Almost contact metric submersions and structure equations

By D. CHINEA (Islas Canarias)

In [2] and [3] we have examined the differential geometric properties of almost contact Riemannian submersions between almost contact metric manifolds (almost contact metric submersions).

In this paper we introduce two new structure equations for almost contact metric submersions and use them to study the influence of a given structure defined in the total space on the fibre submanifolds and the base space, and vice versa. Also, we examine the interrelation between the minimality of the fibres and the influence of a given type of almost contact structure of the total space on the determination of the corresponding structure on the base space, and vice versa.

0. Preliminaries

A $(2n+1)$-dimensional real differentiable manifold M of class C^∞ is said to have a (φ, ξ, η)-structure or an almost contact structure if it admits a field φ of endomorphisms of the tangent spaces, a vector field ξ, and a 1-form η satisfying

$$\eta(\xi) = 1$$
$$\varphi^2 = -I+\eta\otimes\xi$$

where I denotes the identity transformation [9]. Then $\varphi\xi=0$ and $\eta\varphi=0$; moreover the endomorphism φ has rank $2n$, [1].

If a manifold M with a (φ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y)-\eta(X)\eta(Y),$$

where $X, Y\in\chi(M)$, then M is said to have (φ, ξ, η, g)-structure or an almost contact metric structure and g is called a compatible metric, [9]. An immediate consequence is $\eta(X)=g(X, \xi)$. The 2-form Φ on M defined by

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the fundamental 2-form of the almost contact metric structure.

Let M be a manifold with an almost contact structure (φ, ξ, η) and consider the manifold $M\times R$. We denote a vector field on $M\times R$ by $\left(X, a\frac{d}{dt}\right)$ where X is tangent to M, t the coordinate of R and a is a C^∞ function on $M\times R$. S. SASAKI and

Y. HATAKEYAMA [10] define an almost complex structure J on $M\times R$ by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right),$$

and they prove that J is integrable if and only if

$$N + 2d\eta\otimes\xi = 0$$

where N is the Nijenhuis tensor of φ.

Now, if g is a Riemannian metric on the manifold M with (φ, ξ, η)-structure, we define a Riemannian metric on $M\times R$ by

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = g(X, Y) + ab$$

and another by

$$h^0 = e^{2\sigma} h,$$

where $\sigma\colon M\times R\to R$ is defined by $\sigma(x, t)=t$ for all $(x, t)\in M\times R$. Then, the following conditions are equivalent [8]:

i) g is metric compatible with the (φ, ξ, η)-structure.
ii) h is a Hermitian metric on $(M\times R, J)$.
iii) h^0 is a Hermitian metric on $(M\times R, J)$.

J. OUBIÑA obtained in [8] (see also [7]) a classification of the different types of almost contact structure on a manifold M through the types of the associated almost Hermitian structures (J, h) and (J, h^0) on $M\times R$, by using Gray—Hervella's classification of almost Hermitian manifolds [5].

We recall the various classes of almost contact metric structures here involved (for the properties, definitions and examples of the different types of almost contact metric structures, we refer the reader to [1], [8], [7]). Let ∇ and δ denote the Riemannian connection and the coderivative in M, then $(M, \varphi, \xi, \eta, g)$ is said to be

(1) Quasi-K-cosymplectic (qKC) if: $(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y - \eta(Y)\nabla_{\varphi X}\xi = 0$,
(2) Quasi-trans-Sasakian (qtS) if:

$$(\nabla_X\Phi)(Y, Z) + (\nabla_{\varphi X}\Phi)(\varphi Y, Z) + \eta(Y)(\nabla_{\varphi X}\eta)Z = -\frac{1}{n}\left[g(X,Y)\delta\Phi(Z) - \right.$$

$$\left. - g(X, Z)\delta\Phi(Y) - g(X, \varphi Y)\big(\delta\Phi(\varphi Z) - \eta(Z)\delta\eta\big) + g(X, \varphi Z)\big(\delta\Phi(\varphi Y) - \eta(Y)\delta\eta\big)\right]$$

(3) Semi-cosymplectic (sC) if $\delta\Phi = 0$ and $\delta\eta = 0$,
(4) Semi-Sasakian (sS) if $\eta = \frac{1}{2n}\,\delta\Phi$.

A $2r$-dimensional immersed submanifold M of a $(2n+1)$-dimensional almost contact metric manifold $\tilde{M}$ with structure $(\tilde\varphi, \tilde\xi, \tilde\eta, \tilde g)$ is said to be invariant if $\tilde\varphi X$ is tangent to M for any tangent vector X of M.

Any invariant submanifold M with the structure (φ, g) is an almost Hermitian manifold and $\tilde\xi$ is normal to M, where φ and g are the restrictions of $\tilde\varphi$ and $\tilde g$ in M, [12].

Now, let $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta', g')$ be almost contact metric manifolds, with $\dim M = 2m+1$ and $\dim M' = 2m'+1$. A smooth surjective mapping $f\colon M\to M'$ is called a Riemannian submersion [6] if:

(1) f has maximal rank, and
(2) $f_{*/(\ker f_*)^\perp}$ is a linear isometry.

We say that f is an almost contact metric submersion if f is a Riemannian submersion which, additionally, is
(3) An almost contact mapping (i.e., $\varphi' f_* = f_* \varphi$.

Vectors on M which are in the kernel of f_* are tangent to the fibres $(F_{m'} = f^{-1}(m'), m' \in M')$ and are called vertical vectors. Vectors which are orthogonal to the vertical distribution are said to be horizontal. We denote the vertical and horizontal distributions in the tangent bundle of the total space M by $V(M)$ and $H(M)$, respectively. Then $T(M)$ enjoys an orthogonal decomposition: $T(M) = V(M) \oplus H(M)$. The orthogonal projection mappings are denoted $v\colon T(M) \to V(M)$ and $h\colon T(M) \to H(M)$, respectively.

Since the fibres $F_{m'}$ are even-dimensional and invariant submanifolds M, $F_{m'}$ with the structure (φ, g) is almost Hermitian and the vector field ξ is horizontal (we shall suppose that $f_* \xi = \xi'$), [2]. For the properties of the different types of almost contact metric submersions, we refer the reader to [2], [3].

1. The structure equations of an almost contact metric submersion

Let $f\colon (M, \varphi, \xi, \eta, g) \to (M', \varphi', \xi', \eta', g')$ be an almost contact metric submersion. We define a tensor A^* by

$$A^*(X, Y) = A_X \varphi Y - A_{\varphi X} Y,$$

for all horizontal vector fields X and Y, where A is the O'Neill configuration tensor [6], i.e.,

$$A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,$$

for all $E, F \in \chi(M)$, where $\bar{\nabla}$ denotes the Riemannian connection in M.

Let H denote the mean curvature vector field of the fibre submanifolds, $F_{m'}$, of the submersion, let Φ, Φ' and $\hat{\Phi}$ denote the fundamental 2-forms of the total space, base space and of the fibres, respectively, and denote by δ, δ' and $\hat{\delta}$ the coderivatives on said manifolds, respectively. Finally, if X is a horizontal vector field, we denote the vector field $f_* X$ on M' by X_*.

Theorem 1.1. *Let $f\colon M \to M'$ be an almost contact metric submersion, and let E be a vector field on M. Then*

$$\delta\Phi(E) = g(H, \varphi hE) + \delta'\Phi'(hE_*) + \hat{\delta}\hat{\Phi}(vE) + \frac{1}{2} g(\operatorname{tr} A^*, vE)$$

and

$$\delta\eta = -g(H, \xi) + \delta'\eta' \cdot f.$$

Proof. Let $\{E_1, \ldots, E_{m-m'}, \varphi E_1, \ldots, \varphi E_{m-m'}, F_1, \ldots, F_{m'}, \varphi F_1, \ldots, \varphi F_{m'}, \xi\}$ be, a local φ-basis defined on an subset of M, whose horizontal vector fields F_i are basic and the vector fields E_i are vertical. For any vector field E on M, we have,

$$\delta\Phi(E) = -\sum_{i=1}^{m-m'} \left((\nabla_{E_i}\Phi)(E_i, E) + (\nabla_{\varphi E_i}\Phi)(\varphi E_i, E)\right) - \sum_{i=1}^{m'} \left((\nabla_{F_i}\Phi)(F_i, E) + (\nabla_{\varphi F_i}\Phi)(\varphi F_i, E)\right) - (\nabla_\xi \Phi)(\xi, E).$$

Thus, if $E=V$ is vertical, we deduce

$$\delta\Phi(V) = \hat{\delta}\hat{\Phi}(V) + \frac{1}{2}\, g(\operatorname{tr} A^*, V)$$

and if $E=X$ is horizontal ([2])

$$\delta\Phi(X) = g(H, \varphi X) + \delta'\Phi'(X)_* f,$$

so,

$$\delta\Phi(E) = \delta\Phi(hE) + \delta\Phi(vE) = g(H, \varphi hE) + \delta'\Phi'(hE_*)f + \hat{\delta}\hat{\Phi}(vE) + \frac{1}{2} g(\operatorname{tr} A^*, vE).$$

The other relation is deduced from [2, Prop. 1.3 and Theor. 3.1].

The equations of Theorem 1.1 are called the structure equations ofthe submersion. Next, we study the influence of the structure equations on some types of almost contact metric submersion.

If $f\colon M \to M'$ is a quasi-K-cosymplectic submersion: $A_X\varphi Y = -A_Y\varphi X$ for all X, Y horizontals, and $\delta\Phi = \delta'\Phi' = \hat{\delta}\hat{\Phi} = \delta\eta = \delta'\eta' = 0$, then, the structure equations are equivalen to

$$g(H, \varphi X) = 0 \quad \text{for all } X \text{ horizontal, and} \quad g(H, \xi) = 0,$$

and we get

Theorem 1.2. *Let $f\colon M \to M'$ be a quasi-K-cosymplectic submersion. Then the fibres $F_{m'}$ are minimal submanifolds of M.*

If $f\colon M \to M'$ is an almost contact metric submersion and M is quasi-trans-Sasakian, then

$$\delta\Phi(X) = \frac{m}{m'}\, \delta'\Phi'(X^*)f, \quad \text{for all } X \text{ horizontal}$$

$$\delta\Phi(V) = \frac{m}{m-m'-1}\, \hat{\delta}\hat{\Phi}(V), \quad \text{for all } V \text{ vertical,}$$

and

$$\delta\eta = \frac{m}{m'}\, \delta'\eta'.$$

Thus, the structure equations are reduced to

$$\frac{m-m'}{m}\, \delta'\Phi'(X_*)f = g(H, \varphi X) \quad \text{for all } X \text{ horizontal}$$

$$\frac{m'+1}{m-m'-1}\, \hat{\delta}\hat{\Phi}(V) = \frac{1}{2}\, g(\operatorname{tr} A^*, V) \quad \text{for all } V \text{ vertical,}$$

and

$$\frac{m'-m}{m}\, \delta'\eta' = g(H, \xi),$$

so, we obtain,

Theorem 1.3. *Let f: $M \to M'$ be an almost contact metric submersion and M a quasi-trans-Sasakian manifold. Then the fibres $F_{m'}$ of f are minimal submanifolds of M if and only if M' is quasi-K-cosymplectic.*

Finally, if $m-m' \neq 1$ we get

Theorem 1.4. *Let f: $M \to M'$ be an almost contact metric submersion and M a quasi-trans-Sasakian manifold. Then the fibres $F_{m'}$ of f are quasi Kaehler manifolds if and only if* $\operatorname{tr} A^*=0$.

2. The structure equations of almost contact metric semi-submersions

If f: $M \to M'$ is an almost contact metric submersion and M is semi-cosymplectic, then the structure equations of the submersion are

$$g(H, \varphi hE)+\delta'\Phi'(hE_*)+\hat{\delta}\hat{\Phi}(vE)+\frac{1}{2}\,g(\operatorname{tr} A^*, vE) = 0, \quad \text{and} \quad g(H, \xi) = \delta'\eta'\cdot f.$$

We begin our applications of the structure equations by proving a theorem first announced in [2].

Theorem 2.1. *Let f: $M \to M'$ be an almost contact metric submersion and M a semi-cosimplectic manifold. Then M' is semi-cosymplectic if and only if the fibres $F_{m'}$ of f are minimal submanifolds (if and only if f is a harmonic mapping).*

PROOF. Let X be a basic vector field on M. The first of the structure equations implies that,

$$g(H, \varphi X) = -\delta'\Phi'(X_*)f,$$

and the second,

$$g(H, \xi) = \delta'\eta' f.$$

Thus, $H=0$ if and only if $\delta'\Phi'=\delta'\eta'=0$, i.e., if and only if M' is semi-cosymplectic.

Theorem 2.2. *Let f: $M \to M'$ be an almost contact metric submersion and M a semi-cosimplectic manifold. Then the fibres $F_{m'}$ of M are semi-Kaehler if and only if* $\operatorname{tr} A^*=0$.

PROOF. Let V be a vertical vector field on M. Then, the first of the structure equations implies that,

$$\hat{\delta}\hat{\Phi}(V) = -\frac{1}{2}\,g(\operatorname{tr} A^*, V),$$

from which we deduce the theorem.

Also, we have

Theorem 2.3. *Let f: $M \to M'$ be an almost contact metric submersion whose base space M' is semi-cosymplectic and whose fibres are both minimal and semi-Kaehler. Then the total space M is semi-cosymplectic if and only if* $\operatorname{tr} A^*=0$.

This may be reformulated as:

Theorem 2.4. *Let $f\colon M\to M'$ be a harmonic almost contact metric submersion whose base space is semi-cosymplectic and whose fibres are minimal. Then M is semi-cosymplectic if and only if* $\operatorname{tr} A^*=0$.

Now, if $f\colon M\to M'$ is an almost contact metric submersion and M a semi-Sasakian manifold, then the structure equations are,

$$2m\eta(E)=g(H,\varphi hE)+\delta'\Phi'(hE_*)+\hat{\delta}\hat{\Phi}(vE)+\frac{1}{2}g(\operatorname{tr} A^*, vE),\ \text{ and }\ \delta'\Phi' f=g(H,\xi),$$

so, we get.

Theorem 2.5. *Let $f\colon M\to M'$ be an almost contact metric submersion and M a semi-Sasakian manifold. Then the fibres $F_{m'}$ of M are semi-Kaehler if and only if* $\operatorname{tr} A^*=0$.

Theorem 2.6. *Let $f\colon M\to M'$ be an almost contact metric submersion and M semi-Sasakian. Then the fibres $F_{m'}$ of f are minimal submanifolds of M if and only if*

$$\eta'=\frac{1}{2m}\delta'\Phi'.$$

Finally, if we consider in M' the almost contact metric structure $\left(\varphi', \frac{m'}{m}\xi', \frac{m}{m'}\eta', \frac{m^2}{m'^2}g'\right)$, we obtain,

Theorem 2.7. *Let $f\colon M\to M'$ be an almost metric submersion and M a semi-Sasakian manifold. Then M' is semi-Sasakian if and only if the fibres $F_{m'}$ of f are minimal submanifolds (if and only if f is a harmonic mapping).*

Theorem 2.8. *Let $f\colon M\to M'$ be an almost contact metric submersion, M' a semi-Sasakian manifold and the fibres $F_{m'}$ semi-Kaehler and minimal submanifolds. Then the total space M is semi-Sasakian if and only if* $\operatorname{tr} A^*=0$.

References

[1] D. E. Blair, Contact manifolds in Riemannian geometry, *Lecture Notes in Math.* **509** (1976), *Springer, Berlin—Heidelberg—New York.*

[2] D. Chinea, Almost contact metric submersions, *Rend. Circ. Mat. Palermo* **84** (1985), 89—104.

[3] D. Chinea, Quasi-K-cosymplectic submersions, *Rend. Circ. Mat. Palermo* **83** (1984), 319—330.

[4] J. Eells & J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109—160.

[5] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their lineal invariants, *Ann. Mat. Pura Appl.* (IV), **123** (1980), 35—58.

[6] B. O'Neill, The fundamental equations of a submersion, *Michigan Math. J.* **13** (1966), 459—469.

[7] J. A. Oubiña, New classes of almost contact metric structures, *Publicationes Mathematicae (Debrecen).* **32** (1985), 189—193.

[8] J. A. Oubiña, A classification for almost contact structures, *to appear.*

[9] S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure I, *Tohoku Math. J.* **12** (1960), 459—476.

[10] S. SASAKI and Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II, *Tohoku Math. J.* **13** (1961), 281—294,
[11] B. WATSON and L. VANHECKE, The structure equation of an almost semi-Kähler submersion. *Houston J. of Math.* **5** (1979), 295—305.
[12] K. YANO and S. ISHIHARA, Invariant submanifolds of an almost contact manifold, *Kodai Math. Sem. Reo.* **21** (1969), 350—364.

DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA
FACULTAD DE MATEMATICAS
UNIVERSIDAD DE LA LAGUNA
ISLAS CANARIAS — ESPAÑA

(Received July 31, 1985.)