Publ. Math. Debrecen
51 / 1-2 (1997), 21–33

On extended topologies
Sequential operators and Heine criteria

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Abstract. It is the aim of this paper to investigate FU and sequential operators for spaces endowed with some extensions of topologies. The general Heine criteria of continuity of maps are discussed and the problem of extensions of sequentially continuous mappings is studied.

1. Introduction

It is well-known that there exist topological spaces in which the topology cannot be fully described in terms of sequences. However spaces for which sequences suffice are of special interest. The general convergence structures have been studied by many authors from various points of view. Here we study sequential operators for spaces (X, f) where f is a closure operation not defining a topology on X but nevertheless fulfilling some of Kuratowski's closure axioms and hence defining some extension of topology. We define and examine FU (Fréchet-Urysohn) operators and sequential ones. Next the Heine criteria of the continuity of maps between (X, f) spaces are defined and studied. As consequences we obtain characterizations of FU and sequential spaces. Finally, we discuss the problem of extensions of sequentially continuous mappings.

Mathematics Subject Classification: 54A05, 54D10.
Key words and phrases: FU operator, sequential operator, Heine criterion, identity principle, sequentially dense set.

2. Preliminaries

We follow the notation of [3] and [4]. As for prerequisities, the reader is expected to be familiar with [4]. We will denote by $\mathcal{P}(X)$ the power set of X. Any mapping $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is called a c-operation (closure operation) for a set X. The following basic definitions of classes of c-operations are made. A c-operation f is isotonic if $A \subset B \subset X$ implies $f(A) \subset f(B)$; f is an extended topology (f is ET) if f is isotonic and $f(\emptyset) = \emptyset$; f is enlarging if $A \subset f(A)$ for $A \subset X$; f is expansive if f is isotonic and enlarging; f is idempotent if $ff = f$; f is a closure function if f is expansive and idempotent; f is finitely additive (finitely subadditive) if $f(\bigcup_s A_s) = \bigcup_s f(A_s)$ ($f(\bigcup_s A_s) \subset \bigcup_s f(A_s)$) for every finite family $\{A_s\}$; f is a Kuratowski operator if f is finitely additive, enlarging and idempotent. Write $\mathcal{V}_f(x) = \{U \subset X : x \in \operatorname{Int}_f U\}$, $x \in X$, where $\operatorname{Int}_f U = X \setminus f(X \setminus U)$. We say that a pair (X, f) satisfies the condition (AC_1) if, for every $x \in X$, there exists a countable f-net (see [4] for this notion) $\mathcal{B}_f(x)$ of neighborhoods of x. A pair (X, f) is called quasi-metrizable if there exists a function $d : X \times X \longrightarrow \mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$ such that for every $x \in X$ and $A \subset X, x \in f(A)$ iff $d(x, A) = 0$, where $d(x, A) = \inf\{d(x, a) : a \in A\}$ if $A \neq \emptyset$ and $d(x, \emptyset) = 1$ for any $x \in X$ (see [5]).

Theorem 2.1 ([5], Theorem 5.1)**.** *A pair (X, f) is quasi-metrizable iff f is finitely additive and the condition* (AC_1) *is fulfilled.*

Suppose we are given two spaces (X, f) and (Y, g) and a mapping α of (X, f) into (Y, g). The mapping α is said to be continuous if $\alpha(f(A)) \subset g(\alpha(A))$ for all $A \subset X$.

Let $X \neq \emptyset$ be an arbitrary set. We say that a sequence $(x_n) \subset X$ converges to $x \in X$ (and denote this by $x_n \underset{f}{\longrightarrow} x$ or by $x \in \lim_f x_n$) if for every $\emptyset \neq V \in \mathcal{V}_f(x)$ there exists an $N \in \mathbb{N}$ such that $x_n \in V$ for every $n \geq N$. Then a point x is called an f-limit of (x_n). A constant sequence such that $x_n = x$ for every $n \in \mathbb{N}$ is denoted by (x). In the sequel we assume that f is a c-operation for X. Observe that: any sequence $(x_n) \subset X$ converges to $x \in X$ iff $\mathcal{V}_f(x) \subset \{\emptyset, X\}$; if f is isotonic, $\operatorname{card}(X) > 1$, $x \in X$ and if $\emptyset \in \mathcal{V}_f(x)$ then no sequence $(x_n) \subset X$ converges to x. Note that if f is enlarging then $x \in \lim_f(x)$ for every $x \in X$. If $f(X) = X$ then the converse implication holds. Hence if f is enlarging then $(X, \underset{f}{\longrightarrow})$ is an $\mathcal{L}^*$-space in the sense of [6] and maximal $\mathcal{L}$-structure in the sense of [2], and conversely, if $(X, \underset{f}{\longrightarrow})$ is an $\mathcal{L}^*$-space (or maximal $\mathcal{L}$-structure) and $f(X) = X$, then f is enlarging. Suppose f is enlarging and $T_1(\mathcal{V}_f)$ axiom holds. Then for every $x \in X$ we have $\lim_f(x) = \{x\}$. If $f(X) = X$ then the converse property holds. We say that a pair (X, f) satisfies $(T_2)_f$ axiom ($T_2(\mathcal{V}_f)$ axiom) if for every pair of distinct points $x, y \in X$ there exist $U, V \in \mathcal{T}_f$ ($\emptyset \neq U \in \mathcal{V}_f(x)$ and $\emptyset \neq V \in \mathcal{V}_f(y)$) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ (such that $U \cap V = \emptyset$). It is clear that $(T_2)_f$ axiom implies $(T_1)_f$ axiom. Note that for o-metrizable spaces $T_2(\mathcal{V}_f)$ axiom reduces to $\sigma - T_2$ axiom (see [7]). We see at once that $(T_2)_f$ axiom implies $T_2(\mathcal{V}_f)$ axiom; if f is enlarging and idempotent then they are mutually equivalent; if f is enlarging and $T_2(\mathcal{V}_f)$ axiom holds then $T_1(\mathcal{V}_f)$ axiom is satisfied. In the sequel, every f-net (see [4]) not contained in $\{\emptyset, X\}$ will be called a non-trivial f-net. It is easily seen that if $T_2(\mathcal{V}_f)$ axiom holds then every f-net at every point $x \in X$ is non-trivial iff $f(X) = X$.

Let $X \neq \emptyset$ and $(x_n) \subset X$. Consider the following axiom (uniqueness of limits):

$$\text{(UL)} \quad \text{if} \quad x_n \underset{f}{\longrightarrow} x \quad \text{and} \quad x_n \underset{f}{\longrightarrow} y \quad \text{then} \quad x = y.$$

It is easily seen that $T_2(\mathcal{V}_f)$ axiom implies (UL) and if f is finitely subadditive, $f(X) = X$ and for every $x \in X$ there exists a non-trivial countable f-net $\mathcal{B}_f(x)$ (equivalently, $\mathcal{V}_f(x) \neq \{X\}$ for every $x \in X$ and (AC_1) axiom holds, see [5]), then (UL) implies $T_2(\mathcal{V}_f)$ axiom.

3. FU operators

Let f be a c-operation for a set X. A sequential modification f_{seq} of f is defined as follows: $x \in f_{seq}(A)$, $A \subset X$, iff there exists a sequence $(x_n) \subset A$ such that $x_n \underset{f}{\longrightarrow} x$. A c-operation f is called FU operator (Fréchet-Urysohn) and a pair (X, f) an FU-space if $f = f_{seq}$. We see that f_{seq} is finitely additive c-operation (and hence is ET); $x \in f_{seq}(A)$ for every $A \neq \emptyset$ iff $\mathcal{V}_f(x) \subset \{\emptyset, X\}$; if f is enlarging then f_{seq} is enlarging and conversely, if $f(X) = X$ or if f is isotonic, and f_{seq} is enlarging then f is enlarging. If $\operatorname{card}(X) = 1$ then $f_{seq} \neq \emptyset$. If $\operatorname{card}(X) = 1$, f is isotonic and $f \neq \emptyset$ then $f_{seq} \subset f$ and $f_{seq} \neq \emptyset$. If $\operatorname{card}(X) > 1$ and $f = \emptyset$ then $f_{seq} = \emptyset$.

Proposition 3.1. *Suppose $f(X) = X$ or f is isotonic and $f \neq \emptyset$ (if $\operatorname{card}(X) > 1$ the last condition is superfluous). Then $f_{seq} \subset f$. Analogously, if $f(\emptyset) = \emptyset$ and $f(X) = X$ then $f_{seq} \subset p_f$.*

Proof. Let $f(X) = X$. Then $\emptyset \notin \mathcal{V}_f(x)$ for every $x \in X$. We can suppose that $A \neq \emptyset$ and $f_{seq}(A) \neq \emptyset$. Choose $x \in f_{seq}(A)$. Hence $x_n \underset{f}{\longrightarrow} x$ for some sequence $(x_n) \subset A$. If $\mathcal{V}_f(x) = \emptyset$ then $x \in f(A)$ for all $A \subset X$. If $\mathcal{V}_f(x) \neq \emptyset$ then $V \cap A \neq \emptyset$ for every $V \in \mathcal{V}_f(x)$ and so $x \in p_f(A) \subset f(A)$.

Now let $f \neq \emptyset$ be isotonic. One can suppose that $A \neq \emptyset$ and $f_{seq}(A) \neq \emptyset$. Let $x \in f_{seq}(A)$. Then $\emptyset \notin \mathcal{V}_f(x)$. If $\mathcal{V}_f(x) = \emptyset$ then $x \in f(A)$ for every $A \subset X$. Let $\mathcal{V}_f(x) \neq \emptyset$. If $A = X$ then of course $x \in f(X)$. If $A \neq X$ then $x_n \underset{f}{\longrightarrow} x$ for some $(x_n) \subset A$. Suppose $x \notin f(A)$. It follows that $\emptyset \neq X \setminus A \in \mathcal{V}_f(x)$, hence $x_n \in X \setminus A$ for some $N \in \mathbb{N}$ and all $n \geq N$, a contradiction. $\square$

Proposition 3.2. a) *Let $\operatorname{card}(X) = 1$. Then f_{seq} is the finest non-empty* ET *g for X with $x_n \underset{f}{\longrightarrow} x \implies x_n \underset{g}{\longrightarrow} x$.*

b) *If $\operatorname{card}(X) > 1$ then f_{seq} is the finest* ET *g for X with $x_n \underset{f}{\longrightarrow} x \implies x_n \underset{g}{\longrightarrow} x$.*

c) *If, moreover, $f(X) = X$ then f_{seq} is the finest* ET *for X with the same convergent sequences as f.*

Proof. We first show that $x_n \underset{f}{\longrightarrow} x$ implies $x_n \underset{f_{seq}}{\longrightarrow} x$. Suppose $x_n \underset{f}{\longrightarrow} x$. If $f = \emptyset$ and $\operatorname{card}(X) = 1$ then $\mathcal{V}_f(x) = \mathcal{P}(X)$ and $\mathcal{V}_{f_{seq}}(x) = \{X\}$; if $\operatorname{card}(X) > 1$ then $f_{seq} = \emptyset$. Hence in both cases the implication holds. Let $f \neq \emptyset$. We have $\emptyset \notin \mathcal{V}_{f_{seq}}(x)$. If $\mathcal{V}_{f_{seq}}(x) \subset \{X\}$ then for every sequence (x_n), $x_n \underset{f_{seq}}{\longrightarrow} x$. Assume $\mathcal{V}_{f_{seq}}(x) \not\subset \{X\}$ and let $x_n \underset{f_{seq}}{\not\longrightarrow} x$. There exists $X \neq U \in \mathcal{V}_{f_{seq}}(x)$ such that for every $N \in \mathbb{N}$ and some k_N,

$x_{k_N} \notin U$. But $x \notin f_{seq}(X \setminus U)$, hence $x_{k_N} \underset{f}{\not\longrightarrow} x$, a contradiction. Now let $\mathrm{card}(X) = 1$ and let $g \neq \emptyset$ be ET for X such that $x_n \underset{f}{\longrightarrow} x$ implies $x_n \underset{g}{\longrightarrow} x$. If $f = \emptyset$ then $f_{seq} \neq \emptyset$ and consequently $f_{seq} = g$. If $f \neq \emptyset$ then $f_{seq} = g_{seq} \subset g$. Hence we obtain a).

To prove b) assume $\mathrm{card}(X) > 1$. If $f = \emptyset$ then $f_{seq} = \emptyset$ hence $f_{seq} \subset g$. Let $f \neq \emptyset$. If $f_{seq} = \emptyset$ then obviously $f_{seq} \subset g$. If $f_{seq} \neq \emptyset$ then $x \in f_{seq}(A)$ for some $x \in X$ and $A \subset X$. There exists a sequence $(x_n) \subset A$ such that $x_n \underset{f}{\longrightarrow} x$. By assumption, $x_n \underset{g}{\longrightarrow} x$ and so $x \in g_{seq}(A)$. We conclude that $g_{seq} \neq \emptyset$, hence $g \neq \emptyset$. We have to show that $f_{seq} \subset g$. Suppose, on the contrary, there exist $x \in X$ and $A \subset X$ such that $x \in f_{seq}(A)$ and $x \notin g(A)$. There is a sequence $(x_n) \subset A$ such that $x_n \underset{f}{\longrightarrow} x$. It follows $x_n \underset{g}{\longrightarrow} x$. But $X \setminus A \in \mathcal{V}_g(x)$, which yields $x_n \in X \setminus A$ for every $n \geq N$, a contradiction (observe that $A \neq X$ and $A \neq \emptyset$ because $\mathrm{card}(X) > 1$, g is ET and $x_n \underset{g}{\longrightarrow} x$).

It remains to prove c). Let $f(X) = X$ and suppose that $x_n \underset{f_{seq}}{\longrightarrow} x$. From Proposition 3.1 we have $f_{seq} \subset f$ and hence $\mathcal{V}_f(x) \subset \mathcal{V}_{f_{seq}}(x)$. Let $V \in \mathcal{V}_f(x)$ be arbitrary. Then $x_n \in V$ for all $n \geq N$, because $x_n \in U$ for every $U \in \mathcal{V}_{f_{seq}}(x)$ and for all $n \geq N$. We conclude that $x_n \underset{f}{\longrightarrow} x$. □

Theorem 3.3. *Let $f(X) = X$. Then f_{seq} is the unique* FU *operator for X with the same convergent sequences as f.*

Proof. Let g be FU operator for X with the same convergent sequences as f. From Proposition 3.2 we have $f_{seq} \subset g$. To prove that $g \subset f_{seq}$ suppose that $x \in g(A)$ and $x \notin f_{seq}(A)$ for some $A \subset X$. By assumption, $x \in g_{seq}(A)$. But $\emptyset \notin \mathcal{V}_g(x)$, hence there exists a sequence $(x_n) \subset A$ such that $x_n \underset{g}{\longrightarrow} x$. Then $x_n \underset{f}{\longrightarrow} x$ and so $x \in f_{seq}(A)$, a contradiction. □

Applying Proposition 3.2 we obtain

Theorem 3.4. *Suppose $f(X) = X$ and let g be* ET *for X such that $g \subset f$. Then $g \supset f_{seq}$ iff f and g have the same convergent sequences.*

4. Sequential operators

A c-operation f for X is called a sequential operator and a pair (X, f) a sequential space if $\mathcal{F}_f = \mathcal{F}_{f_{seq}}$. Of course every FU operator is a sequential one. Let $q_{(f_{seq})}(A) = \bigcap\{F \subset X : F \supset A, F = f_{seq}(F)\}, A \subset X$. It is clear that $q_{(f_{seq})}$ is a Kuratowski operator, $f_{seq} \subset q_{(f_{seq})}$ and if f is enlarging then $q_{(f_{seq})} \subset q_f$. Also if f is sequential operator then $q_f(\emptyset) = \emptyset$. If f is enlarging and idempotent then $q_{(f_{seq})} \subset f$.

Proposition 4.1. a) *Let f be enlarging. Then $q_{(f_{seq})}$ is the finest idempotent c-operation g for X such that $g(X) = X$ with $x_n \underset{f}{\longrightarrow} x \implies x_n \underset{g}{\longrightarrow} x$.*

b) *If, moreover, f is idempotent then $q_{(f_{seq})}$ is the finest idempotent c-operation g for X such that $g(X) = X$ with the same convergent sequences as f.*

Proof. a) It is clear that $q_{(f_{seq})}(X) = X$. Suppose $x_n \underset{f}{\longrightarrow} x \not\Longrightarrow x_n \underset{q_{(f_{seq})}}{\longrightarrow} x$ for some $(x_n) \subset X$ and $x \in X$. Then there is $V \in \mathcal{V}_{q_{(f_{seq})}}(x)$ such that for every $N \in \mathbb{N}$, $x_{k_N} \notin V$ for some $k_N \geq N$. As $x \notin q_{(f_{seq})}(X \setminus V)$ we have $x \notin f_{seq}(X \setminus V)$. But $(x_{k_N}) \subset X \setminus V$ and $x_{k_N} \underset{f}{\longrightarrow} x$, a contradiction. It remains to show that $q_{(f_{seq})} \subset g$ for every idempotent c-operation g with $g(X) = X$ and $x_n \underset{f}{\longrightarrow} x \implies x_n \underset{g}{\longrightarrow} x$. Choose $A \subset X$ (we can assume that $A \neq \emptyset$ and $A \neq X$) and let $x \in q_{(f_{seq})}(A)$. Then $A \subset f_{seq}(A) \subset g(A)$ and $g(A) \in \mathcal{F}_g \subset \mathcal{F}_{f_{seq}}$. Hence $g(A) = f_{seq}g(A)$ and $x \in g(A)$.

b) By assumptions, $q_{(f_{seq})} \subset f$. Hence $\mathcal{V}_f(x) \subset \mathcal{V}_{q_{(f_{seq})}}(x)$ for every $x \in X$ and so $x_n \underset{q_{(f_{seq})}}{\longrightarrow} x \implies x_n \underset{f}{\longrightarrow} x$. With a) we obtain our assertion. □

Note that under the assumptions of Proposition 4.1b we have $f_{seq} = (q_{(f_{seq})})_{seq}$.

Theorem 4.2. *Suppose f is enlarging and idempotent. Then $q_{(f_{seq})}$ is the unique isotonic (and also* ET*) idempotent sequential operator such that $g(X) = X$ with the same convergent sequences as f.*

Proof. Let $A \in \mathcal{F}_{q_{(f_{seq})}}$. We have $(g_{(f_{seq})})_{seq}(A) \subset A$, because if $x \notin A$ and $x \in (q_{(f_{seq})})_{seq}(A)$ for some $x \in X$ then $X \setminus A \in \mathcal{V}_{g_{(f_{seq})}}(x)$

and $x_n \in X \setminus A$ for $n \geq N$, where $(x_n) \subset A$, which is impossible. Now let $A \in \mathcal{F}_{(q_{(f_{seq})})_{seq}}$. From Proposition 4.1b it follows that $A = f_{seq}(A)$, hence $q_{(f_{seq})}(A) \subset A$ and $A \in \mathcal{F}_{q_{(f_{seq})}}$. Hence $\mathcal{F}_{q_{(f_{seq})}} = \mathcal{F}_{(q_{(f_{seq})})_{seq}}$ and we see that $q_{(f_{seq})}$ satisfies the conditions. We show the uniqueness of $q_{(f_{seq})}$. Let g be isotonic idempotent sequential operator such that $g(X) = X$ with the same convergent sequences as f. From Proposition 4.1b we have $q_{(f_{seq})} \subset g$. By assumptions, $g_{seq} = f_{seq}$. As g is idempotent we have $q_{(g_{seq})} = q_g$. Since g is isotonic, $g \subset q_g$ and hence $g \subset q_{(f_{seq})}$. $\square$

Applying Proposition 4.1b we obtain

Theorem 4.3. *Let f be enlarging and idempotent. Suppose g is idempotent, $g(X) = X$ and $g \subset f$. Then $g \supset q_{(f_{seq})}$ iff g and f have the same convergent sequences.*

The next theorem provides a criterion for (X, f) to be FU-space.

Theorem 4.4. a) *Suppose f is enlarging and idempotent. If f is* FU *operator then $f_{seq} = q_{(f_{seq})}$.*

b) *Suppose f is isotonic sequential operator. If $f_{seq} = q_{f_{seq}}$ then f is* FU *operator.*

Proof. a) We have $f = f_{seq} \subset q_{(f_{seq})} \subset f$. b) As $\mathcal{F}_f = \mathcal{F}_{f_{seq}}$, $q_f = q_{(f_{seq})} = f_{seq}$. By assumptions, f_{seq} is idempotent hence $f_{seq} = f f_{seq}$ and so $X = q_{(f_{seq})}(X) = f_{seq}(X) = f f_{seq}(X) = f(X)$. It follows $f_{seq} \subset f \subset q_f$. Thus $f = f_{seq}$ (and $f = q_{(f_{seq})}$). $\square$

Of course, if f is an idempotent FU operator then f_{seq} is idempotent. Conversely, if f is isotonic and enlarging sequential operator and f_{seq} is idempotent then f is FU. Hence we obtain

Corollary 4.5. a) *A topological sequential space (X, f) is* FU-*space iff f_{seq} is idempotent.*

b) *Let f be idempotent. Then a sequential space (X, f) is* FU *iff f_{seq} is idempotent and $f f_{seq} = f_{seq} f$.*

Finally we establish some sufficient conditions under which f is FU operator.

Proposition 4.6. *Suppose f is finitely additive and $f \neq \emptyset$ or* $\mathrm{card}(X) > 1$ *and let condition* (AC_1) *holds. Then f is* FU *operator.*

Proof. It suffices to show that $f \subset f_{seq}$. Choose $A \subset X$ (we can assume that $A \neq \emptyset$ and $f(A) \neq \emptyset$) and let $x \in f(A)$. Then $\emptyset \notin \mathcal{V}_f(x)$ and $\mathcal{V}_f(x) \neq \emptyset$. By assumption, there exists a countable f-net $\mathcal{B}_f(x) = \{U_n : n \geq 1\}$ such that $U_{n+1} \subset U_n$ for $n \in \mathbb{N}$. Of course, $U_n \cap A \neq \emptyset$ for $n \in \mathbb{N}$. Choose $x_n \in U_n \cap A, n \in \mathbb{N}$, and let $V \in \mathcal{V}_f(x)$ be arbitrary. There exists $U_n \subset V$, hence $x_n \in V$ and $x_m \in U_n \subset V$ for $m \geq n$. It follows that $x \in f_{seq}(A)$ and so $f \subset f_{seq}$. $\square$

Corollary 4.7. *If (X, f) is quasi-metrizable then f is* FU *operator.*

5. Heine criteria

The Heine criterion of the continuity of maps between topological spaces is well-known (see [1], Proposition 1.7). Here we examine this problem from the general point of view and obtain necessary and sufficient conditions under which the space (X, f) is FU or sequential space. We start with the notion of sequential continuity.

Let $\alpha : (X, f) \longrightarrow (Y, g)$ be an arbitrary mapping. The mapping α is said to be sequentially continuous if $x_n \underset{f}{\longrightarrow} x$ implies $\alpha(x_n) \underset{g}{\longrightarrow} \alpha(x)$ for $x \in X$ and $(x_n) \subset X$; in other words if $\alpha(\lim_f x_n) \subset \lim_g \alpha(x_n)$ for every $(x_n) \subset X$. It is clear that if α is sequentially continuous then $\alpha f_{seq}(A) \subset g_{seq}\alpha(A)$ for every $A \subset X$, i.e., α is continuous as a map of (X, f_{seq}) into (Y, g_{seq}).

Proposition 5.1. *Suppose $\alpha : (X, f) \longrightarrow (Y, g)$ where $f \neq \emptyset$ or* $\mathrm{card}(X) > 1$ *and let g be isotonic or α be onto. If α is continuous then α is sequentially continuous.*

A set $F \subset X$ is called sequentially closed if $F \in \mathcal{F}_{f_{seq}}$. It is clear that if f is enlarging and if $\alpha f_{seq} \subset g_{seq}\alpha$ then $\alpha^{-1}(F) \in \mathcal{F}_{f_{seq}}$ for every $F \in \mathcal{F}_{g_{seq}}, F \subset Y$.

Let $\mathcal{W}$ be a property of c-operation g for Y. We consider the following properties $\mathcal{W}$: $\mathcal{W} = is$ means g is isotonic; $\mathcal{W} = fa$: g is finitely additive; $\mathcal{W} = cl$: g is a closure; $\mathcal{W} = Ko$: g is a Kuratowski operator. We say that for (X, f) the Heine criterion $(H_{\mathcal{W}})$ holds if a map α of (X, f) into any (Y, g) where g has the property $\mathcal{W}$ is continuous iff α is sequentially continuous. It is clear that the following implications hold: $(H_{is}) \implies (H_{fa}) \implies (H_{Ko})$ and $(H_{is}) \implies (H_{cl}) \implies (H_{Ko})$. We say that a map $\alpha : (X, f) \longrightarrow (Y, g)$ is t-continuous if $\alpha^{-1}(B) \in \mathcal{T}_f$ for every $B \in \mathcal{T}_g$. One can prove that if f is enlarging and g is isotonic or α is onto then continuity of α implies t-continuity of α. If f is isotonic and g is enlarging and idempotent then the converse implication holds. Replacing in the above definition of $(H_{\mathcal{W}})$ continuity by t-continuity we obtain the t-Heine criterion $(tH_{\mathcal{W}})$.

Theorem 5.2. *Suppose $\alpha : (X, f) \longrightarrow (Y, g)$ and $f(X) = X$. The following conditions are equivalent:*

a) *f is* FU *operator.*
b) *For (X, f) criterion (H_{is}) holds.*
c) *For (X, f) criterion (H_{fa}) holds.*

Proof. a) $\implies$ b): Let α be continuous. Then α is sequentially continuous. Conversely, let α be sequentially continuous. Hence α is continuous as a map of (X, f_{seq}) into (Y, g_{seq}). As g is isotonic, $g_{seq} \subset g$, i.e., $\mathrm{id}_Y : (Y, g_{seq}) \longrightarrow (Y, g)$ is continuous. But $f = f_{seq}$ and $\alpha = \mathrm{id}_Y \circ \alpha : (X, f) \longrightarrow (Y, g)$ is continuous.

c) $\implies$ a): Since $f(X) = X$, $f_{seq} \subset f$. From Proposition 3.2 it follows that id_X is sequentially continuous as a map of (X, f) onto (X, f_{seq}). But f_{seq} is finitely additive, hence by criterion (H_{fa}) id_X is continuous. This gives $f \subset f_{seq}$ and f is FU. $\square$

Note that in implication b) $\implies$ c) the condition $f(X) = X$ is superfluous. If f is a Kuratowski operator we obtain a criterion for topological space to be an FU space.

Theorem 5.3. *Let $\alpha : (X, f) \longrightarrow (Y, g)$.* a) *Suppose f is expansive. If f is a sequential operator then for (X, f) criterion (tH_{cl}) (and also (H_{cl})) holds.*

b) *Let f be enlarging. If for (X, f) criterion (tH_{Ko}) holds then f is a sequential operator.*

c) *Suppose f is enlarging and idempotent. If for (X, f) criterion (H_{Ko}) holds then f is a sequential operator.*

Proof. a) Suppose α is t-continuous. Then α is continuous. As g is isotonic, α is sequentially continuous. Conversely, let α be sequentially continuous. Choose $B \in \mathcal{T}_g$ and let $x \in f_{seq}\alpha^{-1}(B)$. Then there exists a sequence $(x_n) \subset \alpha^{-1}(B)$ such that $\alpha(x) \in \lim_g \alpha(x_n)$. It follows that $\alpha(x) \in g_{seq}(B) \subset g(B) = B$ and so $x \in \alpha^{-1}(B)$. We conclude that $f_{seq}\alpha^{-1}(B) \subset \alpha^{-1}(B)$. Since f is enlarging, $\alpha^{-1}(B) \in \mathcal{F}_{f_{seq}} = \mathcal{F}_f$ and α is t-continuous. Consequently, criterion (tH_{cl}) holds.

b) Since f is enlarging, $\mathcal{F}_f \subset \mathcal{F}_{f_{seq}}$. From Proposition 4.1a it follows that id_X is sequentially continuous as a map of (X, f) onto $(X, q_{(f_{seq})})$, hence id_X is t-continuous. We obtain $\mathcal{F}_{q_{(f_{seq})}} \subset \mathcal{F}_f$. But $\mathcal{F}_{f_{seq}} \subset \mathcal{F}_{q_{(f_{seq})}}$ and $\mathcal{F}_f \subset \mathcal{F}_{f_{seq}}$ and so $\mathcal{F}_f = \mathcal{F}_{f_{seq}}$.

c) Arguing as in b) we have that id_X is continuous as a map of (X, f) onto $(X, q_{(f_{seq})})$, hence $f \subset q_{(f_{seq})}$ and so $f = q_{(f_{seq})}$. This implies that $\mathcal{F}_{f_{seq}} \subset \mathcal{F}_f$. This clearly forces $\mathcal{F}_{f_{seq}} = \mathcal{F}_f$ and so f is a sequential operator. □

We can rephrase the above result as follows.

Theorem 5.4. *Suppose $\alpha : (X, f) \longrightarrow (Y, g)$ and let f be expansive. The following conditions are equivalent:*

a) *f is a sequential operator.*

b) *For (X, f) criterion (tH_{cl}) holds.*

c) *For (X, f) criterion (tH_{Ko}) holds.*

Remark. If f and g are Kuratowski operators the above theorem gives necessary and sufficient conditions under which a topological space is a sequential one (see [1], 1.7 Proposition).

6. Quotient operators

Let f be c-operation for X and $\sim$ an equivalence relation on X. Let us denote by k the natural map $k : X \longrightarrow X/\sim$. A c-operation $f_\sim$ for $X/\sim$ is defined as follows: $(f_\sim)(A) = A \cup k(f(k^{-1}(A)))$ for every $A \subset X/\sim$. It is clear that $f_\sim$ is enlarging; $(f_\sim)(\emptyset) = \emptyset$ iff $f(\emptyset) = \emptyset$. If f is finitely additive or isotonic then so is $f_\sim$; if f is isotonic then k is continuous and sequentially continuous and also it is continuous as a map of (X, f_{seq}) onto $(X/\sim, (f_\sim)_{seq})$, i.e., $k(f_{seq}(A)) \subset (f_\sim)_{seq}(k(A))$ for every $A \subset X$; k is sequentially continuous iff k is continuous as a map of (X, f_{seq}) onto $(X/\sim, (f_{seq})_\sim)$.

Proposition 6.1. *If f is isotonic then $f_\sim$ is the finest enlarging c-operation g for $X/\sim$ such that $k : (X, f) \longrightarrow (X/\sim, g)$ is continuous and sequentially continuous.*

Proof. By assumptions it follows that $k : (X, f) \longrightarrow (X/\sim, f_\sim)$ is continuous, hence $k(f(A)) \subset (f_\sim)(k(A))$ for every $A \subset X$. Let g be enlarging c-operation such that $k : (X, f) \longrightarrow (X/\sim, g)$ is continuous and let $B \subset X/\sim$. We have $k(f(k^{-1}(B))) \subset g(B)$, hence $(f_\sim)(B) \subset B \cup g(B) = g(B)$ and $f_\sim \subset g$. $\square$

7. Identity principle

In this section we discuss the problem of extension of sequentially continuous mappings. We start with two definitions. Let f be c-operation for X and $X_o \subset X$. A set X_o is called sequentially dense in X if $f_{seq}(X_o) = X$. Note that if α is a continuous mapping of (X, f_{seq}) onto (Y, g_{seq}) and $X_o \subset X$ is sequentially dense in X, then $\alpha(X_o)$ is sequentially dense in Y. A set $X_o \subset X$ is called weakly sequentially dense in X if for every $A \in \mathcal{F}_{f_{seq}}$ such that $X_o \subset A \subset X$ we have $A = X$ (this definition fails if $X \notin \mathcal{F}_{f_{seq}}$). It is easily seen that sequential density implies weak sequential density if $X \in \mathcal{F}_{f_{seq}}$.

Theorem 7.1 (Identity principle). a) *Suppose α and β are sequentially continuous as mappings of (X, f) into (Y, g) and let (Y, g) satisfies condition* (UL). *Let $\alpha|_{X_o} = \beta|_{X_o}$ where $X_o \subset X$ is sequentially dense in X. Then $\alpha = \beta$.*

b) *If g is enlarging then the condition* (UL) *is also a necessary one.*

Proof. a) Let $A = \{x \in X : \alpha(x) = \beta(x)\}$, $(x_n) \subset A$ and $x_n \underset{f}{\longrightarrow} x \in X$. By assumptions $\alpha(x_n) \underset{g}{\longrightarrow} \alpha(x)$, $\beta(x_n) \underset{g}{\longrightarrow} \beta(x)$ and $\alpha(x_n) = \beta(x_n)$ for $n \in \mathbb{N}$. Applying (UL) we obtain $\alpha(x) = \beta(x)$, i.e., $x \in A$. It follows that $f_{seq}(A) \subset A$. But $X_o \subset A$, hence $X = f_{seq}(X_o) \subset f_{seq}(A) \subset A \subset X$ and so $A = X$.

b) Let g be enlarging. It suffices to prove that if (Y, g) does not satisfy (UL) then there exist a space (X, f), sequentially continuous mappings $\alpha, \beta : (X, f) \longrightarrow (Y, g)$ and sequentially dense subset $X_o \subset X$ such that $\alpha|_{X_o} = \beta|_{X_o}$ but $\alpha \neq \beta$. Let $X_o = \{a_1, a_2, \dots\}$, $X = \{x, a_1, a_2, \dots\}$,

$x \neq a_i$, $i = 1, 2, \dots$, and let $f(\emptyset) = \emptyset$, $f(X \setminus \{a_k\}) = X \setminus \{a_k\}$, $k = 1, 2, \dots$, $f(\{x, a_1, \dots, a_k\}) = \{a_1, \dots, a_k\}$, $k = 1, 2, \dots$, and $f(A) = X$ otherwise. Then $f_{seq}(X_o) = X$. Since (UL) is not satisfied, there exists a sequence $(y_n) \subset Y$ such that $y_n \underset{g}{\longrightarrow} y$, $y_n \underset{g}{\longrightarrow} z$ and $y \neq z$. Let $\alpha(a_k) = \beta(a_k) = y_k$, $k = 1, 2, \dots, \alpha(x) = y$, $\beta(x) = z$. Then $\alpha|_{X_o} = \beta|_{X_o}$ and $\alpha \neq \beta$. One can easily show that α and β are sequentially continuous. □

Remarks. 1° Note that (X, f) and (Y, g) in general are not $\mathcal{L}$-spaces in sense of [2].

2° Suppose f is enlarging. Then in the above theorem X_o may be weakly sequentially dense ("weak identity principle").

Let $\alpha : (X, f) \longrightarrow (Y, g)$ be sequentially continuous. A mapping α is called an epimorphism if for every (Z, h) and every sequentially continuous mappings $\phi : (Y, g) \longrightarrow (Z, h)$ and $\psi : (Y, g) \longrightarrow (Z, h)$ the condition $\phi\alpha = \psi\alpha$ implies $\phi = \psi$.

Theorem 7.2. *Let $\alpha : (X, f) \longrightarrow (Y, g)$ be sequentially continuous.*

a) *If α is onto then α is an epimorphism.*

b) *Suppose g is isotonic. If α is an epimorphism then α is onto.*

Proof. The proof of a) is immediate. To prove b) suppose α is not onto, i.e., $Y \setminus \alpha(X) \neq \emptyset$. Let $\sim = \alpha(X) \times \alpha(X) \cup \bigcup \{(y, y) : y \in Y \setminus \alpha(X)\}$ and let $Z = Y/\sim$ and $h = g_\sim$. Let $k : (Y, g) \longrightarrow (Z, h)$ be a natural mapping. Fix $x_o \in X$ and put $\phi = k(\alpha(x_o))$ and $\psi = k$. Since g is isotonic and k is onto, ψ is sequentially continuous. Of course, ϕ is also sequentially continuous. It is clear that $\phi\alpha = \psi\alpha$ but $\phi \neq \psi$. □

Remark. Let **IS** denote the category whose objects are isotonic spaces (pairs (X, f), where f is isotonic) and whose morphisms are sequentially continuous mappings. Then Theorem 7.2 can be restated as follows: *Let α be an* **IS**-*morphism. Then α is an epimorphism in* **IS** *iff α is onto.*

Applying the identity principle and arguing similarly as in the proof of Theorem 7.2 we have

Theorem 7.3. a) *Let $\mathcal{U}$ denote a family of spaces satisfying condition* (UL). *Let $\alpha : (X, f) \longrightarrow (Y, g)$ be sequentially continuous, where (X, f) and (Y, g) are in $\mathcal{U}$. If $\alpha(X)$ is sequentially dense in Y then α is an epimorphism in $\mathcal{U}$.*

b) *Suppose g is isotonic. If α is an epimorphism in $\mathcal{U}$ then $\alpha(X) \cup g_{seq}\alpha(X) = Y$.*

Remarks. 1° Let **UES** denote the full subcategory of **IS** whose objects are expansive spaces satisfying condition (UL). From Theorem 7.3 it follows that: *If* $\alpha : (X, f) \longrightarrow (Y, g)$ *then* α *is an epimorphism in* **UES** *iff* $\alpha(X)$ *is sequentially dense in* Y*.*

2° If f is enlarging then in Theorem 7.3 $\alpha(X)$ may be weakly sequentially dense in Y.

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(Received November 27, 1995; revised November 25, 1996)