Remarks on Hyers's theorem

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Abstract. In this note we study the connections between Hyers's theorem on the stability of linear functional equations and amenability. We prove a stronger version of Hyers's theorem on amenable semigroups and state some open problems concerning Hyers's theorem.

In this note C denotes the set of complex numbers and throughout the paper G is a fixed semigroup with identity e. We shall use the operators of left and right translations ${}_yT$ and T_y defined for any function $f\colon G\to C$ by

$${}_yTf(x)=f(yx),\quad T_yf(x)=f(xy)$$

further, the left and right difference operators ${}_y\Delta$ and Δ_y defined by

$${}_y\Delta={}_yT-I,\quad \Delta_y=T_y-I$$

where I denotes the identity operator. For the products ${}_{y_1}\Delta\dots{}_{y_n}\Delta$ and $\Delta_{y_n}\dots\Delta_{y_1}$ we use the notations ${}_{y_1,\dots,y_n}^{\;n}\Delta$ and $\Delta^n_{y_1,\dots,y_n}$, respectively further ${}^0\Delta=\Delta^0=I$. For any integer $n\geqq 0$ we let

$$B_n(G)=\{f:G\to C\,|\,(x,y_1,\dots,y_{n+1})\to\Delta^{n+1}_{y_1,\dots,y_{n+1}}f(x)\ \text{is bounded}\},$$

$$P_n(G)=\{f:G\to C\,|\,\Delta^{n+1}_{y_1,\dots,y_{n+1}}f(x)=0\quad\text{for all}\quad x,y_1,\dots,y_{n+1}\quad\text{in}\quad G\},$$

$$M_n(G)=\{f:G\to C\,|\,\Delta^n_{y,\dots,y}f(x)=n!f(y)\quad\text{for all}\quad x,y\quad\text{in}\quad G\}.$$

Obviously we have $M_n(G)\subseteqq P_n(G)\subseteqq B_n(G)$ for all n. The elements of $P_n(G)$ are called polynomials of degree at most n and the elements of $M_n(G)$ are called monomials of degree n. The elements of $B_0(G)$ are just the bounded complex functions on G, further $P_0(G)$ and $M_0(G)$ can be identified with C. We note, that $B_n(G)$ and $P_n(G)$ are also meaningful for $n=-1$, and $B_{-1}(G)=B_0(G)$, $P_{-1}(G)=\{0\}$.

We introduce a seminorm on $B_n(G)$ for $n\geqq 0$ by

$$|||f|||_n=\sup_{y_1\dots y_{n+1}}|\Delta^{n+1}_{y_1,\dots,y_{n+1}}f(e)+(-1)^nf(e)|.$$

If $n=-1$ we let $|||f|||_{-1}=|||f|||_0$. It is easy to see that $|||f|||_n=0$ if and only if f is a polynomial of degree at most n with $f(e)=0$. Obviously, $|||f|||_0$ is just the sup-norm of f in $B_0(G)$.

Now we formulate a property of G for all n, which relates to Hyers's theorem. We say that G has the property (H_n) if there exists a mapping $F_n\colon B_n(G)\to P_n(G)$ such that

(H1) F_n is linear,

(H2) $F_n(p)=p$ for all p in $P_n(G)$ with $p(e)=0$,

(H3) $I-F_n$ maps $B_n(G)$ into $B_0(G)$ continuously.

This definition is motivated by the fact that Hyers's theorem can be formulated as follows:

Theorem 1. (Hyers, [5].) *If G is a commutative semigroup with identity, then it has property (H_n) for all n.*

(See also [1], [6].) At the same time, not all semigroups with identity have property (H_n). Forti [3] gave an example, which shows that the free group on two generators fails to have property (H_1). Nevertheless, there are lots of noncommutative semigroups having the property (H_n) for all n. To see this we need the notion of amenable semigroups. We say that G is amenable, if there exists a nonnegative linear functional $M: B_0(G)\to C$ which is translation invariant and normalized in the sense $M(1)=1$. Nonnegativity means that $M(f)\geqq 0$ for all $f\geqq 0$, and translation invariance means, that $M({}_yTf)=M(T_yf)=M(f)$ for all y in G and f in $B_0(G)$. Such a functional M is called an invariant mean on G. It is known (see e.g. [4]), that all commutative semigroups amenable, but the free group on two generators fails to be amenable. The problem of characterizing all amenable groups or semigroups is still unsolved (see [4]).

Returning to Hyers's theorem, we proved in [7], that if G is amenable, then it has property (H_n) for all n. It means, we have

Theorem 2. (Székelyhidi, [7].) *If G is an amenable semigroup with identity, then it has property (H_n) for all n.*

Actually, we can prove more, but first we formulate another property of G, which seems to be stronger then (H_n) and may enlighten the relation between amenability and the validity of Hyers's theorem. We say, that G has property (P_n), if there exists a mapping $\Phi_n: B_n(G)\to M_n(G)$ such that

(P1) Φ_n is linear,

(P2) $\Phi_n(m)=m$ for all m in $M_n(G)$,

(P3) $I-\Phi_n$ maps $B_n(G)$ into $B_{n-1}(G)$ continuously,

(P4) Φ_n is translation invariant.

The aim of this work is to prove the following statements:

1. If (P_k) holds for $k=1, \dots, n$ then (H_k) holds for $k=1, \dots, n$.
2. (P_0) is equivalent to amenability.
3. (P_0) implies (P_n) for all n.

Theorem 3. *If G has property (P_k) for $k=1, \dots, n$, then it has property (H_k) for $k=1, \dots, n$.*

Proof. Let f be arbitrary in $B_k(G)$, then by (P_k) we have

$$f=\Phi_k(f)+f_1$$

where f_1 belongs to $B_{k-1}(G)$, and $|||f_1|||_{k-1}\leqq C_1|||f|||_k$. Now, by (P_{k-1}) we have

$$f_1=\Phi_{k-1}(f_1)+f_2$$

with some f_2 in $B_{k-2}(G)$, and $|||f_2|||_{k-2} \leqq C_2|||f|||_k$. Continuing this process we obtain

$$f = \Phi_k(f) + \Phi_{k-1}(f_1) + \ldots + \Phi_1(f_{k-1}) + f_k,$$

where f_k is in $B_0(G)$ and $|||f_k|||_0 \leqq C_k|||f|||_k$. If we define

$$F_k(f) = \Phi_k(f) + \Phi_{k-1}(f_1) + \ldots + \Phi_1(f_{k-1})$$

then we see that F_k maps $B_k(G)$ into $P_k(G)$ linearly, and $I - F_k$ maps $B_k(G)$ into $B_0(G)$ continuously. Hence we have (H1) and (H3). To prove (H2), first we show that $\Phi_k(p)=0$ for all p in $P_{k-1}(G)$. By the results of [7], each polynomial has a unique representation as the sum of monomials. Hence, it is enough to prove that $\Phi_k(m)=0$ for all monomials m of degree smaller than k. Of course, we may suppose, that $k \leqq 1$. First we show that $\Phi_k(1)=0$. Let a be any additive function (that is, any element of $M_1(G)$), then we have

$$\Phi_k(a) = \Phi_k(T_y a) = \Phi_k\big(a + a(y)\big) = \Phi_k(a) + a(y)\,\Phi_k(1).$$

It follows, that $\Phi_k(1) \neq 0$ implies $a=0$ for all a in $M_1(G)$. But this means, that $M_n(G)=\{0\}$ for all $n \geqq 1$, and in particular $\Phi_k(1)=0$. Now suppose that $\Phi_k(m)=0$ for all monomials m of degree smaller than $k-1$, and let m be arbitrary in $M_{k-1}(G)$ $(k \geqq 2)$. Let $a \neq 0$ be any element of $M_1(G)$ (we may suppose by the above considerations, that there exists such an a). Then $m \cdot a$ belongs to $M_k(G)$ and we have

$$\Phi_k(m \cdot a) = \Phi_k\big(T_y(m \cdot a)\big) = \Phi_k(T_y m \cdot a) + a(y) \cdot \Phi_k(T_y m).$$

It follows from [7], but it can also be proved directly, that

$$T_y m = m + p_y + q_y,$$

where p_y is in $M_{k-2}(G)$ and q_y is in $P_{k-3}(G)$. Hence $q_y \cdot a$ is in $P_{k-2}(G)$, and by our assumption we have

$$\Phi_k(m \cdot a) = \Phi_k(m \cdot a) + \Phi_k(p_y \cdot a) + a(y) \cdot \Phi_k(m),$$

that is

$$\Phi_k(p_y \cdot a) + a(y) \cdot \Phi_k(m) = 0,$$

for all y in G. If $y=x$, then $\Phi_k(m) \neq 0$ implies that $a \cdot \Phi_k(m)$ is of degree $k+1$, and this contradicts the fact, that Φ_k maps into $M_k(G)$. This contradiction proves our statement. Now let p be arbitrary in $P_k(G)$ with $p(e)=0$. We express p as a sum of monomials

$$p = m_k + m_{k-1} + \ldots + m_1$$

with m_j in $M_j(G)$ $(j=1, \ldots, k)$. Hence, with the above notations we have

$$\Phi_k(p) = m_k$$

and

$$\Phi_k(p_j) = m_j$$

where $p_j = p_{j+1} - \Phi_{j+1}(p_{j+1})$ for $j=1, \ldots, k-1$, and $p_k = p$. It follows, that

$$F_k(p) = \Phi_k(p) + \Phi_{k-1}(p_{k-1}) + \ldots + \Phi_1(p_1) = m_k + \ldots + m_1 = p,$$

wich was to be proved.

Theorem 4. *G is amenable if and only if it has property* (P_0).

Proof. Let G be amenable, and let M be any invariant mean on G. If we let $\Phi_0=M$, then obviously Φ_0 maps $B_0(G)$ into $M_0(G)=C$, further Φ_0 is linear and translation invariant. Condition (P2) is trivially fulfilled, and we evidently have (P3), as for all f in $B_0(G)$

$$|||f-\Phi_0(f)|||_{-1} = |||f-M(f)|||_0 \leqq |||f||_0+|M(f)| \leqq 2||f||_0$$

as the norm of the normalized linear functional M is obviously 1. Conversely, suppose that G has property (P_0). It means, that there exists a nonzero, translation invariant bounded linear functional Φ_0 on $B_0(G)$. Then obviously the total variation $|\Phi_0|$ (see e.g. [2]) of Φ_0 is a nonzero, nonnegative translation invariant linear functional on $B_0(G)$. As $|\Phi_0|$ is nonzero and nonnegative, we have $|\Phi_0|(1)\neq 0$, and then $(|\Phi_0|(1))^{-1}$. Φ_0 is an invariant mean on G.

Theorem 5. *If G has property* (P_0), *then it has property* (P_n) *for all n.*

Proof. By theorem 4, G is amenable. Let M be any invariant mean on G. Then for all f in $B_n(G)$ the function $k(f)$ defined by

$$k(f)(x) = (-1)^n M_{y_1}\dots M_{y_n}(\Delta^{n+1}_{y_1,\dots,y_n,x} f(e))$$

is bounded, and $f-k(f)$ belongs to $P_n(G)$ (see [7]). Obviously $k(f)(e)=0$, hence we have

$$f = m_n+m_{n-1}+\dots+m_1+k(f)+f(e),$$

where m_k belongs to $M_k(G)$. It is easy to see (and it follows from [7]) that

$$m_n(x) = \frac{1}{n!}\,{}^{\,n}_{x,\dots,x}\Delta(f-k(f))(e)$$

holds for all x in G. Now let

$$\Phi_n(f) = m_n.$$

Obviously Φ_n is linear and $\Phi_n(f)$ is in $M_n(G)$. On the other hand,

$$f-\Phi_n(f) = m_{n-1}+\dots+m_1+k(f)+f(e),$$

which is an element of $B_{n-1}(G)$. If m is in $M_n(G)$, then

$$\Delta^{n+1}_{y_1,\dots,y_n,x} m(e) = 0$$

for all $y_1,\dots,y_n,x$ in G, hence $k(m)=0$ and

$$\Phi_n(m) = m.$$

It means, that (P1) and (P2) are fulfilled. We show, that Φ_n is translation invariant. Indeed, for all x, y in G we have

$$\Phi_n(T_y f)(x) = \frac{1}{n!}\Delta^n_{x,\dots,x}(T_y f)(e)+\frac{1}{n!}(-1)^{n+1}M_{y_1}\dots M_{y_n}(\Delta^n_{x,\dots,x}\Delta^n_{y,\dots,y_n}(T_y f)(e)) =$$

$$= \frac{1}{n!}\Delta^n_{x,\dots,x} f(y)+\frac{1}{n!}(-1)^{n+1}M_{y_1}\dots M_{y_n}(\Delta^n_{x,\dots,x}\Delta^n_{y_1,\dots,y_n} f(y)) = = \Phi_n(f)(x).$$

Similarly, we obtain $\Phi_n({}_yTf)(x)=\Phi_n(f)(x)$.

Finally, to prove continuity in (P3), we notice that

$$f=\Phi_n(f)+p_{n-1}+k(f)+f(e),$$

where p_{n-1} belongs to $B_{n-1}(G)$, and $p(e)=0$. Further

$$|k(f)(x)+f(e)|=\left|(-1)^n M_{y_1}\dots M_{y_n}(\Delta^{n+1}_{y_1,\dots,y_n,x}f(e)+(-1)^n f(e))\right|\leqq |||f|||_n$$

for all x in G, that is $|||k(f)+f(e)|||_0\leqq|||f|||_n$. On the other hand, we obviously have

$$|||k(f)+f(e)|||_{n-1}\leqq C_n|||k(f)+f(e)|||_0,$$

where C_n depends only on n. Now it follows

$$|||f-\Phi_n(f)|||_{n-1}=|||p_{n-1}+k(f)+f(e)|||_{n-1}=|||\,k(f)+f(e)|||_{n-1}\leqq$$
$$\leqq C_n|||k(f)+f(e)|||_0\leqq C_n|||f|||_n,$$

and the theorem is proved.

Corollary 6. *Amenability of G implies that G has the property* (P_n), *and hence the property* (H_n) *for all n.*

Remarks 7. In theorem 3 we were unable to prove that (P_n) is stronger than (H_n) for a *fixed n*. It would be also interesting to know, whether the converse of theorem 5. is true. If so, at least for $n=1$, then amenability would equivalent to (P_1), which is a stronger form of Hyers's original theorem. We can state these questions as open problems:

1. Is it true, that (P_n) implies (H_n)?
2. Is it true, that (P_1) implies amenability?
3. Is it true, that (H_1) implies amenability?

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