On the sums of permanents generated by certain Vandermonde matrices

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Summary. It is well-known that the calculation of permanents is generally a more complicated problem than that of determinants. In this paper we deal with the following problem: Let V be the unitary Vandermonde matrix generated by the nth $(n \geqq 2)$ roots of unity, and let U be the Vandermonde matrix generated by the elements $1, \ldots, p-1$, where p is an odd prime number. Our aim is to calculate certain sums of permanents of matrices, which are included in the Cauchy—Binet expansion of VV^*, and of $(\mathrm{Det}\, U)^{p-2} U \,\mathrm{adj}\, U^*$, respectively.

1. Introduction and the results

Let $n \geqq 2$ be an integer. Let the $n \times n$ matrix $A=(a_{jk})$ with complex entries be given. Denote by A^* the conjugate transpose of A. Let M be the $n \times n$ matrix with all its entries ones. E is the unit matrix.

The permanent of A, denoted by $\mathrm{Per}\, A$, is defined as follows:

$$\mathrm{Per}\, A = \sum_{(i_1, \ldots, i_n)} a_{1i_1} \ldots a_{ni_n},$$

where $(i_1, \ldots, i_n)$ runs over the full symmetric group.

Let β_j, $1 \leqq j \leqq n$, be non-negative integers satisfying the equality $\beta_1 + \ldots + \beta_n = n$. Then $C_{\beta_1 \ldots \beta_n}(A)$ denotes the $n \times n$ matrix, which contains certain columns of A, namely the jth column of A appears β_j times in $C_{\beta_1 \ldots \beta_n}(A)$.

Let $1, \omega_1, \ldots, \omega_{n-1}$ be the nth roots of unity different from one another. Let p be an odd prime number. Let the matrices

$$V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_1 & \omega_2 & \ldots & \omega_{n-1} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 1 & \omega_1^{n-1} & \omega_2^{n-1} & \ldots & \omega_{n-1}^{n-1} \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 3 & \ldots & p-1 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 1 & 2^{p-2} & 3^{p-2} & \ldots & (p-1)^{p-2} \end{pmatrix}$$

of Vandermonde type be given, and let

$$U^{-1} = (\mathrm{Det}\, U)^{p-2}\, \mathrm{adj}\, U^*. \tag{1.1}$$

It is known that

$$VV^* = E, UU^{-1} \equiv E (\mathrm{mod}\, p). \tag{1.2}$$

Let $\beta_1 = k$, and let

$$A_k(n) = \sum \frac{1}{\beta_2! \dots \beta_n!} |\mathrm{Per}\, C_{k\beta_2 \dots \beta_n}(V)|^2,$$

$$B_k = \sum \frac{1}{\beta_2! \dots \beta_{p-1}!} \mathrm{Per}\, C_{k\beta_2 \dots \beta_{p-1}}(U) \cdot \mathrm{Per}\, C_{k\beta_2 \dots \beta_{p-1}}(U^{-1*}),$$

where the summation is extended over all non-negative integers β_j, $2 \leqq j \leqq n$, and $2 \leqq j \leqq p-1$, satisfying the equalities $\beta_2 + \dots + \beta_n = n-k$, and $\beta_2 + \dots + \beta_{p-1} = p-1-k$, respectively.

The aim of the present paper is to prove the following two Theorems:

Theorem 1.1. *For* $0 \leqq k \leqq n$ *the identity*

$$A_k(n) = \frac{(-1)^{n-k} n!}{n^n (n-k)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (n-j)! (-n)^j \tag{1.3}$$

holds.

Theorem 1.2. *For* $0 \leqq k \leqq p-1$ *the congruence*

(1.4)

$$B_k \equiv (-1)^{p-k} \sum_{j=0}^{p-1-k} (-1)^j (p-1)^j [j!(p-1-k-j)!]^{p-2} (p-1-j)! \quad (\mathrm{mod}\, p)$$

holds.

Using theorems of the papers [1] and [3] both Theorems can be generalized. For the proofs we need a Lemma.

Let

$$C = (a_{jk}),\ D = (b_{jk}) \tag{1.5}$$

be $(n+1) \times (n+1)$ upper triangular matrices with entries

$$\begin{cases} a_{j\alpha} = 0, & 0 \leqq \alpha \leqq j-1, \\ a_{jj+s} = \dfrac{1}{s!}; & 0 \leqq s \leqq n-j, \end{cases} \tag{1.6}$$

$$\begin{cases} b_{j\alpha} = 0, & 0 \leqq \alpha \leqq j-1, \\ b_{jj+s} = \dfrac{(-1)^s}{s!}, & 0 \leqq s \leqq n-j, \end{cases} \tag{1.7}$$

respectively.

Lemma 1.1. *D is the inverse of C, i.e.*

$$CD = E.$$

Proof. Let $CD = (c_{jk})$. Since C and D are upper triangular matrices,

$$c_{jk} = 0, \quad 0 \leqq k < j \leqq n. \tag{1.8}$$

If $j \leqq k$, then

$$c_{jk} = \sum_{s=0}^{k-j} a_{jj+s} b_{j+sk} = \sum_{s=0}^{k-j} \frac{(-1)^{k-j-s}}{s!(k-j-s)!}$$

by (1.6) and (1.7). Thus

$$c_{jj} = 1, \quad 0 \leqq j \leqq n. \tag{1.9}$$

If $j<k$, we have

$$c_{jk} = \frac{(-1)^{k-j}}{(k-j)!} \sum_{s=0}^{k-j} (-1)^s \binom{k-j}{s} = 0 \tag{1.10}$$

by the well-known combinatorial identity.

Formulas (1.8), (1.9) and (1.10) give us the statement of the Lemma.

The proofs of Theorems 1.1 and 1.2 can be found in sections 2 and 3, respectively.

2. The proof of Theorem 1.1.

For the purpose of Theorem 1.1 let $A^{(n)}(x, y)$ be the $n \times n$ matrix with entries y, except the main diagonal, in which the entries are x. It is obvious that

$$A^{(n)}(x, y) = yM + (x-y)E. \tag{2.1}$$

Using the variables $z=x-y$ and y instead of x and y the eigenvalues of the matrix (2.1) are the following:

$$\lambda_0 = z + ny, \quad \lambda_k = z, \quad 1 \leqq k \leqq n-1.$$

It is known that the spectral representation

$$A^{(n)}(x, y) = V \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & (0) \\ & (0) & \ddots & \\ & & & \lambda_{n-1} \end{pmatrix} V^* \tag{2.2}$$

holds. Using representation (2.1)

$$\operatorname{Per} A^{(n)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} k!\, y^k z^{n-k}. \tag{2.3}$$

Applying the Cauchy—Binet expansion formula ([2], Theorem 1.3) we obtain by (2.2) that

$$\operatorname{Per} A^{(n)}(x, y) = \sum \frac{\lambda_0^{\beta_0} \lambda_1^{n-\beta_0}}{\beta_1! \dots \beta_n!} |\operatorname{Per} C_{\beta_1 \beta_2 \dots \beta_n}(V)|^2,$$

where the summation is extended over all non-negative integers β_j, $1 \leqq j \leqq n$, satisfying the equality $\beta_1 + \dots + \beta_n = n$. It is obvious that

$$\operatorname{Per} A^{(n)}(x, y) = \sum_{k=0}^{n} \frac{1}{k!} (z+ny)^k z^{n-k} A_k'(n). \tag{2.4}$$

By (2.3) and (2.4) the identity

$$\sum_{k=0}^{n} \binom{n}{k} k!\, y^k z^{n-k} = \sum_{k=0}^{n} \frac{A_k(n)}{k!} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{n-k+\nu} n^{k-\nu} y^{k-\nu}$$

holds in y and in z. After a rearrangement this identity has the form

$$\sum_{k=0}^{n} n^k y^k z^{n-k} \sum_{\nu=k}^{n} \binom{\nu}{k} \frac{A_\nu(n)}{\nu!} = \sum_{k=0}^{n} \binom{n}{k} k!\, y^k z^{n-k}.$$

If $z=1$, we get the polynomial identity

$$\sum_{k=0}^{n} \left[n^k \sum_{\nu=k}^{n} \binom{\nu}{k} \frac{A_\nu(n)}{\nu!} \right] y^k = \sum_{k=0}^{n} \binom{n}{k} k!\, y^k.$$

Identifying the coefficients we have

$$\sum_{\nu=k}^{n} \frac{A_\nu(n)}{(\nu-k)!} = c_k, \quad 0 \leqq k \leqq n, \tag{2.5}$$

where

$$c_k = \frac{n!\, k!}{(n-k)!\, n^k}, \quad 0 \leqq k \leqq n. \tag{2.6}$$

Let v and u be the column vectors with components $A_k(n)$ and c_k, $0 \leqq k \leqq n$, respectively. Then (2.5) may be written in the form of a linear equation system $Cv=u$ with matrix defined by the first formula of (1.5). From Lemma 1.1 we have $v=Du$, where D is defined by the second formula of (1.5). Thus we obtain

$$A_k(n) = \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} c_{k+j}, \quad 0 \leqq k \leqq n. \tag{2.7}$$

Substituting (2.6) into (2.7) we get that

$$A_k(n) = \frac{n!}{n^k (n-k)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (k+j)! \left(-\frac{1}{n}\right)^j,$$

and after a short calculation the solution (1.3).

From the first formula of (1.2) we have

$$\sum_{k=0}^{n} \frac{1}{k!} A_k(n) = 1,$$

and from here we obtain subsequent consequence using (1.3).

Corollary 2.1. *The combinatorial identity*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (n-j)!\, (-n)^j = n^n$$

holds.

In the case of $k=n-1$ we get from (1.3) that $A_{n-1}(n)=0$, i.e.

$$C_{n-1\beta_2\dots\beta_n}(V) = 0 \quad \text{for} \quad \beta_2+\dots+\beta_n = 1.$$

This result can be obtained directly by

$$\sum_{j=0}^{n-1} \omega_j^k = 0, \quad 1 \leqq k \leqq n-1.$$

3. The proof of Theorem 1.2.

In the sequel the following elementary congruences will be applied several times. Let the prime number p be given. Then

$$(p-1)! \equiv -1 \pmod p,$$

$$k^{n-1} \equiv 1 \pmod p \quad \text{if and only if} \quad k \not\equiv 0 \pmod p, \tag{3.1}$$

$$k! \not\equiv 0 \pmod p \quad \text{if and only if} \quad 1 \leqq k \leqq p-1.$$

As in section 2 let p be an odd prime number. Since

$$\operatorname{Det} U = \prod_{k=1}^{p-2} k! \equiv \prod_{k=2}^{p-1} (p-k)^{k-1} \not\equiv 0 \pmod p, \tag{3.2}$$

we have

$$\operatorname{Per} C_{\beta_1\dots\beta_{p-1}}(U^{-1*}) = (\operatorname{Det} U)^{(p-1)(p-2)} \operatorname{Per} C_{\beta_1\dots\beta_{p-1}}(\operatorname{adj} U) \equiv$$
$$\equiv \operatorname{Per} C_{\beta_1\dots\beta_{p-1}}(\operatorname{adj} U) \pmod p$$

by (3.2), and thus we get for $0 \leqq k \leqq p-1$ that

$$B_k \equiv \sum \frac{1}{\beta_2!\dots\beta_{p-1}!} \operatorname{Per} C_{k\beta_2\dots\beta_{p-1}}(U) \operatorname{Per} C_{k\beta_2\dots\beta_{p-1}}(\operatorname{adj} U) \pmod p,$$

where the summation is extended over all non-negative integers β_j, $2 \leqq j \leqq p-1$, satisfying the equation $\beta_2+\dots+\beta_{p-1}=p-1-k$.

Let x and y be arbitrary integers, and let $z=x-y$. Using the polynomial

$$f(\omega) = x+y(\omega+\dots+\omega^{p-2}) = z+y(1+\omega+\dots+\omega^{p-2}),$$

it is obvious that

$$f(1) = f(\omega_0) = z+(p-1)y = \lambda_0,$$

$$f(k) \equiv f(\omega_{k-1}) = z = \lambda_{k-1} \pmod p, \quad 2 \leqq k \leqq p-1.$$

If

$$B(x,y) = U \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & (0) \\ & (0) & \ddots & \\ & & & \lambda_{p-2} \end{pmatrix} U^{-1},$$

we get from Theorem 1 of the paper [1] that

$$B(x,y) \equiv A^{(p-1)}(x,y) \,(\mathrm{mod}\, p),$$

where $A^{(p-1)}(x,y)$ is defined by (2.2). Thus we have

$$\text{(3.3)} \qquad \mathrm{Per}\, B(x,y) \equiv \mathrm{Per}\, A^{(p-1)}(x,y) \,(\mathrm{mod}\, p).$$

Applying the Cauchy—Binet expansion formula ([2], Theorem 1.3) the congruence

$$\mathrm{Per}\, B(x,y) \equiv \sum_{k=0}^{p-1} \frac{1}{k!} (z+(p-1)y)^k z^{p-1-k} B_k \,(\mathrm{mod}\, p)$$

holds. Using (2.4) we obtain

$$\text{(3.4)} \qquad \sum_{k=0}^{p-1} \frac{1}{k!} (B_k - A_k(p-1)) [z+(p-1)y]^k z^{p-1-k} \equiv 0 \;(\mathrm{mod}\, p)$$

by (3.3).

Let $z=1$, and let us substitute the numbers $p, p-1, \ldots, 3, 2$ into y. Then we get the homogeneous linear congruence system

$$\text{(3.5)} \qquad \sum_{k=0}^{p-1} \frac{1}{k!} (B_k - A_k(p-1)) j^k \equiv 0 \;(\mathrm{mod}\, p)$$

for $1 \leqq j \leqq p-1$ from (3.4).

We assert that the congruence

$$\text{(3.6)} \qquad \frac{1}{(p-1)!} (B_{p-1} - A_{p-1}(p-1)) \equiv 0 \;(\mathrm{mod}\, p)$$

holds. Namely

$$\text{(3.7)} \qquad \frac{A_{p-1}(p-1)}{(p-1)!} = (p-1)!.$$

On the other hand the only solution of the linear congruence system

$$U^* \chi \equiv \begin{pmatrix} \mathrm{Det}\, U \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \chi = (x_k)$$

is

$$x_j \equiv (p-1)\, \mathrm{Det}\, U \;(\mathrm{mod}\, p)$$

for $1 \leqq k \leqq p-1$. Thus

$$\text{(3.8)} \qquad \frac{B_{p-1}}{(p-1)!} \equiv (p-1)!(p-1)^{p-1} (\mathrm{Det}\, U)^{p-1} \equiv (p-1)! \;(\mathrm{mod}\, p).$$

From (3.7) and (3.8) we get our statement (3.6).

Taking (3.6) into account we get that (3.5) can be reduced to the homogeneous linear congruence system

$$\sum_{k=0}^{p-2} \frac{1}{k!} \big(B_k - A_k(p-1)\big) j^k \equiv 0 \pmod p, \quad 1 \leqq j \leqq p-1,$$

with matrix U, which has only the trivial solution by (3.2). Thus, using also (3.6),

(3.9) $$B_k \equiv A_k(p-1) \equiv (p-1)^{p-1} A_k(p-1) \pmod p$$

where for $0 \leqq k \leqq p-1$

$$(p-1)^{p-1} A_k(p-1) = (-1)^{p-1-k}(p-1)! \sum_{j=0}^{p-1-k} \binom{p-1-k}{j} (p-1-j)!(-p+1)^j$$

is an integer. From here we get the statement (1.4) of Theorem 1.2 by (3.9) using the elementary congruences (3.1).

Substituting $n=p-1$ into Corollary 2.1 we get the following Corollary using congruences (3.1).

Corollary 3.1. *The congruence*

$$\sum_{k=0}^{p-1} (-1)^{p-k} (k!)^{p-2} \sum_{\nu=0}^{p-1-k} (-1)^{\nu} (p-1)^{\nu} \cdot$$
$$\cdot [\nu!(p-1-k-\nu)!]^{p-2} (p-1-\nu)! \equiv 1 \pmod p$$

holds.

Since $A_{p-2}(p-1)=0$, we have the congruence $B_{p-2} \equiv 0 \pmod p$ by (3.9). Moreover $B_{p-1} \equiv 1 \pmod p$ from (3.8).

References

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