# Two scales of spaces of periodic generalized functions

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## Introduction

Spaces $D'\{m_k\}$ and $D'\{m_k\}$ of periodic ultradistributions are investigated in [1], [2] and [5].

In the first two papers elements of these spaces are characterized by the growth rate of their Fourier coefficients. In the third one the characterization of elements of these spaces as infinite sums of derivatives of the corresponding $L^2(0, 2\pi)$-functions is given.

In this paper we shall give the construction of some new spaces of periodic generalized functions and the characterizations of their elements. Also we shall give the relations between these spaces and the spaces of periodic ultradistributions.

## Basic notions

We shall recall basic notions and notation from [2] and [5].

Let $m_k$ be a sequence of positive numbers for which we assume:

(1) $$m_k^2 \leqq m_{k-1}m_{k+1}, \quad k = 1, 2, \ldots;$$

(2)′ $$\lim_{k\to\infty} \sqrt[k]{m_k} = \infty.$$

(We do not neea the stronger condition supposed in [2]:

(2) $$\sum_{i=1}^{\infty} m_{k-1}/m_k < \infty.)$$

(3) There are constants $A$ and $H$ such that

$$m_{k+1} \leqq AH^k m_k, \quad k = 0, 1, \ldots.$$

Testing function spaces are defined in [2] in the following way:

$D(m_k, L)$, $L>0$, denotes the space of all smooth functions on the unit circle

$K$ such that $\varphi \in D(m_k, L)$ iff

$$\|\varphi\|_{L,\infty} := \sup\left\{\frac{\|\varphi^{(k)}\|_\infty}{L^k m_k};\ k = 0, 1, \dots\right\} < \infty.$$

$$(\|\varphi\|_\infty = \sup\{|(\varphi(t)|;\ t \in [0, 2\pi]\}).$$

$$D\{m_k\} = \operatorname{ind}\lim_{L\to\infty} D(m_k, L),$$

$$D(m_k) = \operatorname{proj}\lim_{L\to 0} D(m_k, L),$$

where these limits are taken in the topological sense.

Condition (2)′ is equivalent to the following one:

$\mathscr{P}$ is a subspace of the space $D(m_k)$.

($\mathscr{P}$ is the space of trigonometrie polynomials)

In this paper we shall observe separately conditions:

(2)′a $\mathscr{P}$ is a subspace of $D\{m_k\}$;

(2)′b $\mathscr{P}$ is subspace of $D(m_k)$.

Since $D(m_k) \subset D\{m_k\}$, clearly, (2)′b implies (2)′a but contrary, it does not hold. For example if

(3) $$m_k = 1, \quad k = 0, 1, \dots \quad \text{or}$$

(4) $$m_k = k, \quad k = 0, 1, \dots$$

with $L > 1$, we obtain that for $D\{m_k\}$ (2)′a holds but (2)′b does not hold.

As in [2] we put

$$\varrho(\lambda) = \sup_k \frac{\lambda^k}{m_k}, \quad \lambda \geqq 1.$$

It follows from [2] that $D(m_k) \not\equiv \mathscr{P}$ if we suppose that $\varrho(\lambda) < \infty$ for every $\lambda \geqq 1$.

If $m_k$ is of the form (3) or (4), [2] does not give characterizations of the space $D\{m_k\}$ because $\varrho(\lambda) = \infty$ for every $\lambda \geqq 1$.

**Proposition 1.** *Condition* (2)′b *is equivalent to* $\varrho(\lambda) < \infty$ *for every* $\lambda \geqq 1$.

**Proof.** Since $\|(e^{ijt})^{(k)}\|_\infty = j^k$ the assertion follows from

$$\sup_k \frac{j^k}{L^k m_k} < \infty, \quad 0 < L < 1 \quad \text{iff} \quad \sup_k \frac{\lambda^k}{m_k} < \infty, \quad \lambda \geqq 1.$$

Hence, if (2)′a holds and (2)′b does not hold for the characterization of the spaces $D'\{m_k\}$ we can only apply Theorem 3 from [5].

## Construction of spaces $\mathscr{A}_{\text{per}}(\exp_p k)$, $p = 1, 2, \dots$

We denote ([5]) by $\mathscr{A}_{\text{per}}$ the space of all functions $\varphi \in L^2(0, 2\pi) \cap C^\infty(0, 2\pi)$ for which

$$\gamma_p(\varphi) := \sup\{\|\varphi^{(k)}\|_2;\ k \leqq p\} < \infty, \quad p = 0, 1, \dots,$$

$$\left(\|\varphi\|_2^2 = \int_0^{2\pi} |\varphi(t)|^2\, dt\right)$$

and

$$\langle \varphi^{(k)}, e^{int} \rangle = (-1)^k (in)^k \langle \varphi, e^{int} \rangle.$$

This space is introduced in [7] as an example of the $\mathscr{A}$-type spaces.

As in [5] we put (in the topological sense)

$$A_{per}\{m_k\} = \text{ind} \lim_{L \to \infty} A_{per}(m_k, L),$$

$$A_{per}(m_k) = \text{proj} \lim_{L \to 0} A_{per}(m_k, L),$$

where

$$A_{per}(m_k, L) = \left\{ \varphi \in A_{per};\ \|\varphi\|_{L,2} = \sum_{k=0}^{\infty} \frac{\|\varphi^{(k)}\|_2}{L^k m_k} < \infty \right\}.$$

**Proposition 2.** *If* $m_k \geqq Ck!$ *for some* $C > 0$, *then*

(i) $$A_{per}(k!) = D(k!) \hookrightarrow D(m_k);$$

(ii) $$A_{per}\{k!\} = D\{k!\} \hookrightarrow D\{m_k\},$$

*where* $A \hookrightarrow B$ *means "A is continuously embedded in B as a dense subspace".*

PROOF. The assertions follows directly from [5, Theorem 1] and from the fact that $\mathscr{P}$ is a dense subspace of mentioned spaces.

We shall use the following notation:

$$\exp_p k = \underbrace{\exp(\exp(\ldots(\exp}_{p} k)\ldots)).$$

In [4] we generalize spaces of $\mathscr{A}$ and $\mathscr{A}'$-type by introducing the spaces $\exp_p \mathscr{A}$ and $\exp_p \mathscr{A}'$. If we apply results from [4] to the space $L^2(0, 2\pi)$ and $\mathscr{R} = i \frac{d}{dt}$ we obtain the definition of the space $\mathscr{A}_{per}(\exp_p k)$ and several properties of these spaces.

*Definition 1.* $\mathscr{A}_{per}(\exp_p k)$, $p$ is a fixed natural number, is the space of all $\varphi$ from $\mathscr{A}_{per}$ for which

$${}_p\gamma_k(\varphi) := \sum_{n=-\infty}^{\infty} |a_n|^2 (\exp_p n)^{2k} < \infty, \quad k = 0, 1, \ldots \quad \left(\varphi \stackrel{2}{=} \sum_{n=-\infty}^{\infty} a_n e^{int}\right).$$

**Proposition 3.** (i) $\mathscr{A}_{per}(\exp_p k) \hookrightarrow \mathscr{A}_{per}(\exp_{p-1} k) \hookrightarrow \mathscr{A}_{per} \hookrightarrow \mathscr{E}$, $p = 2, 3,$ *where* $\mathscr{E}$ *is the space* $C^\infty(0, 2\pi)$ *with the usual sequences of norms.*

(ii) *If* $\varphi \in \mathscr{A}_{per}(\exp_p k)$ *then*

$$E_p^k \varphi = \sum_{m_1=0}^{\infty} \frac{k^{m_1}}{m_1!} \sum_{m_2=0}^{\infty} \frac{m_1^{m_2}}{m_2!} \cdots \sum_{m_p=0}^{\infty} \frac{m_{p-1}^{m_p}}{m_p!} i^{m_p} \varphi^{(m_p)}$$

*also belongs to* $\mathscr{A}_{per}(\exp_p k)$ *and the mapping* $E_p^k\colon \mathscr{A}_{per}(\exp_p k) \to \mathscr{A}_{per}(\exp_p k)$ *is continuous.*

(iii) *The sequence of norms* ${}_p\gamma_k$, $k=0, 1, \dots$ *on* $\mathscr{A}_{\text{per}}(\exp_p k)$ *is equivalent to the following sequence of norms:*

$$ {}_p\theta_k(\varphi) := \sum_{m_1=0}^{\infty} \frac{k^{m_1}}{m_1!} \sum_{m_2=0}^{\infty} \frac{m_1^{m_2}}{m_2!} \dots \sum_{m_p=0}^{\infty} \frac{m_{p-1}^{m_p}}{m_p!} \|\varphi^{(m_p)}\|_2, \quad k = 1, 2, \dots . $$

The proof of this proposition follows from the corresponding assertions in [4].

Clearly, $\varphi_\nu = \sum_{n=-\nu}^{\nu} a_n e^{int}$ converges to $\varphi = \sum_{n=-\infty}^{\infty} a_n e^{int}$ in $\mathscr{A}_{\text{per}}(\exp_p k)$ for any $\varphi \in \mathscr{A}_{\text{per}}(\exp_p k)$. Thus we obtain that $\mathscr{P}$ is a dense subspace of $\mathscr{A}_{\text{per}}(\exp_p k)$, $p=1, 2, \dots$ .

**Proposition 4.** *For* $p=2, 3, \dots$ *and* $\alpha>0$, $\mathscr{A}_{\text{per}}(\exp_p k) \subset D\big((k!)^\alpha\big)$.

Proof. We have

$$ \sup_n \frac{\lambda^n}{(n!)^\alpha} = \sup_n \left( \frac{(\lambda^{1/\alpha})^n}{n!} \right)^\alpha \leq \exp(\alpha \lambda^{1/\alpha}) \leq C_\alpha \exp(\exp \lambda) $$

where $C_\alpha > 0$ is a suitable constant. This implies the assertion.

Thus we see that spaces $\mathscr{A}_{\text{per}}(\exp_p k)$, $p=1, 2, \dots$ makes a scale of spaces which may not be constructed by the methods given in [1] and [2]. The corresponding dual spaces $\mathscr{A}'_{\text{per}}(\exp_p k)$ $p=1, 2, \dots$ satisfy the following assertions:

**Proposition 5.** (i) $\mathscr{E}' \hookrightarrow \mathscr{A}'_{\text{per}} \equiv D'(k!) \hookrightarrow \dots \mathscr{A}'_{\text{per}}(\exp_{p-1} k) \hookrightarrow \mathscr{A}'_{\text{per}}(\exp_p k) \hookrightarrow \dots$ $\dots \hookrightarrow \mathscr{P}'$.

(ii) *If* $f \in \mathscr{A}'_{\text{per}}(\exp_p k)$ *then there exists a sequence of complex numbers* $b_n$, $n=0, \pm 1, \dots$, *such that*

$$ f = \sum_{n=-\infty}^{\infty} b_n e^{int}, \quad b_n = \langle f, e^{-int} \rangle, \quad n = 0, \pm 1, \dots, \tag{5} $$

*where the series converges weakly in* $\mathscr{A}'_{\text{per}}(\exp_p k)$.

(iii) *The series on the right side of* (5) *converges weakly in* $\mathscr{A}_{\text{per}}(\exp_p k)$ *iff there exists a non-negative integer r such that*

$$ \sum_{n=-\infty}^{\infty} |b_n|^2 \exp(-2r(\exp_{p-1} n)) < \infty. $$

(iv) *If* $f \in \mathscr{A}'_{\text{per}}(\exp_p k)$ *then there exist a sequence* $f_{(m_1, \dots, m_p)}$, $m_1=0, 1, \dots, \dots$, $m_p=0, 1, \dots$, *from* $L^2(0, 2\pi)$ *and a non-negative integer r such that*

$$ f = \sum_{m_1=0}^{\infty} \frac{r^{m_1}}{m_1!} \sum_{m_2=0}^{\infty} \frac{m_1^{m_2}}{m_2!} \dots \sum_{m_p=0}^{\infty} \frac{m_{p-1}^{m_p}}{m_p!} f_{(m_1, \dots, m_p)}^{(m_p)}, \tag{6} $$

$$ \sup \{ \|f_{(m_1, \dots, m_p)}\|_2, \; m_1 = 0, 1, \dots, \dots, m_p = 0, 1, \dots \} < \infty. \tag{7} $$

*Conversely, if a sequence* $f_{(m_1, \dots, m_p)}$ *from* $L^2(0, 2\pi)$ *satisfies* (7), *then a unique element from* $\mathscr{A}_{\text{per}}(\exp_p k)$ *is defined by the series on the right of* (6) *(in the sense of weak convergence).*

Proof. (i) follows from Proposition 4 (i) and the other assertions follow from the corresponding assertion in [4].

### Construction of spaces $\mathscr{A}_{\text{per}}\{\exp_p k\}$, $p=1, 2, \ldots$

Now we shall define spaces $\mathscr{A}_{\text{per}}\{\exp_p k\}$ and we shall give several properties of these spaces.

*Definition 2.* $\mathscr{A}_{\text{per}}\{\ \}=\text{ind} \lim_{L\to\infty} \mathscr{A}_{\text{per}}(k!, L)$ (in the topological sense).

**Proposition 6.** $\mathscr{A}_{\text{per}}\{\ \}\equiv D\{k!\}$.

Proof. This follows from the inequalities

$$\|\varphi\|_{L,2} \leqq C_1\|\varphi\|_{2L,\infty}; \quad \|\varphi\|_{L,\infty} \leqq C_2\|\varphi\|_{L/H,2}$$

where $C_1$ and $C_2$ are suitable constant and $H$ is from (3), which are proved in [5, Theorem 1].

By computing $\varrho(\lambda)$ for $m_k=k!$, [2, Theorem 1] implies:

**Proposition 7.** *$\varphi\in L^2(0, 2\pi)$ belongs to $\mathscr{A}_{\text{per}}\{\ \}$ iff for some non-negative integer $k$*

$$\sum_{n=-\infty}^{\infty} |a_n(\varphi)|^2 n^{2/k} < \infty.$$

*Definition 3.* $\mathscr{A}_{\text{per}}\{\exp_p k\}$ is a subspace of $\mathscr{A}_{\text{per}}\{\ \}$ such that $\varphi\in\mathscr{A}_{\text{per}}\{\exp_p k\}$ if for some non-negative integer $k$

$${}_p\gamma_{1/k}(\varphi) = \sum_{n=-\infty}^{\infty} |a_n|^2 (\exp_p n)^{2/k} < \infty.$$

We supply this space by the inductive topology.

**Proposition 8.** (i) $\mathscr{A}_{\text{per}}\{\exp_p k\} \hookrightarrow \mathscr{A}_{\text{per}}\{\exp_{p-1} k\} \hookrightarrow \mathscr{A}_{\text{per}}\{\ \} \hookrightarrow \mathscr{E}$.
(ii) *If $\varphi\in\mathscr{A}_{\text{per}}\{\exp_p k\}$ then*

$$\varphi \to E_p^{1/k}\varphi = \sum_{m_1=0}^{\infty} \frac{(1/k)^{m_1}}{m_1!} \cdots \sum_{m_p=0}^{\infty} \frac{m_{p-1}^{m_p}}{m_p!}\, i^{m_p}\varphi^{(m_p)}$$

*is a continuous mapping from $\mathscr{A}_{\text{per}}\{\exp_p k\}$ into the same space.*
(iii) *The sequence of norms ${}_p\gamma_{1/k}$, $k=1, 2, \ldots$ and ${}_p\theta_{1/k}$ are equivalent.*

Proof. (ii) and (iii) can be proved in the same way as in Proposition 3. The proof that $\mathscr{A}_{\text{per}}\{\ \}\hookrightarrow\mathscr{E}$ can be deduced from the definition of $D\{k!\}$ by using the fact that for any compact subset of $(0, 2\pi)$ there exists the corresponding compact subset on the unit circle.

In the same way as in Proposition 4 and can prove:

**Proposition 9.** $\mathscr{A}_{\text{per}}\{\exp_p k\} \hookrightarrow D\{(k!)^\alpha\}$ *for $p=2, 3, \ldots$ and $\alpha>0$.*
The same remark given after Proposition 4 holds for the scale of space $\mathscr{A}_{\text{per}}\{\exp_p k\}$.

**Proposition 10.** (i)$\mathscr{E}' \hookrightarrow \mathscr{A}'_{\text{per}}\{\ \} \equiv D'\{k!\} \hookrightarrow \mathscr{A}'_{\text{per}}\{\exp_{p-1} k\} \hookrightarrow \mathscr{A}'_{\text{per}}\{\exp_p k\} \hookrightarrow$
$\hookrightarrow \ldots \hookrightarrow \mathscr{P}'$.

(ii) *If* $f \in \mathcal{A}'_{\mathrm{per}}\{\exp_p k\}$ *then there exists a sequence of complex numbers* $b_n$, $n=0, \pm 1, \pm 2, \ldots$ *such that (in the sense of weak topology).*

$$f = \sum_{n=-\infty}^{\infty} a_n e^{\mathrm{int}}, \quad b_n = \langle f, e^{-\mathrm{int}} \rangle, \quad n = 0, \pm 1, \ldots. \tag{8}$$

(iii) *The series on the right side of* (8) *converges weakly in* $\mathcal{A}'_{\mathrm{per}}\{\exp_p k\}$ *iff for every non-negative integer r*

$$\sum_{n=-\infty}^{\infty} |b_n|^2 (\exp_p n)^{-2/r} < \infty.$$

(iv) *If* $f \in \mathcal{A}'_{\mathrm{per}}\{\exp_p k\}$ *then there exists a sequence* $f_{(m_1, \ldots, m_p)}$ $m_1 = 0, 1, \ldots, \ldots m_p = 0, 1, \ldots$ *from* $L^2(0, 2\pi)$ *such that*

$$f = \sum_{m_1=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} f^{(m_p)}_{(m_1, \ldots, m_p)}, \tag{9}$$

$$\sum_{m_1=0}^{\infty} r^{m_1} m_1! \sum_{m_2=0}^{\infty} \frac{m_2!}{m_1^{m_2}} \cdots \sum_{m_p=0}^{\infty} \frac{m_p!}{m_{p-1}^{m_p}} \|f_{(m_1, \ldots, m_p)}\|_2 < \infty \tag{10}$$

*for every non-negative integer r.*

*Conversely if a sequence* $f_{(m_1, \ldots, m_p)}$ *from* $L^2(0, 2\pi)$ *satisfies* (10), *then a unique element from* $\mathcal{A}_{\mathrm{per}}\{\exp_p k\}$ *is defined by the sequence on the right side of* (9).

PROOF. We shall only prove (iv) since (i), (ii) and (iii) can be proved by standard arguments.

We denote by $\mathcal{A}_{\mathrm{per}}\{\exp_p k, 1/r\}(\bar{\mathcal{A}}_{\mathrm{per}}\{\exp_p k, 1/r\})$ the subspace of $\mathcal{A}_{\mathrm{per}}\{\ \}$ such that $\varphi \in \mathcal{A}_{\mathrm{per}}\{\exp_p k, 1/r\}(\varphi \in \bar{\mathcal{A}}_{\mathrm{per}}\{\exp_p k, 1/r\})$ if ${}_p\theta_{1/r}(\varphi) < \infty$. $({}_p\gamma_{1/k}(\varphi) < \infty.)$ One can easily prove that the mappings $i_r$: $\bar{\mathcal{A}}_{\mathrm{per}}\{\exp_p k, 1/r\} \to \bar{\mathcal{A}}_{\mathrm{per}}\{\exp_p, 1/r+1\}$ are compact. Since the sequences ${}_p\theta_{1/r}$ and ${}_p\gamma_{1/r}$, $r = 1, 2, \ldots,$ are equivalent and the product of a continuous and a compact mapping is compact we obtain that $\mathcal{A}_{\mathrm{per}}\{\exp_p k\}$ is an inductive limit of an injective compact sequence of $(B)$-spaces. Now the rest of the proof is the same as the proof of Theorem 3 from [5].

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