About eigensolutions of abstract differential equations with mixed conditions

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Introduction

Let's take Operational Calculus $\mathrm{CO}(L^0, L^1, S, Tq, sq, q, Q)$, where L^0, L^1 are linear spaces, $S\colon L^1 \to L^0$ (onto) is a linear operation called derivative.

Linear operations $T(q)\colon L^0 \to L^1$, $s(q)\colon L^1 \to \operatorname{Ker} S$, such that $ST(q) = \mathrm{id}_{L^0}$, $T(q)S = \mathrm{id}_{L^1} - s(q)$ are called integral and limit condition. Index $q \in Q$ defines uniqueily integral and limit condition (definition and properties of the Operational Calculus see [3]).

We assume, that $\operatorname{Ker} S$ is algebra with unit.

Linear space $\Xi(X)$ is a set of results that is fractions of the form $\frac{f}{u}$, where f is an element of the linear space X, while $U\colon X \to X$ is an endomorphism and at the same time an injection belonging to a commutative subgroup $\pi(X)$ (see [3]).

If $A\colon X \to X$ is a commutative endomorphism with injections $U \in \pi(X)$, then operation $\mu = \frac{A}{U}$ definition in the results space $\Xi(X)$ with the formula

$$\mu \frac{f}{V} = \frac{Af}{UV} \tag{1}$$

is called an operator.

In particular operator $p = \frac{\mathrm{id}_{L^0}}{Tq}$ is the so called operator of HEAVISIDE (see [3]).

In this work we will use the formula

$$S^n x = p^n x - p^n x_0 - p^{n-1} x_1 - \ldots - p x_{n-1}, \tag{2}$$

$$x_i = s(q) S^i x, \quad i = 0, 1, \ldots, n-1,$$

if only $x \in L^n$ (proof see [3]).

In the space $L^k \subset L^0$, $k \in N$ we introduce the relation of equivalence $\frac{\|k\|}{\bar{q}}$:

$$x \frac{k}{\bar{q}} y \Leftrightarrow \bigwedge_{0 \le i \le k-1} s(q) S^i (x - y) = 0 \quad \text{(see [4], [7])}. \tag{3}$$

Set $\mathscr{C}_q^k := L^k / \frac{k}{\bar{q}}$, whose elements are abstract's classes

(4) $$[x]_q^k = \left[\sum_{i=0}^{k-1} T^i(q)s(q)S^ix\right]_q^k$$

is called a set of Taylor's jets (see [4], [7]). Operation $t^i(q)$ with the property

$$t^i(q)c = [i!T^i(q)c]_q^k, \quad c \in \operatorname{Ker} S$$

is called an abstract variable. Hence

$$\xi := [x]_q^k = \sum_{i=0}^{k=1} \frac{t^i(q)}{i!} x_i, \quad x_i = s(q)S^ix \in \operatorname{Ker} S.$$

CHAPTER I.
EXPOTENTIAL FUNCTIONS. HEAVISIDE'S THEOREMS*

§ 1. Logarithm. Expotential functions and expotential polynomials.

Df. 1.1. Commutative endomorphism R with derivative S and with limit condition $s(q)$ is called a logarithm, if

(1.1.1) $(I - T(q)R)f = 0$ entails $f = 0$ for $f \in L^0$, or what gives the same result

(1.1.2) $$Sf = Rf, \quad s(q)f = 0 \quad \text{implies} \quad f = 0.$$

Df. 1.2. If there exist elements $f \neq 0$ satisfying condition $s(q)f = 0$ and equation $Sf = Rf$, then endomorphism $R \neq 0$ is called eigenendomorphism and element f eigenelement. When $R = \varrho \operatorname{id}_{L^0}$, when number ϱ is called eigenvalue.

Theorem 1.1. *If there exists a solution of equation*

(1.1.3) $$Sx = Rx, \quad s(q)x = c, \quad c \in \operatorname{Ker} S, \ R\text{-logarithm},$$

then there exists only one.

PROOF. Let's assume, that x_1, x_2, $x_1 \neq x_2$ are solutions of the equation (1.1.3). So

$$Sx_1 = Rx_1, \quad s(q)x_1 = c$$

$$Sx_2 = Rx_2, \quad s(q)x_2 = c.$$

Substracting these equations and applying linear character of logarithm R and then using (1.1.2) we obtain contraction with our assumption, what was to be proved.

Let R be logarithm and let a solution of equation (1.1.3) exist. We define this solution

(1.1.4) $$x = e^{Rt(q)}c$$

* This chapter is written on the basis of [3] § 6.

Df. 1.3. Function

$$e^{Rtq}\colon \mathrm{Ker}\, SL^1 \tag{1.1.5}$$

given by the formula

$$c = e^{Rtq}c \tag{1.1.6}$$

is called exponential function.

Theorem 1.2. *If result* c) $I-TqR$ *is an element of space* L^0, *then exponential function can be defined with operator* $I(I-TqR=P)$ $P-R$ *in the following way:*

$$e^{Rtq}c = \frac{c}{I-TqR} = \frac{p}{p-R}c. \tag{1.1.7}$$

PROOF. Let $x=e^{Rtq}c$. Multiplying both sides of the equation (1.1.3) by integral Tq and applying limit condition, we get

$$x-c = TqRx.$$

Dividing by $I-TqR$ is possible because R is logarithm. Hence we have

$$x = \frac{c}{I-TqR}. \tag{1.1.8}$$

Multiplying numerator and denominator of the fraction by p we get

$$x = \frac{p}{p-R}c, \tag{1.1.9}$$

what was to be proved.

Theorem 1.3. *Let's assume, that together with the expotential function*

$$ce^{Rtq}c = \frac{c}{I-TqR}$$

as a result belonging to space L^0, *further results*

$$\frac{T^m(q)\,c}{(I-T(q)R)^{m+1}} = \frac{p}{(p-R)^{m+1}}c, \quad m = 1, 2, \ldots \tag{1.1.10}$$

are also elements of space L^0. *Than all these elements belong to space* L^∞. *Defining*

$$\frac{t^m(q)}{m!}e^{Rt(q)}c := \frac{T^m(q)\,c}{(I-T(q)R)^{m+1}}, \tag{1.1.11}$$

we have also

$$\begin{aligned} S\left(\frac{t^m(q)}{m!}e^{Rt(q)}c\right) &= \frac{t^{m-1}(q)}{(m-1)!}e^{Rt(q)}c + \frac{t^m(q)}{m!}e^{Rt(q)}Rc, \\ s\left(\frac{t^m(q)}{m!}e^{Rt(q)}c\right) &= 0 \quad for \quad m = 1, 2, \ldots. \end{aligned} \tag{1.1.12}$$

PROOF. Let's notice the following identity:

$$\frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}=\frac{T^{m-1}(q)c}{(I-T(q)R)^{m}}+\frac{T^{m}(q)Rc}{(I-T(q)R)^{m+1}}. \tag{1.1.13}$$

From the assumption it follows, that the right side is an element of space L^0. In such a case

$$T(q)\frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}\in L^1,$$

i.e.

$$\frac{T^{m}(q)c}{(I-T(q)R)^{m+1}}\in L^1$$

for every natural m. Through induction we get that all these elements are the elements of space L^∞.

Of course

$$S\left(T(q)\frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}\right)=\frac{T^{m-1}(q)c}{(I-T(q)R)^{m-1}} \tag{1.1.14}$$

and

$$s\left(T(q)\frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}\right)=0, \tag{1.1.15}$$

so we have proved that element

$$T(q)\frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}$$

is an element of space L^1. Applying definition 1.1.11 we get

$$\frac{T^{m}(q)c}{(I-T(q)R)^{m+1}}=\frac{t^m(q)}{m!}e^{Rt(q)}c,$$

$$\frac{T^{m-1}(q)c}{(I-T(q)R)^{m}}=\frac{t^{m-1}(q)}{(m-1)!}e^{Rt(q)}c,$$

$$\frac{T^{m}(q)Rc}{(I-T(q)R)^{m+1}}=\frac{t^m(q)}{m!}e^{Rt(q)}Rc.$$

From identity (1.1.13) and formulas (1.1.14), (1.1.15) we obtain formulas (1.1.12). We get (1.1.10) multiplying the numerator and denominator of the left side of the identity by p^{m+1}.

Results $\frac{t^m(q)}{m!}e^{Rt(q)}c$ are called expotential functions if they belong to space L^0.

Df. 1.4. Linear combination of arbitrary expotential functions is called expotential polynomial.

It is then of the form

$$W(t(q)) = \sum_{j=1}^{m} \sum_{k_j=0}^{n} \frac{t^{k_j}(q)}{k_j!} e^{R_j t(q)} c_{jk_j}. \tag{1.1.16}$$

On the basis of (1.1.10) expotential polynomial has also the form

$$V(p) = \sum_{j=1}^{m} \sum_{k_j=0}^{n_j} \frac{p}{(p-R_j)^{k_j+1}} c_{jk_j} \in \Xi(L^0). \tag{1.1.17}$$

§ 2. Heaviside's theorem about expotential functions of the operator p.

Theorem 2.1. (The first Heaviside's theorem). *In case when the degree of the numerator does not exceed the degree of the denominator and when polynomial $H(p)$ can be presented in the form*

$$H(p) = B_m(p-R_1)(p-R_2)\cdot \ldots \cdot(p-R_m), \tag{1.2.1}$$

the rational function of operator p

$$F(p)\,c := \frac{K(p)}{H(p)}\,c = \frac{A_n p^n + A_{n-1} p^{n-1} + \ldots + A_1 p + A_0}{B_m p^m + B_{m-1} p^{m-1} + \ldots + B_1 p + B_0}\,c, \tag{1.2.2}$$

can be defined by the ordinary expotential functions with the formula

$$F(p)c = \frac{K(0)}{H(0)}\,c + \sum_{k=1}^{m} \frac{K(R_k)}{B_m R_k H'(R_k)}\, e^{R_k t(q)} c, \tag{1.2.3}$$

where $A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_m \in \pi(X)$, $R_1, R_2, \ldots, R_m$ are logarithms differents from zero, $H'(R_k) = (R_k - R_1)\cdot \ldots \cdot (R_k - R_{k-1})(R_k - R_{k+1}) \cdot \ldots \cdot (R_k - R_m)$.

Proof. From the assumption we have

$$\frac{K(p)}{pH(p)} = \frac{K(p)}{B_m p(p-R_1)\cdot \ldots \cdot (p-R_m)} = \frac{K_0}{p} + \frac{K_1}{p-R_1} + \ldots + \frac{K_m}{p-R_m},$$

because in this case the degree of the numerator is lower from the degree of denominator and there exists distribution into ordinary proper fractions. Hence we have

$$K(p) = B_m K_0(p-R_1)\cdot \ldots \cdot (p-R_m) + B_m K_1 p(p-R_2)\cdot \ldots \cdot (p-R_m) + \ldots +$$
$$+ B_m K_m p(p-R_1)\cdot \ldots \cdot (p-R_{m-1})$$

and hence

$$K_k = \frac{K(R_k)}{B_m R_k (R_k - R_1)\cdot \ldots \cdot (R_k - R_{k-1})(R_k - R_{k+1})\cdot \ldots \cdot (R_k - R_m)} = \frac{K(R_k)}{B_m R_k H'(R_k)}$$

$$\text{for} \quad k = 1, 2, \ldots, m.$$

All these calculations are permissible, if it is allowed to devide by R_k and $R_i - R_j$; We have also

$$K(0) = B_m K_0(-R_1)\cdot \ldots \cdot (-R_m) = K_0 H(0).$$

hence

$$K_0 = \frac{K(0)}{H(0)}.$$

So

$$F(c) = \frac{K(p)}{H(p)}c = \left(K_0 + K_1\frac{p}{p-R_1} + \ldots + K_m\frac{p}{p-R_m}\right)c = K_0 c + \sum_{j=1}^{m} K_j e^{R_j t(q)} c,$$

what was to be proved.

Theorem 2.2. (The second Heaviside's theorem). *Let's admit for denominator $H(p)$ a canonical distribution with multiple roots and with zero root.*

$$\text{(1.2.4)} \qquad H(p) = B_m p^{\alpha_0}(p-R_1)^{\alpha_1}\ldots(p-R_l)^{\alpha_l}, \quad \sum_{j=0}^{l}\alpha_j = m.$$

Then

$$\text{(1.2.5)} \qquad F(p)c = \frac{K(p)}{H(p)}c = \sum_{r=0}^{\alpha_0} A_{0r}\frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!}c + \sum_{k=1}^{l}\sum_{r=1}^{\alpha_k} A_{kr}\frac{t_{(q)}^{\alpha_k+1-r}}{(\alpha_k+1-r)!}e^{R_k t(q)}c,$$

where

$$\text{(1.2.6)} \qquad A_{0r} = \Big|_{p=0}\frac{1}{r!}\frac{d^r}{dp^r}\left[\frac{p^{\alpha_0}K(p)}{H(p)}\right],$$

$$\text{1.2.7)} \qquad A_{kr} = \Big|_{p=R_k}\frac{1}{(r-1)!}\frac{d^{r-1}}{dp^{r-1}}\left[\frac{(p-R_k)^{\alpha_k}K(p)}{pH(p)}\right].$$

Proof. Similarly as in the proof of theorem 2.1 we look for distribution of function $\frac{F(p)}{p}c$ into common fractions, obtaining

$$\frac{K(p)}{pH(p)}c = \frac{A_{00}}{p^{\alpha_0+1}}c + \frac{A_{01}}{p^{\alpha_0}}c + \ldots + \frac{A_{0\alpha_0}}{p}c + \frac{A_{11}}{(p-R_1)^{\alpha_1}}c + \frac{A_{12}}{(p-R_1)^{\alpha_1-1}}c + \ldots +$$

$$+ \frac{A_{1\alpha_1}}{(p-R_1)^1}c + \ldots + \frac{A_{l1}}{(p-R_l)^{\alpha_l}}c + \ldots + \frac{A_{l\alpha_l}}{(p-R_l)^1}c.$$

Multiplying the last identity by p^{α_0+1} and then derivating r-fold with regard to variable p we obtain

$$\frac{d^r}{dp^r}\left[\frac{p^{\alpha_0}K(p)}{H(p)}\right] = r!\,A_{0r} + (r+1)!\,pA_{0,r+1} + \ldots + \frac{\alpha_0!}{(\alpha_0-r)!}p^{\alpha_0-r}A_{0,\alpha_0} +$$

$$+ (\alpha_0+1)p^{\alpha_0}\left[\frac{A_{11}}{(p-R_1)^{\alpha_1}} + \ldots + \frac{A_{l\alpha_l}}{(p-R_l)^1}\right] + p^{\alpha_0+1}\frac{d^r}{dp^r}\left[\frac{A_{11}}{(p-R_1)^{\alpha_1}} + \ldots + \frac{A_{l\alpha_l}}{(p-R_l)^1}\right].$$

Substituting $p=0$ we get formula (1.2.6). Similarly, multiplying the considered identity by $(p-R_k)^{\alpha_k}$, then deriviting it r-fold and substituting $p=R_k$ we get identity (1.2.7).

Remembering, that

$$\frac{1}{p^k} c = \frac{t^k(q)}{k!} c, \quad \frac{p}{(p-R_k)^{j_k+1}} c = \frac{t^{j_k}(q)}{j_k!} e^{R_k t(q)} c$$

we get (1.2.5).

We will use formula (1.2.5) under the assumption that it is allowed to divide by endomorphisms R_i and $R_j - R_k$.

CHAPTER II.
EIGENSOLUTIONS OF THE ABSTRACT DIFFERENTIAL LINEAR EQUATIONS OF THE ORDER N AND THE SYSTEM OF EQUATIONS OF THE FIRST ORDER

§ 1. The existence of the eigensolutions of the abstract differential linear equation with mixed conditions.

Let's consider linear differential equations of the order n

$$a_n S^n x + a_{n-1} S^{n-1} x + \dots + a_1 Sx + a_0 x = \lambda x, \qquad x \in L^n, \quad \lambda, a_i \in \pi(L^1). \tag{2.1.1}$$

Onto solution x we impose conditions on the mixed type

$$\sum_{j=1}^{n} \sum_{\beta=0}^{r_j} a_{\beta j}^{(\gamma)} x_{\beta j} = 0, \quad \gamma = 1, 2, \dots, w, \ a_{\beta j}^{(\gamma)} \in \operatorname{Ker} S, \tag{2.1.2}$$

where

$$x_{\beta j} := \sigma(t_j) S^{\beta} x = |_{t(q)=t_j}((S^{\beta} x), \quad \beta = 0, 1, \dots, r_j; \ j = 1, \dots, N, \tag{2.1.3}$$

while

$$t_j \underset{\tau_0}{\overset{m}{=}} t_0^{(j)} + \frac{\tau(\tau_0)}{1!} t_1^{(j)} + \dots + \frac{\tau^{m-1}(\tau_0)}{(m-1)!} t_{m-1}^{(j)}, \quad t_i^{(j)} \in \operatorname{Ker} S_\tau, \ i = 0, 1, \dots, m-1, \tag{2.1.4}$$

is Taylor's jet of the class $\mathscr{C}_{\tau_0}^m$, where S_τ is arbitrary derivative (see [4], [7]).

Applying formulas (2) and (2.1.1) we get

$$a_n(p^n x - p^n x_0^0 - p^{n-1} x_1^0 - \dots - p x_{n-1}^0) + a_{n-1}(p^{n-1} x - p^{n-1} x_0^0 - - p^{n-2} x_i^0 - \dots - p x_{n-2}^0) + \dots + a_1(px - px_0^0) + a_0 x = \lambda x. \tag{2.1.6}$$

Hence

$$x = \sum_{i=0}^{n-1} \frac{W_i(p)}{W(p, \lambda)} x_i^0, \quad x_i^0 = s(q) S^i x, \ i = 0, 1, \dots, n-1, \tag{2.1.7}$$

where

$$W(p, \lambda) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + (a_0 - \lambda) \tag{2.1.8}$$

$$W_i(p) = a_n p^{n-i} + a_{n-1} p^{n-i+1} + \dots + a_{i+1} p \quad \text{for} \quad i = 0, 1, \dots, n-1. \tag{2.1.9}$$

Formula (2.1.7) can be applied when the roots of polynomial $W(p, \lambda)$ are logarithms of derivative S.

Let $0, R_1, R_2, \ldots, R_l$ be the roots of the equation $W(p, \lambda)=0$ with multiplication factors recpectively $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_l$. Applying the Second Heaviside's Theorem we get

$$x = \sum_{i=0}^{n-1} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i)} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} + \sum_{k=1}^{i} \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \frac{t_{(q)}^{\alpha_k+1-r}}{(\alpha_k+1-r)!} e^{R_k t(q)} \right] x_i^0, \tag{2.1.10}$$

where

$$A_{0r}^{(i)} = \Big|_{p=0} \frac{1}{r!} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0} W_i(p)}{W(p, \lambda)} \right] \tag{2.1.11}$$

$$A_{kr}^{(i)} = \Big|_{R=p_k} \frac{1}{(r-1)!} \frac{d^{r-1}}{dp^{r-1}} \left[\frac{(p-R_k)^{\alpha_k} W_i(p)}{pW(p, \lambda)} \right].$$

It is easy to notice that

$$\sigma(t_j) S^{\beta} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} = \begin{cases} \dfrac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} & \text{for} \quad \alpha_0+1-r-\beta \geqq 0 \\ 0 & \text{for} \quad \alpha_0+1-r-\beta < 0 \end{cases},$$

similarly for

$$\frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} e^{R_k t(q)}.$$

Acting with operation σ onto general solution (2.1.10) we have

$$x_{\beta j} = \sum_{i=0}^{n-1} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i)} \frac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} + \sum_{k=1}^{i} \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \sigma(t_j) \left\{ \sum_{\nu=0}^{\beta} \binom{\beta}{\nu} \times \right.\right.$$
$$\left.\left. \times \left(\frac{t_{(q)}^{\alpha_k+1-r-\nu}}{(\alpha_k+1-r-\nu)!} e^{R_k t(q)} + \frac{t_{(q)}^{\alpha_k+1-r}}{(\alpha_k+1-r)!} R_k^{\beta-\nu} e^{R_k t(q)} \right) \right\} \right] x_i^0. \tag{2.1.12}$$

Substituting the so defined $x_{\beta j}$ into equations (2.1.2) we get the system

$$\sum_{j=1}^{N} \sum_{\beta=0}^{r_j} a_{\beta j}^{(j)} \sum_{i=0}^{n-1} \left[A_{0r}^{(i)} \frac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} + \sum_{k=1}^{i} \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \sigma(t_j) \left\{ \sum_{\nu=0}^{\beta} \binom{\beta}{\nu} \times \right.\right.$$
$$\left.\left. \times \left(\frac{t_{(q)}^{\alpha_k+1-r-\nu}}{(\alpha_k+1-r-\nu)!} e^{R_k t(q)} + \frac{t_{(q)}^{\alpha_k+1-r}}{(\alpha_k+1-r)!} R_k^{\beta-\nu} e^{R_k t(q)} \right) \right\} \right] x_i^0 = 0, \tag{2.1.13}$$

System (2.1.13) is a system of homogeneous W equations with unknown $x_0^0, x_1^0, \ldots$ $\ldots, x_{n-1}^0$.

If the order of the matrix $\mathfrak{a}$ of this system is smaller than n, then equation (2.1.1) with conditions (2.1.12) has eigensolutions defined with the formula (2.1.10), where constants $x_0^0, x_1^0, \ldots, x_{n-1}^0$ are solutions of the system (2.1.3), where the order of the matrix is equals r (then we write $r(\mathfrak{a})=r$), if there exists a reversible determinant of the order r formed from the elements of this matrix while all the determinants of the order higher than r are not revercible (see [8] collorary on page 359). Let's consider in particular the case where polynomial $W(p, \lambda)$ has non-zero roots with multiplication one: $R_1, R_2, \ldots, R_n$, while condition (2.1.2) is of the form

$$\text{(2.1.14)} \qquad \sum_{\beta=0}^{r_1} A_\beta^{(\gamma)} x_{\beta 1} = \sum_{\beta=0}^{r_2} B_\beta^{(\gamma)} x_{\beta 2}; \; x_{\beta j} = |_{t(q)=t_j}(S^\beta x), \; j = 1, 2; \; \gamma = 1, 2, \ldots, W.$$

Applying to the formula the first Heaviside's theorem we get

$$\text{(2.1.15)} \qquad x = \sum_{i=0}^{n-1} \Big[b_i + \sum_{k=1}^{n} d_{ik} e^{R_k t(q)}\Big] x_i^0,$$

where

$$b_i := \frac{W_i(0)}{W(0, \lambda)}, \quad d_{ik} := \frac{W_i(R_k)}{a_k R_k W'(R_k)}.$$

Because from the assumption we have that $W(0, \lambda)$ is reversible (we write $W(0, \lambda) \in \mathrm{Inv}$), and for every $i = 0, 1, \ldots, n-1$, $W_i(0) = 0$, hence $b_i = 0$.

Taking denotation

$$\text{(2.1.16)} \qquad c_k := \sum_{i=0}^{n-1} d_{ik} x_i^0$$

finaly we have formula

$$\text{(2.1.17)} \qquad x = \sum_{k=1}^{n} e^{R_k t(q)} c_k.$$

Substituting the obtained general solution to (2.1.14) we get system of homogeneous equations

$$\text{(2.1.18)} \qquad \sum_{\beta=0}^{\max(r_1, r_2)} \sum_{k=1}^{n} R_k^\beta (A_\beta^{(\gamma)} e^{R_k t_1} - B_\beta^{(\gamma)} e^{R_k t_2}) c_k = 0, \quad r = 1, 2, \ldots, W,$$

with unknowns $c_1, c_2, \ldots, c_n$. Condition

$$\text{(2.1.19)} \qquad r(\mathfrak{a}) < n,$$

where

$$\mathfrak{a} = [a_{ij}]_{\substack{1 \le i \le W \\ 1 \le j \le n}}, \; a_{ij} = \sum_{\beta=0}^{\max(r_1, r_2)} R_j^\beta (A_\beta^{(i)} e^{R_j t_1} - B_\beta^{(i)} e^{R_i t_2})$$

is necessary and sufficient for equation (2.1.1) with conditions (2.1.14) to have eigensolutions defined with the formula (2.1.17) where constants satisfy system (2.1.18).

Example 1. Let $L^0 = L^1 = L^2 = C(N)$ be a space of sequences of real numbers.

Derivative $S\colon C(N) \to C(N)$, integral $T(k_0)$ and limit condition $s(k_0)$ are defined with the formulas

$$S\{x_k\} = \{x_{k+1} - x_k\}$$

$$T(k_0)\{x_k\} = \begin{cases} 0 \quad \text{for} \quad k = k_0 \\ x_{k_0} + x_{k_0+1} + \ldots + x_{k-1} \quad \text{for} \quad k_0 < k \\ -x_{k_0-1} - x_{k_0-2} - \ldots - x_k \quad \text{for} \quad k_0 > k \end{cases}$$

$$s(k_0)\{x_k\} = \{x_{k_0}\} \quad \text{(see [5])}.$$

Let's consider equation

$$(*) \qquad S^2\{x_k\} = \lambda\{x_k\}$$

with conditions

$$(**) \qquad \begin{cases} x_{k_1} - x_{k_2+1} = 0 \\ 3x_{k_1} + 3x_{k_2} - x_{k_1+1} - x_{k_2+1} = 0 \end{cases}$$

which are conditions of the type (2.1.14) because they can be written in the form

$$(***) \qquad \begin{cases} \sigma(k_1)\{x_k\} - \sigma(k_2)\{x_k\} - \sigma(k_2)\,S\{x_k\} = 0 \\ 2\sigma(k_1)\{x_k\} - \sigma(k_1)\,S\{x_k\} + 2\sigma(k_2)\{x_k\} - \sigma(k_2)\,S\{x_k\} = 0 \end{cases}$$

where operation $\sigma(k_i)$ means substitution of natural number k_i instead of variable k.

Polynomial $W(p, \lambda) = p^2 - \lambda$ has non-zero roots $R_1 = \sqrt{\lambda}$ and $R_2 = -\sqrt{\lambda}$. Because in this model

$$e^{Rt(q)}c = (1+R)^k c$$

hence applying (2.1.17) we get general solution of the equation $(*)$

$$x_k = (1+\sqrt{\lambda})^k c_1 + (1-\sqrt{\lambda})^k c_2,$$

and hence

$$\sigma(k_i)\{x_k\} = (1+\sqrt{\lambda})^{k_i} c_1 + (1-\sqrt{\lambda})^{k_i} c_2$$

$$\sigma(k_i)\,S\{x_k\} = [(1+\sqrt{\lambda})^{k_i} c_1 - (1-\sqrt{\lambda})^{k_i} c_2]\sqrt{\lambda}, \quad i = 1, 2.$$

Substituting previous formulas to conditions $(***)$ we get the following system of homogenenous equations with unknown c_1 and c_2:

$$[(1+\sqrt{\lambda})^{k_1} - (1+\sqrt{\lambda})^{k_2+1} c_1 + [1-\sqrt{\lambda})^{k_1} - (1-\sqrt{\lambda})^{k_2+1}]\,c_2 = 0$$

$$(2-\sqrt{\lambda})(1+\sqrt{\lambda})^{k_1} + (1+\sqrt{\lambda})^{k_2}]\,c_1 + (2+\sqrt{\lambda})\,[(1-\sqrt{\lambda})^{k_1} + (1-\sqrt{\lambda})^{k_2} c_2 = 0.$$

Thys systems has non-zero solutions if

$$\binom{***}{*} \quad \begin{vmatrix} (1+\sqrt{\lambda})^{k_1} - (1+\sqrt{\lambda})^{k_2+1} & (1-\sqrt{\lambda})^{k_1} - (1-\sqrt{\lambda})^{k_2+1} \\ (2-\sqrt{\lambda})\,[(1+\sqrt{\lambda})^{k_1} + (1+\sqrt{\lambda})^{k_2}] & (2+\sqrt{\lambda})\,[(1-\sqrt{\lambda})^{k_1} + (1-\sqrt{\lambda})^{k_2}] \end{vmatrix} = 0$$

Conditions $\binom{***}{*}$ defines how eith given λ we must take values k_1 and k_2 so that equation $(*)$ with conditions $(**)$ should have eigensolutions.
And so for example if $\lambda = 4$, then we get connection

$$(3^{k_1} - 3^{k_2+1})\,[(-1)^{k_1} + (-1)^{k_2}] = 0,$$

and hence we have

$$k_2 = k_1 - 1, \quad k_1 = 1, 2, \ldots, \quad \vee \quad k_2 = k_1 + 2l + 1$$

In this case the eigensolutions of the equation ($*$) with conditions

$$\sigma(k_1)\{x_k\}-\sigma(k_1-1)\{x_k\}-\sigma(k_1-1)S\{x_k\} = 0$$

$$2\sigma(k_1)\{x_k\}-\sigma(k_1)\{x_k\}+2\sigma(k_1-1)\{x_k\}-\sigma(k_1-1)S\{x_k\} = 0$$

are elements

$$\{x_k\} = \{3^k\alpha\}, \quad \alpha\in R.$$

§ 2. Existence of eigensolutions of a system of differential equations of the first order with moxed conditions.

Let's consider a system of differential equations of the form

(2.2.1)

$$Sx_1+a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n = \lambda_1 x_1$$

$$Sx_2+a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n = \lambda_1 x_2$$

$$Sx_n+a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n = \lambda_n x_n,$$

$$x_i\in L^1, \quad i = 1, 2, \ldots, n,$$

where coefficients of this system and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are endomorphisms commutative with each other.

To obtain eigensolutions on the system (2.2.1) we will impose conditions of the mixed type

(2.2.2) $$\sum_{k=1}^{N}\sum_{p=1}^{n}\sum_{\varrho=1}^{\varrho_{pk}} \alpha_{W\varrho}^{(p,k}\sigma(t_k)S^{\varrho-1}x_p = 0, \quad W = 1, 2, \ldots, m,$$

where operation $\sigma(t_k)$ is defined by formula (2.1.3). As

(2.2.3) $$Sx_i = px_i-px_{i0},$$

where

(2.2.4) $$x_{i0} = s(q)x_i\in\operatorname{Ker} S,$$

hence from the system (2.2.1) we get

(2.2.5)

$$(p+a_{11}-\lambda_1)x_1+a_{12}x_2+\ldots+a_{1n}x_n = px_{10}$$

$$a_{21}x_1+(p+a_{22}-\lambda_2)x_2+\ldots+a_{2n}x_n = px_{20}$$

$$a_{n1}x_1+a_{n2}x_2+\ldots+(p+a_{nn}-\lambda_n)x_n = px_{n0}.$$

Solution of this system are elements

(2.2.6) $$x_j = \frac{\Delta_j(p, \Lambda)}{\Delta(p, \Lambda)}, \quad j = 1, 2, \ldots, n,$$

where

(2.2.7) $$\Delta(p, \Lambda) := \begin{vmatrix} p+a_{11}-\lambda_1 & a_{12}\dots a_{1n} \\ a_{21} & p+a_{22}-\lambda_2 \dots a_{2n} \\ a_{n1} & a_{n2}\dots p+a_{nn}-\lambda_n \end{vmatrix}.$$

(2.2.8) $$\Delta_j(p, \Lambda) = \begin{vmatrix} p+a_{11}-\lambda_1 & a_{12}\dots a_{1j-1} & px_{10} & a_{1,j+1}\dots a_{1n} \\ a_{21} & p+a_{22}-\lambda_2\dots a_{2,j-1} & px_{20} & a_{2,j+1}\dots a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2}\dots a_{n,j-1} & px_{n0} & a_{n,j+1}\dots p+a_{nn}-\lambda_n \end{vmatrix}$$

We can apply formula (2.2.6) if the roots of the equation $\Delta(p, \Lambda)=0$ $R_1, R_2, \dots, R_1$, are logarithms.

Applying to $\Delta_j(p, \Delta)$ theorem of Laplace we obtain

(2.2.9) $$x_j = \sum_{i=1}^{n} \frac{p\Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} x_{i0}, \quad j = 1, 2, \dots, n,$$

where Δ_{ij} is a respective *co*-factor.

Elements x_j defined by the formula (2.2.9) are of the form (2.1.7).
Applying formula (2.1.10) we have

(2.2.10) $$x_j = \sum_{i=1}^{n} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i,j)} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} + \sum_{s=1}^{i} \sum_{r=1}^{\alpha_s} A_{sr}^{(i,j)} \frac{t_{(q)}^{\alpha_s+1-r}}{(\alpha_s+1-r)!} e^{R_s t(q)} \right] x_{i0},$$

where

$$A_{0r}^{(i,j} = |_{p=0} \frac{1}{r!} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0+1}\Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} \right]$$

(2.2.11)

$$A_{sr}^{(i,j)} = |_{p=R_s} \frac{1}{(r-1)!} \frac{d^{r-1}}{dp^{r-1}} \left[\frac{(p-R_s)^{\alpha_s}\Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} \right], \quad \sum_{s=0}^{i} \alpha_s = n.$$

Hence and from (2.2.2) we finally get the system

$$\sum_{k=1}^{N} \sum_{j=1}^{n} \sum_{\varrho=1}^{\varrho_{jk}} \sum_{i=1}^{n} \left[\sum_{r=0}^{\alpha_0} \alpha_{W\varrho}^{(j,k)} A_{0r}^{(i,j)} \frac{t_k^{\alpha_0-\varrho-r+2}}{(\alpha_0-\varrho-r+2)!} + \right.$$

(2.2.12)

$$\left. + \sum_{s=1}^{\iota} \sum_{r=1}^{\alpha_s} \alpha_{W\varrho}^{(j,k)} A_{sr}^{(i,j)} \sigma(t_k) S^{\varrho-1} \left(\frac{t_{(q)}^{\alpha_s+1-r}}{(\alpha_s+1-r)!} e^{R_s t(q)} \right) \right] x_{i0} = 0, \quad w = 1, 2, \dots, \omega.$$

It is a system of homogeneous ω equations with n unknowns $x_{10}, x_{20}, \dots, x_{n0}$ and if only order of the matrix of this system is smaller than n then the system of equations (2.2.1) with conditions (2.2.2) has eigensolutions (2.2.10), where constants $x_{10}, x_{20}, \dots$ $\dots, x_{n0}$ are solutions of the system (2.2.12).

EXAMPLE 2. Let

$$S=\frac{d}{dt}:C^1(R)\to C^0(R)$$

$$T(q)f=\int_{t_0}^{t} f(\tau)\,d\tau,\ f\in C^0(R),\quad t_0\in(a,b)$$

$$s(q)x(t)=x(t_0)=x_0.$$

Let's consider system of equations

$$(*)\qquad \begin{aligned} x_1'(t)-x_1(t)-x_2(t)&=\lambda_1x_1(t)\\ x_2'(t)+5x_1(t)+x_2(t)&=\lambda_2x_2(t)\end{aligned}$$

with conditions

$$4x_1(t_1)+x_1'(t_1)-x_2'(t_2)=0$$

$$x_1'(t_1)+2x_1''(t_1)-x_2'(t_2)-x_2''(t_2)=0$$

$$x_1(t_1)+x_2(t_1)-2x_1(t_2)+2x_2'(t_2)-x_1''(t_2)=0.$$

To simplify the calculations we will make analysis in case $\lambda_1=\lambda_2=0$.

Applying (2.2.3) we get the system

$$(p-1)x_1-x_2=px_{10}$$

$$5x_1+(p+1)x_2=px_{20},$$

for which

$$\Delta(p,0)=p^2+4.$$

Hence we have

$$R_1=2i,\quad R_2=-2i.$$

On the bases of formula (2.2.10) and passing from expotential functions to trigonometric functions finely we get general solution of a given system in the form

$$x_1=\left\{\left(\cos 2t+\frac{1}{2}\sin 2t\right)x_{10}+\left(\frac{1}{2}\sin 2t\right)x_{20}\right\}$$

$$x_2=\left\{\left(-\frac{5}{2}\sin 2t\right)x_{10}+\left(\cos 2t-\frac{1}{2}\sin 2t\right)x_{20}\right\}.$$

Substituting the obtained solution to $(**)$ we get the system

$$(5\cos 2t_1+5\cos 2t_2)x_{10}+(2\sin 2t_1+\cos 2t_1+2\sin 2t_2+\cos 2t_1)x_{20}=0$$

$$-6\sin 2t_1-7\cos 2t_1+5\cos 2t_2-10\sin 2t_2)x_{10}+$$

$$+(\cos 2t_1-4\sin 2t_1+5\cos 2t_2)x_{20}=0$$

$$(\cos 2t_1-2\sin 2t_1-8\cos 2t_2+\sin 2t_2)x_{10}+$$

$$+(\cos 2t_1-3\sin 2t_2-2\cos 2t_2)x_{20}=0,$$

which has non-zero solution if all the determinants of the second order created from the matrix of this system vanish, i.e. if

$$\begin{vmatrix} 5\cos 2t_1+5\cos 2t_2 & 2\sin 2t_1+\cos 2t_1+2\sin 2t_2+\cos 2t_2 \\ -6\sin 2t_1-7\cos 2t_1+5\cos 2t_2-10\sin 2t_2 & \cos 2t_1-4\sin 2t_1+5\cos 2t_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 5\cos 2t_1+5\cos 2t_2 & 2\sin 2t_1+\cos 2t_1+2\sin 2t_2+\cos 2t_2 \\ \cos 2t_1-2\sin 2t_1-8\cos 2t_2+\sin 2t_2 & \cos 2t_1-3\sin 2t_2-2\cos 2t_2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} -6\sin 2t_1-7\cos 2t_1+5\cos 2t_2-10\sin 2t_2 & \cos 2t_1-4\sin 2t_1+5\cos 2t_2 \\ \cos 2t_1-2\sin 2t_1-8\cos 2t_2+\sin 2t_2 & \cos 2t_1-3\sin 2t_2-2\cos 2t_2 \end{vmatrix} = 0.$$

Hence after conversions we get equations

$$4\cos 2(t_2-t_1)+3\sin 2(t_2-t_1)+4 = 0$$

$$2\cos 2(t_2-t_1)-18\sin 2(t_2-t_1)+2 = 0$$

$$11\cos 2(t_2-t_1)+5\sin 2(t_2-t_1)+11 = 0,$$

which the common solution is

$$t_2-t_1 = \frac{\pi}{2}+k\pi, \quad k\in Z_0.$$

It means that the system $(*)$ with conditions

$$4x_1(t_1)+x_1'(t_1)-x_2'\left(t_1+\frac{\pi}{2}+k\pi\right) = 0$$

$$x_1'(t_1)+2x_1''(t_1)-x_2'\left(t_1+\frac{\pi}{2}+k\pi\right)-x_2''\left(t_1+\frac{\pi}{2}+k\pi\right) = 0$$

$$x_1(t_1)+x_2(t_1)-2x_1\left(t_1+\frac{\pi}{2}+k\pi\right)+2x_2'\left(t_1+\frac{\pi}{2}+k\pi\right)-x_1''\left(t_1+\frac{\pi}{2}+k\pi\right) = 0$$

has eigensolutions for arbitrary $t_1\in R$ and $k\in Z$, which are functions of the form

$$x_1 = \{(\cos 2t-\sin 2t)c\}$$

$$x_2 = \{(-3\cos 2t-\sin 2t)c\}, \quad c\in R.$$

References

[1] W. J. Arnold, Równania rózinczkowe zwyczajne. *PWN, Warszawa*, 1975. Tłumaczenie z rosyjskiego.

[2] R. Bittner, Operational calculus in linear spaces. *Studia Math.* **20** (1961), 1—18.

[3] R. Bittner, Rachunek operatorów w przestrzeniach liniowych. *PWN, Warszawa*, 1974.

[4] R. Bitter and Z. Kobus — Taylor's jets. Manifolds of solutions of differential equations in the operational calculus. *Preprint No 35, May 1980. University of Gdańsk.*

[5] R. Bittner, E. Mieloszyk, About eigenvalues of differential equations in the operational calculus. *Zeszyty Naukowe Politechniki Gdańskiej. Matematyka XI* Nr. 285 s. 87—99. *Gdańsk*, 1978 r.

[6] R. Bittner and Z. Smentek, Taylor's formula for many derivatives. Integrals. *Demonstratio Mathematica* **12**. *Technical University Warsaw*, 1—16.

[7] Z. Kobus, H. Wysocki — Pochodna Frecheta odwzorowań kiełków Taylora. Odwracalność kiełków Taylora. *Zeszyty Naukowe WSMW Gdynia* 1983, 95—113.

[8] S. Lang — Algebra. *PWN Warszawa* 1973 r.

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