Isochronism problems for nonlinear second order differential equations

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The isochronism problem is concerned with the differential equation

(1) $$y''+f(y)=0 \quad ('=d/dt),$$

where the function $f(y)$ is continuous, $yf(y)>0$ if $y\neq 0$. Suppose that the function $F(y)$ defined by

(2) $$F(y)=\int_0^y f(\eta)\,d\eta$$

satisfies the limit relation $\lim_{|y|\to\infty} F(y)=\infty$. Then every solution of (1) is oscillatory and periodic. The isochronism problem is the following: under which conditions on $f(y)$ will every solution of (1) have the same period? Clearly, in the case $f(y)=cy$ with some constant $c>0$ every solution has the period $2\pi/\sqrt{c}$. But there are still infinitely many other functions $f(y)$ which have the same property. A good treatment of this problem and also the references can be found in M. URABE's book [4]. The situation changes if we require the isochronism not of the complete periods but of the quarter periods, i.e. the time between the zeros and extremants should be the same for every solution. Then there is the only possibility $f(y)=c_1y$ for $y'>0$ or $f(y)=c_2y$ for $y'<0$.

In general we should distinguish four quarter periods according to the possibilities $y>0,\ y'>0$ or $y>0,\ y'<0$ or $y<0,\ y'>0$ or $y<0,\ y'<0$. In what follows we shall treat only the first of these. The other cases can be treated similarly.

Here we extend our problem to more general differential equations

(3) $$y''+\lambda f(y)\,g(y')=0,$$

where the function $f(y)$ behaves in the same manner as above, and the function $g(z)$ satisfies the following conditions:

(i) $g(z)$ is continuous and positive on $R\setminus\{0\}$,

(ii) the limits $\int_{\pm 0} z/g(z)\,dz$ exist,

(iii) the function $G(z)=\int_0^z \zeta/g(\zeta)\,d\zeta$ satisfies the relations

$$\lim_{|z|\to\infty} G(z)=\infty,$$

and finally λ is a positive parameter. Moreover we suppose that every solution of (3) is oscillatory.

Let $y=y(t)$ be a solution of (3). Multiplying (3) by $y'/g(y')$ and integrating we obtain

$$G(y'(t))+\lambda F(y(t)) = \text{const.} \tag{4}$$

Suppose that $y(0)=0$ and $y'(0)>0$. Then there is a value $\tau>0$ such that $y'(\tau)=0$ and $y'(t)>0$ on $0\leqq t<\tau$. Let $y(\tau)=a$ then clearly $\tau=\tau(a, \lambda)$ and by (4)

$$G(y'(t))+\lambda F(y(t)) = \lambda F(a),$$

hence

$$y' = G_+^{-1}(\lambda(F(a)-F(y(t)))),$$

where the function $G_+^{-1}(z)$ is the inverse function of $G(z)$ on R^+. Hence we obtain

$$\tau(a, \lambda) = \int_0^{\tau(a,\lambda)} \frac{y'(t)\,dt}{G_+^{-1}(\lambda(F(a)-F(y(t))))} = \int_0^a \frac{d\eta}{G_+^{-1}(\lambda(F(a)-F(\eta)))}.$$

Finally by making use of the substitution $\xi=F(\eta)$ we find

$$\tau(a, \lambda) = \int_0^{F(a)} \frac{d\xi}{f(F_+^{-1}(\xi))\cdot G_+^{-1}(\lambda(F(a)-\xi))}, \tag{5}$$

where $F_+^{-1}(y)$ denotes the inverse function of $F(y)$. Now we can formulate our main result.

Theorem 1. *If the quarter period of the solutions of* (3) *given by* (5) *depends only on λ and is independent of a then there are positive constants n, α, β such that $f(y)=\alpha y^n$, $g(z)=\beta z^{1-n}$ if $y, z>0$.*

Proof. Let $\varphi(\xi), \psi(\xi)$ be introduced by

$$\varphi(\xi) = \frac{1}{f(F_+^{-1}(\xi))}, \quad \psi(\xi) = \frac{1}{G_+^{-1}(\xi)},$$

then by the condition of Theorem 1 we can write $\tau(a, \lambda)=\tau(\lambda)$ and by (5)

$$\tau(\lambda) = \int_0^x \varphi(\xi)\psi(\lambda(x-\xi))\,d\xi \tag{6}$$

for all $x=F(a)>0$. Let the function $\Psi(\xi)$ be defined by

$$\Psi(\xi) = \tau(1)\psi(\lambda\xi)-\tau(\lambda)\psi(\xi). \tag{7}$$

Then from (6) it follows that

$$\int_0^x \varphi(\xi)\Psi(x-\xi)\,d\xi = 0 \quad \text{for all} \quad x>0. \tag{8}$$

The integral in (8) is of convolution type and it is well known (see J. Mikusiński [3]) that a convolution (8) can be zero only if one of its factors is zero. In our case

$\varphi(\xi)>0$ for $\xi>0$ hence $\Psi(\xi)=0$, i.e.

$$\tau(1)\psi(\lambda\xi) = \tau(\lambda)\psi(\xi). \tag{9}$$

Putting $\xi=1$ into (9) we get

$$\frac{\psi(\lambda)}{\psi(1)} = \frac{\tau(\lambda)}{\tau(1)} = \vartheta(\lambda). \tag{10}$$

By (9) the function $\vartheta(\lambda)$ satisfies the functional equation

$$\vartheta(\lambda\xi) = \vartheta(\lambda)\vartheta(\xi). \tag{11}$$

The function $\psi(\lambda)$ in (10) is continuous for $\lambda>0$ hence the function $\vartheta(\lambda)$ is also continuous and the only solutions of (11) is (see J. Aczél [1])

$$\vartheta(\lambda) = \lambda^{-\nu} \quad \text{with some} \quad \nu\in R. \tag{12}$$

From the definition of the function $\psi(\xi)$ we know $\lim_{\xi\to 0}\psi(\xi)=\infty$ hence the value of ν in (12) is positive. On the other hand the function $\psi(\xi)$ must be integrable hence there remains the only possibility $0<\nu<1$ in (12).

Making use of the relations (10), (12) we can write (6) as

$$\tau(1) = \psi(1)\int_0^x \varphi(\xi)(x-\xi)^{-\nu}\,d\xi. \tag{13}$$

Since the function $\varphi(\xi)$ is uniquely determined by this convolution integral, we can try to solve it in the form

$$\varphi(\xi) = \varphi(1)\xi^{-\mu} \quad \text{with some} \quad \mu\in R. \tag{14}$$

Then by the substitution $\xi=xs$ in (13) we obtain the relations

$$\mu+\nu = 1, \quad 0<\mu,\nu<1, \tag{15}$$

$$\tau(1) = \psi(1)\varphi(1)\int_0^1 \frac{ds}{s^{\mu}(1-s)^{\nu}}.$$

Now we are able to determine the functions $f(y)$, $g(z)$. By our definition and by (10), (12) we know $\psi(\xi)=1/G_+^{-1}(\xi)=\psi(1)\xi^{-\nu}$, hence $G_+^{-1}(\xi)=\xi^{\nu}/\psi(1)$, therefore $G(\xi)=[\psi(1)\xi]^{1/\nu}$. Due to (iii) $dG(z)/dz=z/g(z)$ hence

$$g(z) = \frac{\nu}{[\psi(1)]^{1/\nu}}\, z^{2-\frac{1}{\nu}}. \tag{16}$$

By (14) and by the definition of $\varphi(\xi)$ we have $f(F_+^{-1}(\xi))=\xi^{\mu}/\varphi(1)$. Let $y=F_+^{-1}(\xi)$. Then $\xi=F(y)$ and $f(y)=F^{\mu}(y)/\varphi(1)$. Since $f(y)=dF(y)/dy$ we obtain from (15)

$$\frac{y}{\varphi(1)} = \int_0^y F^{-\mu}(s)f(s)\,ds = \frac{F^{\nu}(y)}{\nu},$$

hence $F(y)=[\nu/\varphi(1)]^{1/\nu}y^{1/\nu}$, consequently

$$f(y)=\frac{1}{\nu}\left[\frac{\nu}{\varphi(1)}\right]^{\frac{1}{\nu}}y^{\frac{1}{\nu}-1}.$$

Let $n=1/\nu-1$, then by (15) $n>0$ and the functions $f(y), g(z)$ are of the required form which proves Theorem 1.

Let us observe that according to our Theorem 1 the function $H(y,z)=f(y)g(z)$ satisfies the following relation

$$H(cy, cz)=cH(y,z)\quad\text{for all}\quad c>0. \tag{17}$$

i.e. it is a homogeneous function of first degree.

A differential equation of the form

$$y''+h(y,y')=0 \tag{18}$$

is called a half-linear differential equation if the function $h(y,z)$ is defined, continuous and satisfies the homogenity relation (17) for all $(y,z)\in R\times(R\setminus\{0\})$. This definition is a somewhat weakened form of the one given by I. Bihari [2] who assumed the continuity of $h(y,z)$ on the whole R^2. To show an interesting property of the solutions of the half-linear differential equations first we give the definition of perfect isochronism.

Definition. Let l_0 be a halfray through the origin of the plane $y'y$. Let $y_1(t)$, $y_2(t)$ be two solutions of (18) with the initial conditions $(y_i'(t_0), y_i(t_0))\in l_0$, $i=1,2$. The differential equation (18) has the property of perfect isochronism if the points $(y_i'(t), y_i(t))$, $i=1,2$ are on a halfray going through the origin for all $t\geqq t_0$.

Then we have the following result:

Theorem 2. *If a differential equation of the form* (18) *has the property of perfect isochronism then it is half-linear.*

Proof. Let $y_i(t)$, $i=1,2$ be solutions of (18) such that the points $(y_i'(t), y_i(t))$, $i=1,2$ for $t\geqq t_0$ are on the same halfray going through the origin. Then there is a positive function $c(t)$ for $t\geqq t_0$ such that

$$\begin{aligned} y_2(t)&=c(t)\,y_1(t),\\ y_2'(t)&=c(t)\,y_1'(t). \end{aligned} \tag{19}$$

Suppose $y_1(t)\neq 0$. Since $y_i(t), y_i'(t)$ are differentiable the function $c(t)$ is also differentiable. Differentiating the first relation in (19) we obtain $y_2'=c'y_1+cy_1'=c'y_1+y_2'$, hence $c'y_1=0$ or $c'(t)=0$, i.e. $c(t)=c=\text{const}>0$. If we differentiate the second relation in (19) we find $y_2''=cy_1''$ or by (18)

$$h(cy_1, cy_1')=ch(y_1, y_1'),$$

hence the function $h(y,z)$ is a homogenenous function of first degree. Thereby the proof is complete.

Naturally there arises the problem of the isochronism of the quarter periods of the nonlinear differential equations

$$y''+\lambda h(y, y') = 0, \quad \lambda > 0.$$

Is it true that if the quarter periods depend only on λ for all $\lambda > 0$ then the function $h(y, z)$ is a homogenous function of first degree? The expected affirmative answer would be a generealization of Theorem 1.

References

[1] Aczél, J.: Lectures on functional equations and theire applications, *Acad. Press, N. Y.*, 1966, (Chapter 2).

[2] Bihari, I.: Ausdehnung der Sturmschen Oszillations und Vergleichssätze auf die Lösungen gewisser nicht-linearen Differentialgleichungen zweiter Ordnung, *Publ. Math. Inst. Hung. Acad. Sci.*, **2** (1957), 159—173.

[3] Mikusiński, J.: Operational calculus, *Pergamon Press/Panstw. Wyd. Nauk, Warshawa*, 1959, (Chapter 2, Theorem of Titchmarsh).

[4] Urabe, M.: Nonlinear autonomous oscillations. *Analytical theory, Acad. Press, N. Y.*, 1966, (Chapter 13, Inverse problems...).

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