On a theorem of Pisot

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Let F be the field of the rational numbers, or an imaginary quadratic field. The algebraic integers in F form a discrete lattice. For a complex z let $\|z\|$ denote the distance from z to the nearest algebraic integer in F. $\|z\|$ is zero only for integers of F.

The aim of this paper is to prove the following

Theorem 1. *Let $\alpha_1, \dots, \alpha_n$ be distinct algebraic numbers, $|\alpha_j| \geqq 1$ $(j=1, \dots, n)$, $p_1(x), \dots, p_n(x)$ be nonzero polynomials with complex coefficients. Then the relation*

$$\lim_{k=\infty} \Big\| \sum_{i=1}^{n} p_i(k)\alpha_i^k \Big\| = 0 \tag{1}$$

holds if and only if the following assertions are true:

a) *The numbers α_i are algebraic integers.*

b) *The coefficients of $p_i(x)$ are elements of the algebraic extension $F(\alpha_i)$.*

c) *If α_i and α_j are conjugate elements over F, and the corresponding polynomials have the form*

$$p_i(x) = \sum_{u=0}^{t_i} c_u^{(i)} x^u, \quad p_j(x) = \sum_{u=0}^{t_j} c_u^{(j)} x^u,$$

then p_i and p_j have the same degree, $c_u^{(i)}$ and $c_u^{(j)}$ are conjugate elements over F too, and for any such isomorphism τ which is the identical mapping on F and $\tau(\alpha_i)=\alpha_j$, the relations

$$\tau(c_u^{(i)}) = c_u^{(j)} \qquad (u = 0, 1, \dots, t_i = t_j)$$

hold.

d) *All the conjugates of the α_i-s not occurring in the sum $\sum_{i=1}^{n} p_i(k)\alpha_i^k$ have absolute value less than one.*

e) *The sums*

$$\sum_{i=1}^{n}{}^{*} \operatorname{Tr}\big(p_i(k)\alpha_i^k\big)$$

are algebraic integers in F for every large k $\big(\operatorname{Tr}(\alpha)$ denotes the sum of conjugates of α over $F.\big)$ The asterisk in the sum denotes, that the summation is taken over non-conjugate α_i-s.

This assertion is a generalization of a theorem due to PISOT ([1], [2], [3]).

Remark 1. The condition, that all traces $\mathrm{Tr}\,(p_i(k)\alpha_i^k)$ are integers for large $k-s$, is not necessary for the relation

$$\left\|\sum p_i(k)\,\alpha_i^k\right\| \to 0.$$

It holds for example

$$\left\|\frac{1}{4}(1+\sqrt{2})^k+\frac{1}{4}(3+\sqrt{6})^k\right\| \to 0, \quad \text{as} \quad k\to\infty,$$

but $\mathrm{Tr}\left(\frac{1}{4}(1+\sqrt{k})^k\right)$ and $\mathrm{Tr}\left(\frac{1}{4}(3+\sqrt{6})^k\right)$ are not integers, since

$$\mathrm{Tr}\,(((2u+1)+\sqrt{2v})^k) \equiv 2 \bmod 4, \quad \text{if} \quad v$$

is odd.

Remark 2. The relations $\lim\limits_{k=\infty}\|z_k\|=0$ and

$$\lim_{k=\infty}\left\|z_k+\sum p_j(k)\,\beta_j^k\right\| = 0$$

are equivalent, when the absolute values of the β_j-s are less than one, independently of the property, that the β_j-s are algebraic or transcendental's.

If the relation

$$\lim_{k=\infty}\left\|\sum p_i(k)\,\alpha_i^k\right\| = 0$$

is true and the α_i-s with $|\alpha_i|\geqq 1$ are algebraic, then the conclusions $a-e$ of theorem 1 hold for these α_i-s, and inversely, if $a-e$ hold for $|\alpha_i|\geqq 1$, then we have

$$\lim_{k\to\infty}\left\|\sum p_i(k)\,\alpha_i^k\right\| = 0.$$

It is easy to see that the first part of the theorem is true. The sums

$$d_k = \sum{}^* \mathrm{Tr}\,(p_i(k)\,\alpha_i^k)$$

are algebraic integers in F according to the property e). It follows from b)—d), that

$$\left\|\sum p_i(k)\,\alpha_i^k\right\| \leqq \left|d_k-\sum p_i(k)\,\alpha_i^k\right| \leqq \sum_{|\alpha_i^{(j)}|<1} |p_i^{(j)}(k)\,\alpha_i^{(j)k}|,$$

and so the relation

$$\lim_{k=\infty}\left\|\sum_{i=1}^{n} p_i(k)\,\alpha_i^k\right\| = 0$$

is true.

The proof of the second part of the theorem based on the following generalization of a lemma due to FATOU [5].

Lemma 1. *Assume that the polynomials $p(x)$ and $q(x)$ have no common root, $q(0)=1$, furthermore the Taylor-coefficients of the function $p(x)/q(x)$ at the zero are algebraic integers in F. Then the coefficients of $p(x)$ and $q(x)$ are algebraic integers in F too.*

At first we prove the lemma.

Let

$$\frac{p(x)}{q(x)} = \sum_{k=0}^{\infty} c_k x^k,$$

$$p(x) = \sum_{k=0}^{n} p_k x^k, \; p_n \neq 0,$$

$$q(x) = \sum_{k=0}^{m} q_k x^k, \; q_m \neq 0.$$

Comparing the coefficients we have

$$\begin{aligned} &p_i - c_i q_0 - c_{i-1} q_1 - \dots - c_0 q_i = 0, \quad \text{for} \quad i \leqq \min(n, m), \\ (2) \qquad &p_i - c_i q_0 - c_{i-1} q_1 - \dots - c_{i-m} q_m = 0, \text{ for } \quad m \leqq i \leqq n, \\ &0 - c_i q_0 - c_{i-1} q_1 - \dots - c_{i-m} q_m = 0, \quad \text{for} \quad i > n. \end{aligned}$$

If the p_i^*, q_i^* are solutions of the equations in (2), than for the polynomials

$$p^*(x) = \sum_{i=0}^{n} p_i^* x^i,$$

and

$$q^*(x) = \sum_{i=0}^{m} q_i^* x^i$$

hold the equations

$$p^*(x) = \mu p(x), \quad q^*(x) = \mu q(x),$$

with a constant μ, since $p(x)$ and $q(x)$ are relatively primes. So we have, that the solutions of the equations in (2) form a one-dimensional linear variety.

Since $q(0)=1$, there are constans γ_i, γ_i', that

$$p_i = \gamma_i q_0, \quad q_i = \gamma_i' q_0$$

The γ_i-s and the γ_i'-s are elements of the field F, because of the Gaussian elimination, so the p_i-s and q_i-s are elements of the field F, because $q_0=1$.

It is remained to prove, if the coefficients of $p(x)$, $q(x)$ and $p(x)/q(x)$ are integral in F, then q_0, the constant term, is a divisor of all the coefficients of $q(x)$. In that case the coefficients of $p(x)$ are multiples of q_0 too.

Since the polynomials with coefficients in a field form a Euclidean ring, there are polynomials, having integral coefficients, for which

$$p(x)\,u(x) + q(x)\,v(x) = b,$$

holds, where b is an algebraic integer in F. If $p(x)/q(x)$ has integral coefficients, then so has it $b/q(x)$.

If q_0 does not divide all the q_i-s, then there exists a prime ideal P of F, for which P^k is a divisor of q_0, but is not a divisor of q_i for at least one i.

Let k_i denote the greatest exponent, for which P^{k_i}/q_i.

Let k_u be the smallest k_i, and let j be the smallest index, for which $k_u = k_j$.

In F there exists an (integral) ideal A, for which AP^{k_j} is a principal ideal (γ), and there exists an ideal B, for which BA is a principal ideal (β), and A, B are relatively prime, and they are relatively prime to P too. In that case it is true that

$$(\beta q_i) = BA \cdot P^{k_j} P^{k_i - k_j} Q_i = (P^{k_j} A)(BP^{k_i - k_j} Q_i),$$

where the Q_i-s are (integral) ideals. Hence we have $\beta q_i = \gamma \cdot \beta_i$, where β_i is an algebraic integer in F. It is true that

$$b/(q_0 + q_1 x + \ldots + q_m x^m) = \beta b/\gamma(\beta_0 + \beta_1 x + \ldots + \beta_m x^m),$$

where γ is a divisor of $\beta b : \beta b = \gamma g$, ($g$ is integral in F). So, there is a rational fraction, the nominator of which is an integer b' in F, the denominator is a polynomial, having integral coefficients in F, and the Taylor-coefficients of the fraction are integers in F. There is a prime ideal P, for which P divides $\beta_0, \beta_1, \ldots, \beta_{j-1}$, but does not divide β_j.

We prove that this is impossible.

For all integers b let r_b denote the exponent to which P appears in the unique factorization of (b).

Let us consider all those integers b for which the Taylor-coefficients of $b/(\beta_0 + \ldots + \beta_m x^m)$ are integers in F. Let b^* be such an integer for which r_b takes on the minimal value.

Let

$$b^*/(\beta_0 + \beta_1 x + \ldots + \beta_m x^m) = c_0 + c_1 x + \ldots + c_k x^k + \ldots$$

From (2) for $i=j$ we have

$$\beta_0 c_j + \beta_1 c_{j-1} + \ldots + \beta_j c_0 = 0.$$

Since P is a divisor of β_i for $i = 0, 1, \ldots, j-1$ but not of β_j, therefore c_0 is a multiple of P.

From (2) with $i = j+1$ we have

$$\beta_0 c_{j+1} + \beta_1 c_j + \ldots + \beta_j c_1 + \beta_{j+1} c_0 = 0.$$

P divides $\beta_0, \ldots, \beta_{j-1}$ and c_0, but does not divide β_j, so P is a divisor of c_1. By induction it follows, that P divides all the c_i-s.

There exists an ideal A, for which AP is a principal ideal (γ), and there exists an ideal B, such that BA is a principal ideal (β), A and B are relatively prime, and they are relatively prime to P.

Taking into account the equalities

$$(\beta b^*) = BA\, P\tilde{B} = (AP)\, B\tilde{B} \quad \text{and}$$

$$(\beta c_i) = BA\, PC_i = (AP)(BC_i),$$

where $\tilde{B}$ and the C_i-s are integer ideals, we get that γ divides all βc_i-s and βb^*, and that the Taylor-coefficients of the fraction

$$\frac{\beta b^*}{\gamma} \Big/ (\beta_0 + \beta_1 x + \ldots + \beta_m x^m)$$

are integers in F. But it contradicts to the choice of the integer b^*, since $r_{b_1} \leqq r_b - 1$ for $b_1 = \beta b^*/\gamma$. By this the proof of Lemma 1 is finished.

At present we begin the proof of the second part of the Theorem 1.

Let $f_i(x)$ be a minimal polynomial of α_i over F with integral coefficients.

Let t_i be the maximal degree of the polynomials $p_j(x)$, for which α_j is a conjugate of α_i over F.

We consider the product of the polynomials $(f_i(x))^{t_i+1}$ for all non-conjugates α_i.

Let $F(x)$ denote this product:

$$F(x) = a_0 + a_1 x + \ldots + a_T x^T.$$

The a_i-s are integral in F.

If z_k denote the sum

$$z_k = \sum_{i=1}^{n} p_i(k)\alpha_i^k,$$

then we have

$$a_0 z_k + a_1 z_{k+1} + \ldots + a_T z_{k+T} = 0, \quad \text{for} \quad k \geqq 0.$$

Let us denote by E_k the integer in F nearest from z_k, and r_k be defined by

$$z_k = E_k + r_k \tag{6}$$

In this way we have

$$a_0 E_k + \ldots + a_T E_{k+T} = -(a_0 r_k + \ldots + a_T r_{k+T}) \tag{7}$$

The left hand side of (7) is an integer in F, the right hand side is tending to zero as k tends to infinity. Since the integers in F form a discrete lattice, it is true that

$$a_0 r_k + \ldots + a_T r_{k+T} = 0 \tag{8}$$

when k is large enough. Let k_0 be the smallest natural number for which (8) holds whenever $k \geqq k_0$.

So the sequence r_k satisfies a linear recurrence relation. The characteristical polynomial of the sequence is the product of the minimal polynomials of the α_i-s, so the roots of the characteristical polynomial of the sequence are the conjugates of the α_i-s. In this case there are suitable polynomials $q_i^{(j)}$, that the representation

$$r_k = \sum_{i,j} q_i^{(j)}(k)(\alpha_i^{(j)})^k \tag{9}$$

is valid, where the $\alpha_i^{(j)}$-s are the conjugates of the α_i and the degrees of the polynomials $q_i^{(j)}$ are at most t_i.

Since r_k tends to zero, therefore $q_i^{(j)}$ is identically zero for $|\alpha_i^{(j)}| \geqq 1$.

The polynomials $p_i(x)$ and $q_i^{(j)}(x)$ have a representation in the form

$$p_i(x) = \sum_{u=1}^{t_i+1} c_u^{(i)} \binom{x - k_0 + u - 1}{u - 1} \tag{10}$$

$$q_i^{(j)}(x) = \sum_{u=1}^{t_i+1} d_u^{(i,j)} \binom{x - k_0 + u - 1}{u - 1} \tag{11}$$

Let the following rational function be considered

$$(12)\qquad g(x) = \sum_{i=1}^{n}\sum_{u=1}^{t_i+1}\frac{C_u^{(i)}\alpha_i^{k_0}}{(1-\alpha_i x)^u} - \sum_i\sum_j\sum_{u+1}^{t_i+1}\frac{d_u^{(i,j)}(\alpha_i^{(j)})^{k_0}}{(1-\alpha_i^{(j)}x)^u}.$$

Since the equation

$$\frac{1}{(1-\alpha x)^n} = \sum_{v=0}^{\infty}\binom{v+n-1}{n-1}\alpha^v x^v$$

holds, we have from (10), (11), (12) and (13)

$$(14)\qquad g(x) = \sum_{v=0}^{\infty}\left(\sum_{i=1}^{n}\left(\sum_{u=1}^{t_i+1}C_u^{(i)}\binom{v+u-1}{u-1}\right)\alpha_i^{v+k_0} - \right.$$

$$\left. -\sum_{i=1}^{n}{}^{*}\sum_j\left(\sum_{u=1}^{t_i+1}d_u^{(i,j)}\binom{v+u-1}{u-1}\right)(\alpha_i^{(j)})^{v+k_0}\right)x^v =$$

$$= \sum_{k=k_0}^{\infty}\left(\sum_{i=1}^{n}p_i(k)\alpha_i^k - \sum_{i=1}^{n}{}^{*}\sum_j q_i^{(j)}(k)(\alpha_i^{(j)})^k\right)x^{k-k_0}.$$

The terms in (14) with $|\alpha_i^{(j)}|\geqq 1$ are identically zero, and the asterisk notes, that the sum is restricted to non-conjugates α_i.

It follows from (6) and (9), that

$$(15)\qquad g(x) = \sum_{k=k_0}^{\infty}E_k x^{k-k_0}.$$

The function $g(x)$ is a rational function, the Taylor-coefficients of which are integral in F.

The coefficients of the denominator

$$q(x) = \prod_i{}^{*}\prod_j(1-\alpha_i^{(j)}x)^{t_i+1}$$

are integral in F in consequence of the lemma, so the coefficients of the reciprocal polynomial of $q(x)$

$$\tilde{q}(x) = x^T q\left(\frac{1}{x}\right)$$

are integral in F too. So we proved, that the α_i-s are algebraic integers, because the leading coefficient of $\tilde{q}(x)$ is one. The roots of $\tilde{q}(x)$, except the α_i-s, accuring in the sum $\sum p_i(k)\alpha_i^k$, have absolute values less than one.

So we finished the proof of the assertions a) and d) of theorem 1. Now we begin to prove b and c. It is sufficient to prove, that the coefficients $c_u^{(i)}$ in (10) are elements of the field $F(\alpha_i)$. First we prove this for the highest coefficients.

The nominator of $g(x)$ is a polynomial $p(x)$ with integral coefficient. From (12) we have

$$(16) \qquad c_{t_i+1}^{(i)} = \frac{1}{\alpha_i^{k_0}} \lim_{x=1/\alpha_i} \frac{p(x)}{q(x)} \cdot (1-\alpha_i x)^{t_i+1} =$$

$$= \frac{p(1/\alpha_i)\alpha_i^{T-k_0-t_i-1}}{(f_i'(\alpha_i))^{t_i+1} \prod\limits_{j\neq i}^{*} f_j(\alpha_i)^{t_j+1}},$$

Analogously we have

$$(17) \qquad -d_{t_i+1}^{(i,j)} = \frac{p(1/\alpha_i^{(j)})(\alpha_i^{(j)})^{T-k_0-t_i-1}}{(f_i'(\alpha_i^{(j)}))^{t_i+1} \prod\limits_{k\neq i}^{*} f_k(\alpha_i^{(j)})^{t_k+1}}$$

It follows from (16) and (17), that $-d_{t_i+1}^{(i,j)}$ are conjugates of $c_{t_i+1}^{(i)}$, and $c_{t_i+1}^{(j)}$ is a conjugate of $c_{t_i+1}^{(i)}$, if α_j is a conjugate of α_i. So the degrees of p_i and q_i are equals if α_i and α_j are conjugate.

$$(18) \qquad \tilde{g}(x) = \sum_i \left(\frac{c_{t_{j+1}}^{(i)} \alpha_j^{k_0}}{(1-\alpha_j x)^{t_j+1}} - \sum_j \frac{d_{t_{i+1}}^{(i,j)} (\alpha_i^{(j)})^{k_0}}{(1-\alpha_i^{(j)} x)^{t_i+1}} \right) \quad (\alpha_j = \text{conj. of } \alpha_i)$$

forms a rational function, having coefficients in F.

The difference

$$g_1(x) = g(x) - \tilde{g}(x)$$

is a rational function with coefficients in F too. From (18) we have the representation for $q_1(x)$

$$g_1(x) = \sum_{i=1}^{n} \sum_{u=1}^{t_i} \frac{c_u^{(i)} \alpha_i^{k_0}}{(1-\alpha_i x)^u} - \sum_i \sum_j \sum_{u=1}^{t_i} \frac{d_u^{(i,j)} (\alpha_i^{(j)})^{k_0}}{(1-\alpha_i^{(j)})^u}.$$

So we can see the assertions for the coefficients $c_{t_i}^{(i)}$ too. By repeating the argument used earlier we get, that all the $c_u^{(i)}$-s have the desired properties. So the proof of b) and c) is finished. The Taylor-coefficients of $g(x)$ are the E_k-s, where the E_k-s are integers in F. It follows from the precedings and from the Taylor expansion of the functions staying in the right side of (12), that the E_k-s have the representations

$$E_k = \sum\nolimits^{*} \operatorname{Tr}\left(p_i(k)\alpha_i^k\right),$$

so we proved the assertion e) too.

The proofs of Theorem 1 and the Lemma 1 are modifications of the proofs in [5].

In Theorem 1 we assumed that the α_i-s are algebraic. Leaving that assumption we can prove

Theorem 2. *Let z_k denote the sequence*

$$z_k = \sum_{i=1}^{n} p_i(k)\alpha_i^k,$$

where the α_i-s are different complex numbers. If the series

$$\sum_{k=0}^{\infty} \|z_k\|^2$$

is convergent, then the α_i-s with $|\alpha_i| \geqq 1$ are algebraic numbers.

From the convergence of the series

$$\sum \|z_k\|^2$$

follows, that $\|z_k\| \to 0$, so the properties of Theorem 1 are valid for the α_i-s with $|\alpha_i| \geqq 1$, that one can see from Remark 2. stated after Theorem 1.

The proof of Theorem 2 is based on the lemmas in [3] and on their generalizations.

Lemma 2. *A sequence (z_k) satisfies a linear recurrence relation if and only if the determinants*

$$\Delta_k = \begin{vmatrix} z_0 & z_1 & \dots & z_k \\ z_1 & z_2 & \dots & z_{k+1} \\ z_k & z_{k+1} & \dots & z_{2k} \end{vmatrix}$$

vanish, if k is large enough.

One can find the proof of Lemma 2 in [3].

Let m be the smallest natural number for which

$$\Delta_k = 0, \quad \text{if} \quad k \geqq m.$$

Let $D_0, D_1, \dots, D_m$ denote the minors of Δ_m. Then with

$$\delta_i = -\frac{D_{m-i}}{D_m}$$

the relation

$$z_{k+m} = \delta_1 z_{k+m-1} + \dots + \delta_m z_k$$

holds.

So if the z_k-s are elements of a field F, then the δ_i-s are elements of the field F too.

For the proof of Theorem 2 the next theorem is useful.

Theorem 3. *Let z_k be a sequence of complex numbers, and A_k be a sequence of integers in F, and let moreover the series*

$$\sum_{k=0}^{\infty} |z_k - A_k|^2$$

be convergent.

If the sequence z_k satisfies a linear recurrence relation, then the sequence A_k satisfies a linear recurrence relation too (but not necessarily the same).

The proof of the Theorem 3 is the same as the proof of the Theorem 8.4. in [3].

Let A_k be the nearest algebraic integer in F from z_k and let r_k be defined by the relation

$$z_k = A_k + r_k. \tag{19}$$

Since the sequence z_k satisfies a linear recurrence relation, so the sequence A_k does too. The A_k-s are element in F, we have a linear recurrence relation for the A_k-s:

$$\beta_0 A_{k+m} + \beta_1 A_{k+m-1} + \ldots + \beta_m A_k = 0 \tag{20}$$

where the β_i-s are integers in F, according to the note after Lemma 2 and $\beta_0 \neq 0$.

Since r_k tends to zero for $k \to \infty$, so we have

$$\beta_0 z_{k+m} + \beta_1 z_{k+m-1} + \ldots + \beta_m z_k \to 0$$

for $k \to \infty$.

It follows, that

$$\sum_{i=1}^{n} \left(\beta_0 \alpha_i^m p_i(k+m) + \beta_1 \alpha_i^{m-1} p_i(k+m-1) + \ldots + \beta_m p_i(k)\right) \alpha_i^k \to 0. \tag{21}$$

The factor of the α_i^k is a polynomial in k, the leading coefficient of which is the product of the number

$$\beta_0 \alpha_i^m + \beta_1 \alpha_i^{m-1} + \ldots + \beta_m$$

and of the leading coefficient of $p_i(k)$.

The left hand side of (21) tends to zero, then the equation

$$\beta_0 \alpha_i^m + \beta_1 \alpha^{m-1} + \ldots + \beta_m = 0$$

holds for $|\alpha_i| \geqq 1$, as we saw earlier. So we proved Theorem 2.

The assumption on the convergence of the series $\sum_{k=0}^{\infty} \|z_k\|^2$ one can replace by the assumption, that $\|z_k\|$ tends to zero faster as $1/\sqrt{k}$.

It holds the

Theorem 4. *Let*

$$z_k = \sum_{i=1}^{n} p_i(k) \alpha_k^k,$$

where α_i are different complex numbers, $p_i(k)$ are polynomials. If c is a positive number small enough depending on the α_i-s and on the degrees of the polynomials, and

$$\|z_k\| \leqq \frac{c}{\sqrt{k+1}}, \tag{22}$$

then the α_i-s with $|\alpha_i| \geqq 1$ are algebraic numbers and so the properties in Theorem 1 hold.

The Theorem 4 is a generalization of a theorem due to Gelfond. [6].

Let A_k and r_k be as in (15). We prove, that the determinant

$$\Delta_n = \begin{vmatrix} A_0 & A_1 & \dots A_k \\ A_1 & A_2 & \dots A_{k+1} \\ \dots & \dots & \dots \\ A_k & A_{k+1} & \dots A_{2k} \end{vmatrix}$$

tends to zero for $k \to \infty$.

Let $\beta_1, \dots, \beta_T$ be the coefficients of the polynomial

$$\Pi (x-\alpha_i)^{t_i+1},$$

where t_i is the degree of the $p_i(x)$. Then we have

$$z_{k+T} + \beta_1 z_{k+T-1} + \dots + \beta_T z_k = 0.$$

Let ε_k be defined by the relation

$$\varepsilon_k = A_k + \beta_1 A_{k-1} + \dots + \beta_T A_{k-T}.$$

From (19) it follows that

$$\varepsilon_k = -(r_k + \beta_1 r_{k-1} + \dots + \beta_T r_{k-T}).$$

So we have

$$|\varepsilon_k| \leqq \Big(1 + \sum_{j=1}^{T} |\beta_j|\Big) \frac{c}{\sqrt{k-T+1}} \quad \text{for} \quad k \geqq T. \tag{23}$$

Let η_k be defined in the following manner:

$$\eta_k = \varepsilon_k + \beta_1 \varepsilon_{k-1} + \dots + \beta_T \varepsilon_{k-T}.$$

Then we have

$$|\eta_k| \leqq \Big(1 + \sum_{j=1}^{T} |\beta_j|\Big)^2 \frac{c}{\sqrt{k-2T+1}} \quad \text{for} \quad k \geqq 2T.$$

It follows by elementary transformations

$$\Delta_k = \begin{vmatrix} A_0 \dots A_k \\ A_1 \quad A_{k+1} \\ \dots\dots\dots \\ \\ A_k \dots A_{2k} \end{vmatrix} = \begin{vmatrix} A_0 & A_{T-1} & \varepsilon_T & \varepsilon_k \\ \dots & \dots & \dots & \dots \\ A_{T-1} & A_{2(T-1)} & \varepsilon_{2T-1} & \varepsilon_{k+T-1} \\ \varepsilon_T & \varepsilon_{2T-1} & \eta_{2T} & \eta_{k+T} \\ \dots & \dots & \dots & \dots \\ \varepsilon_k & & \eta_{T+k} & \eta_{2k} \end{vmatrix}.$$

We define A and c_2 by the relations

$$A = \max_{0 \leqq j \leqq 2(T-1)} |A_j|,$$

$$c_2 = \Big(1 + \sum_{j=1}^{T} |\beta_j|\Big)^2.$$

Let c be such small that

(24) $$c_2 c < A.$$

So we have

(25)
$$\Delta_k^2 \leqq A^{2T}(k+1)^T \prod_{l=T}^{k} \left(2c_2^2 c^2 \left(\frac{1}{l-T+1} + \dots + \frac{1}{l+k-2T+1}\right)\right) \leqq$$
$$\leqq A^{2T}(2c_2^2c^2)^{k-T+1}(k+1)^T \log(k-T+2) \cdot \prod_{l=T+1}^{k} \log\left(\frac{l+k-2T+2}{l-T}\right) \leqq$$
$$\leqq A^{2T}(2c_2^2c^2)^{k-T+1}(k+1)^T \log(k-T+1) \cdot \frac{(k+1)^{k-T-1}}{(k-T)!}.$$

If the inequality
$$2ec_2^2c^2 < 1$$
holds, then it follows from the Stirling's formula that Δ_k tends to zero.

But the Δ_k-s are algebraic integers in F, so Δ_k is zero, if k is large enough.

If Δ_k is zero for large k, the series z_k satisfies a linear recurrence relation, and so the α_i-s with $|\alpha_i| \geqq 1$ are algebraic numbers, as we saw in the proof of Theorem 2.

References

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(Received December 17, 1984.)