Operational calculus in algebras

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In this paper on the basis of *Bittner operational calculus*

$$CO(L^0, L^1, S, T(q), s(q), Q)$$

— see [1], [2] we introduce a new operational calculus $CO(L^0, L^1, S_p, T_p(q), s_p(q), Q)$ and the multiplication $\circ$ such that the derivative S_p, and the limit condition $s_p(q)$ satisfy conditions (1) and (2).

Let an operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$ be given, where $L^1 \subset L^0$, L^1, L^0 are commutative algebras with unity 1, and with the multiplication $\cdot$, such that for $f, g \in L^1$

$$S(f\cdot g) = (Sf)\cdot g + f\cdot(Sg), \tag{1}$$

$$s(q)(f\cdot g) = (s(q)f)\cdot(s(q)g). \tag{2}$$

Theorem 1. *If there exists a solution* $u \in \mathrm{Inv}$ *of the abstract differential equation*

$$Su = pu \tag{3}$$

with the condition

$$s(q)u = u_0, \quad u\in L^1, \quad p\in L^0, \quad u_0\in \mathrm{Ker}\, S, \tag{4}$$

then abstract differential equation (3) *with condition* (4) *has only one solution.*

PROOF. If equation (3) with condition (4) had two solutions u, v we would get

$$Su = pu, \quad Sv = pv$$

and

$$s(q)u = u_0, \quad s(q)v = u_0$$

i.e.

$$S(u-v) = p(u-v), \quad s(q)(u-v) = 0$$

because operations S and $s(q)$ are linear operations. On the basis of theorems with [8] $u-v=0$, then $u=v$. (Theorem 1 is also true when multiplication $\cdot$ is non-commutative.)

Definition 1. We will say that there exists an element $u \overset{\text{def}}{=} E_1^{T(q)p}$ if and only if $E_1^{T(q)p}$ is a solution of the abstract differential equation

$$Su = pu \tag{5}$$

with condition

(6) $$s(q)u = \mathbf{1}, \quad \text{where} \quad u \in L^1, \quad p \in L^0 \quad (\mathbf{1} \in \operatorname{Ker} S - \text{see [7]})$$

and $E_1^{T(q)p} \in \mathrm{Inv}$.

Corollary 1. *If there exists an element $E_1^{T(q)p}$, then formula*

(7) $$E_1^{T(q)p} E_1^{-T(q)p} = \mathbf{1}$$

is true

PROOF. Let $v = \mathbf{1} - E_1^{T(q)p} E_1^{-T(q)p}$. Finding Sv and $s(q)v$ we will get $Sv=0$, $s(q)v=0$. From the last tuo facts it follows that $v=0$, i.e. formula (7) is true.

Theorem 2. *If there exists an element $E_1^{T(q)p}$ then three operations:*

(8) $$S_p u \stackrel{\text{df}}{=} Su + pu,$$

(9) $$T_p(q)f \stackrel{\text{df}}{=} [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p},$$

(10) $$s_p(q)u \stackrel{\text{df}}{=} (s(q)u) \cdot E_1^{-T(q)p}$$

satisfy axioms of operational calculus, where $u \in L^1, f \in L^0$. Operation S_p is a derivative, operation $T_p(q)$ is an integral, operation $s_p(q)$ is a limit condition.

PROOF. Operations S_p, $T_p(q)$, $s_p(q)$ are linear operations. Applying the axioms of operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$ and theorems about derivative, integral and limit condition in algebras (see [3]) we will get

$$\begin{aligned} S_p T_p(q) f &= S_p\{[T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p}\} = \\ &= S\{[T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p}\} + \\ &+ p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= f \cdot E_1^{T(q)p} \cdot E_1^{-T(q)p} - p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} + \\ &+ p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = f, \end{aligned}$$

i.e. $$S_p T_p(q) f = f \quad \text{for} \quad f \in L^0.$$

$$\begin{aligned} T_p(q) S_p u &= \{T(q)[(Su + pu) \cdot E_1^{T(q)p}]\} \cdot E_1^{-T(q)p} = \\ &= \{T(q)[(Su) \cdot E_1^{T(q)p}]\} \cdot E_1^{-T(q)p} + \\ &+ [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= u \cdot E_1^{T(q)p} \cdot E_1^{-T(q)p} - [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} - \\ &- [s(q)(u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} + \\ &+ [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= u - (s(q)u) \cdot E_1^{-T(q)p}, \end{aligned}$$

i.e. $$T_p(q) S_p u = u - s_p(q) u \quad \text{for} \quad u \in L^1,$$

so operations S_p, $T_p(q)$, $s_p(q)$ satisfy the axioms of Bittner operational calculus.

Corollary 2. *If L^1, L^0 are non-commutative algebras then theorem 2 is true for $p=\alpha\mathbf{1}$, $\alpha\in R$.*

Corollary 3. *For a given derivative S_p, integral $T_p(q)$ and limit condition $s_p(q)$ the operation*

$$\tilde{S}_p u \overset{\text{df}}{=} a\cdot(S_p u), \quad a\in L^\circ, \quad a\in Inv, \quad u\in L^1$$

is a derivative, the operation

$$\tilde{T}_p(q)f \overset{\text{df}}{=} T_p(q)(a^{-1}\cdot f), \quad f\in L^\circ$$

is an integral, the operation

$$\tilde{s}_p(q)u \overset{\text{df}}{=} s_p(q)u, \quad u\in L^1$$

is a limit condition (compare also [4]*).*

Theorem 3. *If $a_1, a_2\in L^0$, $a_1\in \mathrm{Inv}$ and if there exists an element $E_1^{T(q)(a_1^{-1}a_2)}$ then the abstract differential equation*

$$a_1\cdot Su + a_2\cdot u = f \tag{11}$$

with condition

$$s(q)u = u_0, \quad \textit{where} \quad u\in L^1, \quad f\in L^0, \quad u_0\in \mathrm{Ker}\, S \tag{12}$$

has only one solution defined by formula

$$u = [T(q)(a_1^{-1}\cdot f\cdot E_1^{T(q)(a_1^{-1}a_2)})]\cdot E_1^{-T(q)(a_1^{-1}a_2)} + u_0\cdot E_1^{-T(q)(a_1^{-1}a_2)}. \tag{13}$$

Proof. The proof of the theorem follows directly from theorem 2 and corollary 3, and from the theorems of the operational calculus.

Definition 2. If there exists an element $E_1^{T(q)p}$ then for the elements $x, y\in L^0$ we will define the multiplication $x\circ y$ by the formula

$$x\circ y \overset{\text{df}}{=} E_1^{T(q)p}\cdot x\cdot y. \tag{14}$$

Properties of the multiplication $\circ$.

1. For the multiplication $\circ$ unity $\mathbf{1}_0$ is defined by the formula

$$\mathbf{1}_0 = E_1^{-T(q)p}. \tag{15}$$

2. Element $x\in L^0$ has an inverse x^{-1} for the multiplication $\cdot$ if and only if element $x\in L^0$ has an inverse $x^{-1_\circ}$ for the multiplication $\circ$.

$$x^{-1_0} = E_1^{-T(q)p}\cdot E_1^{-T(q)p}\cdot x^{-1}. \tag{16}$$

3. The multiplication $\circ$ satisfies condition (1) for the derivative S_p and condition (2) for the limit condition $s_p(q)$. (Condition (2) usually is not satisfied for $s_p(q_1)$, $q_1\neq q$.)

4. If the multiplication is defined by formula (14) then operations S_p, $T_p(q)$, $s_p(q)$ satisfy the theorems from chapter II. paragraph 1 of the paper [3].

Examples.

A. Using the operational calculus

$$CO\left(C^0((a, b), R),\ C^1((a, b), R),\ \frac{d}{dt},\ \int_{t_0}^{t},\ \big|_{t=t_0},\ R\right)$$

we may define the derivative S_p, the integral $T_p(t_0)$ and the limit condition $s_p(t_0)$ by the following formulas

$$S_p\{u(t)\} \stackrel{\mathrm{df}}{=} \left\{\frac{du(t)}{dt} + p(t)u(t)\right\},$$

$$T_p(t_0)\{f(t)\} \stackrel{\mathrm{df}}{=} \left\{e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} f(\tau) e^{\int_{t_0}^{\tau} p(\xi)d\xi} d\tau\right\} \quad \text{(see [6])}$$

$$s_p(t_0)\{u(t)\} \stackrel{\mathrm{df}}{=} \left\{u(t_0) e^{-\int_{t_0}^{t} p(\tau)d\tau}\right\},$$

where $u = \{u(t)\} \in C^1((a, b), R)$, $f = \{f(t)\}$, $p = \{p(t)\} \in C^0((a, b), R)$.

The multiplication $\circ$ defined by formula (14) has the following form

$$x \circ y = \{x(t)\} \circ \{y(t)\} \stackrel{\mathrm{df}}{=} \left\{e^{\int_{t_0}^{t} p(\tau)d\tau} x(t)y(t)\right\},\ x, y \in C^0((a, b), R)$$

B. In case of the operational calculus with the directional derivative

$$S\{u(x_1, x_2, \ldots, x_n)\} \stackrel{\mathrm{df}}{=} \left\{\sum_{i=1}^{n} b_i \frac{\partial u(x_1, \ldots, x_n)}{\partial x_i}\right\},$$

the integral

$$T(x_n^0)\{f(x_1, x_2, \ldots, x_n)\} \stackrel{\mathrm{df}}{=}$$

$$\stackrel{\mathrm{df}}{=} \left\{\frac{1}{b_n} \int_{x_n^0}^{x_n} f\left(x_1 - \frac{b_1}{b_n}(x_n - \tau),\ x_2 - \frac{b_2}{b_n}(x_n - \tau),\ \ldots,\ x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau),\ \tau\right) d\tau\right\}$$

and the limit conditions

$$s(x_n^0)\{u(x_1, x_2, \ldots, x_n)\} \stackrel{\mathrm{df}}{=}$$

$$\stackrel{\mathrm{df}}{=} \left\{u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0),\ x_2 - \frac{b_2}{b_n}(x_n - x_n^0),\ \ldots,\ x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0),\ x_n^0\right)\right\}$$

where $\quad u \in L^1 \stackrel{\mathrm{df}}{=} C^2(R^{n-1} x \langle x_n^1 x_n^2\rangle, R)$, $\quad f \in L^0 \stackrel{\mathrm{df}}{=} C^1(R^{n-1} x \langle x_n^1, x_n^2\rangle, R)$,

$$x_n^0 \in \langle x_n^1 x_n^2\rangle,\ b_i \in R \quad \text{for} \quad i = 1, 2, \ldots, n,\ b_n \neq 0 \quad \text{(see [4])}$$

the multplication $\circ$ defined by formula (14) has the following form

$$x\circ y=\{x(x_1,x_2,\dots,x_n)\}\circ\{y(x_1,x_2,\dots,x_n)\}\stackrel{\text{df}}{=}$$

$$\stackrel{\text{df}}{=}\left\{e^{\frac{1}{b_n}\int\limits_{x_n^0}^{x_n}p\left(x_1-\frac{b_1}{b_n}(x_n-\tau),\,x_2-\frac{b_2}{b_n}(x_n-\tau),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau),\,\tau\right)d\tau}x(x_1,x_2,\dots,x_n)\cdot\right.$$

$$\left.\cdot\, y(x_1,x_2,\dots,x_n)\right\}.$$

The derivative S_p, the integral $T_p(x_n^0)$ and the limit condition $s_p(x_n^0)$ are defined by the formulas

$$S_p\{u(x_1,x_2,\dots,x_n)\}\stackrel{\text{df}}{=}$$

$$\stackrel{\text{df}}{=}\left\{\sum_{i=1}^{n}b_i\frac{\partial u(x_1,x_2,\dots,x_n)}{\partial x_i}+p(x_1,x_2,\dots,x_n)u(x_1,x_2,\dots,x_n)\right\},$$

$$T_p(x_n^0)\{f(x_1,x_2,\dots,x_n)\}\stackrel{\text{df}}{=}$$

$$\stackrel{\text{df}}{=}\left\{e^{-\frac{1}{b_n}\int\limits_{x_n^0}^{x_n}p\left(x_1-\frac{b_1}{b_n}(x_n-\tau),\,x_2-\frac{b_2}{b_n}(x_n-\tau),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau),\,\tau\right)d\tau}\cdot\right.$$

$$\cdot\frac{1}{b_n}\int\limits_{x_n^0}^{x_n}f\left(x_1-\frac{b_1}{b_n}(x_n-\tau),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau),\,\tau\right)\cdot$$

$$\left.\cdot\, e^{\frac{1}{b_n}\int\limits_{x_n^0}^{\tau}p\left(x_1-\frac{b_1}{b_n}(x_n-\xi),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\xi),\,\xi\right)d\xi}d\tau\right\},$$

$$s_p(x_n^0)\{u(x_1,x_2,\dots,x_n)\}\stackrel{\text{df}}{=}$$

$$\stackrel{\text{df}}{=}\left\{u\left(x_1-\frac{b_1}{b_n}(x_n-x_n^0),\,x_2-\frac{b_2}{b_n}(x_n-x_n^0),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-x_n^0),\,x_n^0\right)\cdot\right.$$

$$\left.\cdot\, e^{-\frac{1}{b_n}\int\limits_{x_n^0}^{x_n}p\left(x_1-\frac{b_1}{b_n}(x_n-\tau),\,\dots,\,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau),\,\tau\right)d\tau}\right\}$$

where $u\in L^1$, $f,p\in L^0$.

C. Into the space $C(N)$ of real sequences $a=\{a_k\}$ let us introduce the derivative $S=P$ according to the formula

$$P\{a_k\}\stackrel{\text{df}}{=}\{a_{k+1}\}.$$

Introducing to the space $C(N)$ the multiplication of the sequences $a=\{a_k\}$, $b=\{b_k\}$ according to the formula

$$a*b\stackrel{\text{df}}{=}\left\{\sum_{i=0}^{k}\binom{k}{i}a_ib_{k-i}\right\}$$

we may prove that condition

$$S(a*b)=(Sa)*b+a*(Sb)$$

is satisfied (see [2]).

The limit condition s corresponding to the derivative $S=P$ has the form

$$s\{a_k\} \stackrel{\text{df}}{=} \begin{cases} a_0 & \text{for} \quad k=0 \\ 0 & \text{for} \quad k=1,2,\dots \end{cases} \quad \text{(see [2])}.$$

On the basis of theorem 2 we can define the derivative S_p, the integral T_p and the limit condition s_p.

Operations S_p, T_p, s_p are defined by the formulas

$$S_p\{a_k\} \stackrel{\text{df}}{=} \{a_{k+1}+\{p_k\}*\{a_k\}\},$$

$$T_p\{a_k\} \stackrel{\text{df}}{=} \{T(\{a_k\}*\{E_1^{T\{p_k\}}\})\}*\{E_1^{-T\{p_k\}}\},$$

$$s_p\{a_k\} \stackrel{\text{df}}{=} \{s\{a_k\}\}*\{E_1^{-T\{p_k\}}\}, \ \{a_k\}, \{p_k\} \in C(N),$$

where

$$T\{u_k\} \stackrel{\text{df}}{=} \begin{cases} 0 & \text{for} \quad k=0 \\ u_{k-1} & \text{for} \quad k=1,2,\dots, \quad \{u_k\} \in C(N) \end{cases}$$

and

$$\{E_1^{T\{p_k\}}\} \stackrel{\text{df}}{=} \{v_k\}.$$

The sequence $\{v_k\}$ is defined by the recurrent formula

$$\begin{cases} v_0 = 1 \\ v_{k+1} = \sum_{i=0}^{k} \binom{k}{i} v_i p_{k-i}, \quad k \geqq 1. \end{cases}$$

The multiplication $\circ$ has the form

$$\{x_k\} \circ \{y_k\} \stackrel{\text{df}}{=} \{E^{T\{p_k\}}\}*\{x_k\}*\{y_k\}.$$

Series $\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$ convergent in interval $|x|<R$ and sequences $\{a_k\}$ of real numbers are isomorphic. The derivative d/dt is equivalent to operation P (see [2]).

Cauchy's multiplication of the series is evidently equivalent to the multiplication $*$ of the sequences.

If $\{p_k\}=(-1, 0, 0, \dots)$ then $S_p=\Delta$ then for the operation Δ there esists a multiplication defined by the formula (16), where the sequence $\{E_1^{T(-1,0,0,\dots)}\} \stackrel{\text{df}}{=} \{v_k\}$ is defined by the formula

$$\{v_k\} = \{(-1)^k\}.$$

From the examples presented it follows how many types of equations can be solved: ordinary differential equations, partial differential equations difference equations.

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Reference

[1] R. Bittner, Algebraic and analytic properties of solutions of abstract differential equations. *Rozprawy Matemat.* **41** (1964), 1—63.
[2] R. Bittner, Rachunek operatorów w przestrzeniach liniowych. *Warszawa* 1974.
[3] R. Bittner—E. Mieloszyk, About eigenvalues of differential equations in the operational calculus. *Zeszyty Naukowe PG, Matematyka* **11** (1979), 87—99.
[4] R. Bittner—E. Mieloszyk, Application of the operational calculus to solving non-homogeneous linear partial differential equations of the first order with real coefficients. *Zeszyty Naukowe PG, Matematyka* **12** (1982), 33—45.
[5] R. Bittner—E. Sadowska, Derivative, integral and limit condition in non abelian grups. *Zesyty Naukowe UG, Matematyka* **1** (1972), 15—20.
[6] J. Dimovski, Operational calculus for the general linear differential operator of the first order. *Compt. rend. Acad. bulg. Sci.* **26** (1973), 1579—1582.
[7] E. Mieloszyk, Partial difference equation *(in preparation)*.
[8] E. Mieloszyk, About solving some abstract linear differential equation *(in preparation)*.
[9] D. Przeworska-Rolewicz, Operational calculus and algebraic analysis. *Proc. Conf. Varna,* Sept. 30—Oct. 6, 1975, *Sofia* 1979, 178—181.
[10] E. Sadowska, On the solvability of the differential equation $Sx=f$ of the first order. *Zeszyty Naukowe UG, Matematyka* **1** (1972), 117—123.

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