0$.

Definition. If h belongs to L_1, we define

$$\{h\}^1 q(X) = \int_{K_1}\!\!\ldots\int h(X^1-U^1)q(x_1, u_2, \ldots, u_n)\,du_2\ldots du_n$$

for all X in J and q in Q. If h belongs to L_i, where $2\leqq i\leqq n$, we define $\{h\}^i$ similarly.

Theorem. *If h belongs to L_i then $\{h\}^i$ is a perfect operator.*

Proof. We give the proof for $i=1$. If we define $f(X)=h(X^1)$, then

$$\{f\}D_1q(X) = \int_0^{x_n}\!\!\ldots\int_0^{x_1} f(X-U)q_1(U)\,du\ldots du_n =$$

$$= \int_0^{x_n}\!\!\ldots\int_0^{x_1} h(X^1-U^1)q_1(U)\,du_1\ldots du_n =$$

$$= \int_0^{x_n}\!\!\ldots\int_0^{x_2} h(X^1-U^1)\Big[\int_0^{x_1} q_1(U)\,du_1\Big]\,du_2\ldots du_n =$$

$$= \int_0^{x_n}\!\!\ldots\int_0^{x_2} h(X^1-U^1)q(x_1, u_2, \ldots, u_n)\,du_2\ldots du_n$$

for all X in J and all q in Q. So, $\{h\}^1=D_1\{f\}$; it follows that $\{h\}^1$ is a perfect operator.

Theorem. *If f and f_i are in L, then*

$$\{f_i\} = D_i\{f\}-\{f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)\}^i$$

for $i=1, \ldots, n$.

Proof. We give the proof for $i=1$. For each q in Q we can integrate by parts to obtain

$$\{f_1\}q(X) = \int_0^{x_n}\!\!\ldots\int_0^{x_1} f_1(U)q(X-U)\,du_1\ldots du_n =$$

$$= -\int_0^{x_n}\!\!\ldots\int_0^{x_2} f(0, u_2, \ldots, u_n)q(x_1, x_2-u_2, \ldots, x_n-u_n)\,du_2\ldots du_n +$$

$$+\int_0^{x_n}\!\!\ldots\int_0^{x_1} f(U)q_1(X-U)\,du_1\ldots du_n = -\{f(0, x_2, \ldots, x_n)\}^1q(X)+\{f\}D_1q(X) =$$

$$= [D_1\{f\}-\{f(0, x_2, \ldots, x_n)\}^1]q(X).$$

We say that a perfect operator A is *invertible* if there is a perfect operator B such that AB is the identity. In this case we write $B=A^{-1}$. As in [3], the perfect operator $D_1+\ldots+D_n$ is invertible; its inverse is given by the equation

$$(D_1+\ldots+D_n)^{-1}q(X)=\int_0^\infty q(x_1-t,\ldots,x_n-t)\,dt \quad (X \text{ in } J)$$

and the equation

$$(D_1+\ldots+D_n)^{-1}\{f\}=\Big\{\int_0^\infty f(x_1-t,\ldots,x_n-t)\,dt\Big\}$$

holds for all f in L.

We conclude by showing that the space $\mathscr{D}'_+$ is algebraically isomorphic to the space P of perfect operators.

Theorem. *The mapping* $F\mapsto\{F\}$ *from* $\mathscr{D}'_+$ *into* P *is linear and one-to-one.*

PROOF. The linearity follows from the bilinearity of convolution. Suppose $\{F\}=0$. For $\varphi\in\mathscr{D}_-$ there exist $x_i<a_i$ such that $\varphi(U)=0$ if some $u_i\geqq x_i$. If we define $p(U)=\varphi(X-U)$, then $p\in Q$ and

$$\langle F,\varphi\rangle = F*p(X)=\{F\}p(X)=0.$$

Since φ in $\mathscr{D}_-$ was arbitrary, it follows that $F=0$.

Corollary. *If f and g are in L and* $\{f\}=\{g\}$, *then* $f=g$ *almost everywhere.*

Theorem. *The mapping* $F\mapsto\{F\}$ *is a linear bijection of* $\mathscr{D}'_+$ *onto* P.

PROOF. It only remains to prove the surjectivity. Let A be a perfect operator. Let $q_1,q_2,q_3,\ldots$ be a "delta sequence" in Q. For any φ in $\mathscr{D}_-$ there exist, as before, $x_i<a_i$ such that $\varphi(U)=0$ if some $u_i\geqq x_i$. If we define $p(U)=\varphi(X-U)$, then $p\in Q$ and

$$Ap(X)=\lim_{n\to\infty}Aq_n*p(X)=\lim_{n\to\infty}\langle Aq_n(U),\,p(X-U)\rangle=\lim_{n\to\infty}\langle Aq_n(U),\,\varphi(U)\rangle.$$

Since the sequence $\langle Aq_n,\varphi\rangle$ converges for all φ in $\mathscr{D}_-$, we infer from the sequential completeness of $\mathscr{D}'_+$ the existence of F in $\mathscr{D}'_+$ such that

$$\langle F,\varphi\rangle=\lim_{n\to\infty}\langle Aq_n,\varphi\rangle \quad (\text{all } \varphi \text{ in } \mathscr{D}_-).$$

Now, for any q in Q, we have

$$\{F\}q(X)=\langle F(U),\,q(X-U)\rangle=\lim_{n\to\infty}\langle Aq_n(U),\,q(X-U)\rangle=$$

$$=\lim_{n\to\infty}Aq_n*q(X)=\lim_{n\to\infty}q_n*Aq(X)=Aq(X).$$

Thus, $A=\{F\}$.

Bibliography

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