

Minimal Banach spaces and atomic representations

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Introduction

Atomic representations, i.e. the characterization of arbitrary elements in a given Banach space as (absolutely) convergent sums of elements of a particular simple form, to be called the 'atoms', play an important role in the modern treatment of Banach spaces of functions (distributions), in particular, in connection with the modern theory of real Hardy spaces (cf. [2]). In the present note it is to be shown that one has for the Segal algebra $S_0(G)$, which is defined for arbitrary locally compact abelian groups and which may be considered as a useful tool for harmonic analysis (cf. [7], [8], [9]), the following kind of atomic representation: Given any $f_0 \in S_0(G)$, $f_0 \neq 0$ (e.g. $f_0 \in L^1(G)$ with compactly supported Fourier transform \hat{f}_0) one has:

$$S_0(G) = \left\{ f \mid f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{t_n} f_0, y_n \in G, t_n \in \hat{G}, a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

This characterization is a consequence of a minimality property of $S_0(G)$, and related results for other Wiener-type spaces can be derived in a similar way. As a consequence a characterization of the Schwartz space $\mathfrak{S}(R^m)$ through suitable atomic representations of a similar kind is obtained. At the end related characterizations for certain homogenous Besov spaces of order zero are mentioned, with an application in the theory of Hardy spaces.

1. Notations and Preliminaries

In the sequel G denotes a locally compact abelian group, with *character group* \hat{G} . For $x \in G$ and $t \in \hat{G}$ we write $\langle x, t \rangle = t(x)$. For (measurable) functions f on G the *translation operators* and *character multiplications* are given by

$$L_y f(x) := f(x - y) \quad \text{and} \quad M_t f(x) := \langle x, t \rangle f(x)$$

respectively. These operators act isometrically on the Lebesgue spaces $(L^p(G), \| \cdot \|_p)$. For $1 \leq p < \infty$ the space $\mathfrak{K}(G)$ of continuous functions with compact support is dense in $L^p(G)$, and $C^0(G)$ is identified with the closure of $\mathfrak{K}(G)$ in $L^\infty(G)$. More generally, we shall consider *BF-spaces* on G , i.e. Banach spaces $(B, \| \cdot \|_B)$ which are continuously embedded in the topological vector space $L^1_{loc}(G)$ of locally integrable functions on G . Such spaces are called [*strongly*] *translation* or *character invariant* if

$L_y B \subseteq B$ for all $y \in G$ or $M_t B \subseteq B$ for all $t \in \hat{G}$ [and $\|L_y f\|_B = \|f\|_B$ or $\|M_t f\|_B = \|f\|_B$ for all $f \in B$]. Strongly translation invariant BF-spaces with *continuous translation* (i.e. with $\|L_y f - f\|_B \rightarrow 0$ for $y \rightarrow 0$, for all $f \in B$) are called *homogeneous Banach spaces* on G , and *Segal algebras* if furthermore $B \subseteq L^1(G)$ as a dense subspace. For general invariant BF-spaces one only knows (by the closed graph theorem) that $L_y(M_t)$ defines a bounded operator and that $y \rightarrow \|L_y\|_B$ ($\|M_t\|_B$) (these symbols are used for the operator norm on B) defines a submultiplicative function on G resp. \hat{G} . A submultiplicative measurable function w on G (i.e. with $w(x+y) \leq w(x)w(y)$ for all $x, y \in G$) is called a *weight function* on G if one has also $w(x) \geq 1$ for all $x \in G$ (cf. [17], III. 7). For any weight function the space $L_w^1(G) := \{f | f w \in L^1(G)\}$ is a Banach algebra with respect to convolution, called *Beurling algebra*, with the norm $\|f\|_{1,w} := \|f w\|_1$. Consequently $A_w(\hat{G}) := \{\hat{f} | f \in L_w^1(G)\}$, its image under the *Fourier transform*, is a homogenous Banach space on \hat{G} as well as a pointwise Banach algebra (in $C^0(\hat{G})$) with the norm $\|\hat{f}\|_{A_w} := \|f\|_{1,w}$ (denoted by $\mathcal{F}_w^1(\hat{G})$ in [17]). Since $G = (\hat{G})^\wedge$ according to Pontrjagin's duality $A_v(G)$ is well defined for any weight v on \hat{G} . Since $A_w(\hat{G})$ is regular (as an algebra of continuous functions on G) if and only if w satisfies the *Beurling—Domar condition BD* we shall be exclusively interested in such weights (cf. [17]; VI. 3; BD): $\sum_{n=1}^\infty n^{-2} \log w(nx) < \infty$ for all $x \in G$.

Since $A_w(\hat{G})$ is even a Wiener algebra (in the sense of [17], Chap. II) any character invariant ideal in $L_w^1(G)$ is dense in $L_w^1(G)$. As a consequence of the Wiener—Lévy theorem any dense ideal B in $L_w^1(G)$ (in particular, any Segal algebra, being a dense ideal in $L^1(G)$) contains $A_w^K(G) := \{f | f \in L_w^1(G), \text{supp } \hat{f} \text{ compact in } \hat{G}\}$. If, furthermore, B is a BF-space with respect to some norm, then one can show that $A_w^K(G)$ is dense in $(B, \|\cdot\|_B)$ if and only if B is an *essential Banach ideal* in $L_w^1(G)$ (i.e. $L_w^1 * B$ is dense in B) or — more conveniently to check — if and only if translation is continuous in B (cf. [12], § 1 for assertion in this direction). Furthermore, the norms $\|\cdot\|_B$ and $\|\cdot\|_{1,w}$ are equivalent on $\{f | f \in L_w^1(G), \text{supp } \hat{f} \subseteq Q\}$ for any relatively compact set $Q \subseteq \hat{G}$. (cf. [17], VI., 2.2. iv)).

2. An atomic decomposition for $S_0(G)$

Let us recall some facts concerning the Segal algebra $S_0(G)$ as introduced for arbitrary locally compact abelian groups in [7]. There it has been shown that $S_0(G)$ is the smallest among all strongly character invariant Segal algebras, or equivalently, the smallest (nontrivial) homogenous Banach space on G which is at the same time a Banach module over the Fourier algebra $A(G)$ (cf. [6]). Among others the following characterization of $S_0(G)$ is derived therefrom (see [7], Theorem 2).

$$S_0(G) = \left\{ f | f = \sum_{n=1}^\infty g_n * M_{t_n} f_0, t_n \in \hat{G}, g_n \in L^1(G), \sum_{n=1}^\infty \|g_n\|_1 < \infty \right\},$$

for any non-zero $f_0 \in S_0(G)$ (e.g. $f_0 \in L^1(G)$, $\text{supp } \hat{f}_0$ compact), and the following norm is equivalent to the original norm:

$$\|f\|_{S_0} := \inf \left\{ \sum_{n=1}^\infty \|g_n\|_1, \dots \right\},$$

the infimum being taken over all admissible representations as above. From this representation we shall obtain the atomic representation announced in the introduction.

Theorem 1. *Let G be a lca group, and let $f_0 \in S_0(G)$, $f_0 \neq 0$ be given. Then one has*

$$S_0(G) = \left\{ f \mid f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{t_n} f_0, y_n \in G, t_n \in \hat{G}, \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

and the expression

$$\|f\|_{(S_0)} := \inf \left\{ \sum_{n=1}^{\infty} |a_n|, f = \sum_{n=1}^{\infty} \dots \right\},$$

the infimum being taken over all representations of f of the above form, defines an equivalent norm on $S_0(G)$.

PROOF. Write S_{f_0} for the space of all f having an 'atomic' representation (involving f_0) of the above kind. The sum being (automatically) absolutely convergent in $S_0(G)$ (hence uniformly and in $L^1(G)$) it is no problem to verify that $(S_{f_0}, \|\cdot\|_{(S_0)})$ is a Banach space which is continuously embedded in $S_0(G)$.

In order to show the converse inclusion we shall derive — starting with the above representation — the existence of a 'discrete atomic representation' for any given $f \in S_0(G)$. For $f_1 := f$ we choose a representation

$$f_1 = \sum_n g_n * M_{t_n} f_0 \quad \text{with} \quad \sum_n \|g_n\|_1 \leq 2\|f_1\|_{S_0}.$$

It is clear that one may even suppose $g_n \in \mathfrak{R}(G)$ for $n \geq 1$ because $g \in L^1(G)$ may be written as absolutely convergent series of elements in $\mathfrak{R}(G)$. It follows

$$\left\| f_1 - \sum_{n=1}^{n(1)} g_n * M_{t_n} f_0 \right\|_{S_0} \leq \frac{1}{8} \|f_1\|_{S_0} \quad \text{for some} \quad n(1) \geq 1.$$

Since the convolution product $g_n * M_{t_n} f_0$ can now be interpreted as a vector-valued Riemann integral (with values in $S_0(G)$) with bounded, continuous integrand, one finds for each n , $1 \leq n \leq n(1)$, sequences $(z_k^{(n)})_{k=1}^{k(n)}$ in G and $(b_k^{(n)})_{k=1}^{k(n)}$ such that

$$\left\| g_n * M_{t_n} f_0 - \sum_{k=1}^{k(n)} b_k^{(n)} L_{z_k} M_{t_n} f_0 \right\|_{S_0} < \|f_1\|_{S_0} / 8n(1),$$

and

$$\sum_{k=1}^{k(n)} |b_k^{(n)}| \leq \|g_n\|_1 \quad \text{for} \quad 1 \leq n \leq n(1).$$

Combining these estimates one obtains for

$$f_2 := f_1 - \sum_{n=1}^{n(1)} \sum_{k=1}^{k(n)} b_k^{(n)} L_{z_k} M_{t_n} f_0$$

$$\|f_2\|_{S_0} \leq \frac{1}{4} \|f_1\|_{S_0} \quad \text{and} \quad \sum_{n=1}^{n(1)} \sum_{k=1}^{k(n)} |b_k^{(n)}| \leq \sum_{n=1}^{n(1)} \|g_n\|_1 \leq 2\|f_1\|_{S_0}.$$

Relabelling this finite sum we may write

$$\|f_2\|_{S_0} = \left\| f_1 - \sum_{k=1}^{k_1} c_k^1 L_{y_k^1} M_{t_k^1} f_0 \right\|_{S_0} \leq \frac{1}{4} \|f_1\|_{S_0},$$

and

$$\sum_{k=1}^{k_1} |c_k^1| \leq 2 \|f_1\|_{S_0}.$$

We then proceed by induction. Given f_1, \dots, f_m we repeat the above procedure by choosing suitable sequences $(y_k^m)_{k=1}^{k_m}$ in G , $(t_k^m)_{k=1}^{k_m}$ in \hat{G} and $(c_k^m)_{k=1}^{k_m}$ in C such that one has for

$$f_{m+1} := f_m - \sum_{k=1}^{k_m} c_k^m L_{y_k^m} M_{t_k^m} f_0$$

$$\|f_{m+1}\|_{S_0} \leq \frac{1}{4} \|f_m\|_{S_0} \leq \frac{1}{4} 2^{-2m} \|f_1\|_{S_0},$$

and

$$\sum_{k=1}^{k_m} |c_k^m| \leq 2 \|f_m\|_{S_0} \leq 2^{-m} \|f_1\|_{S_0}.$$

It follows that

$$f = f_1 = \sum_{m=1}^{\infty} \sum_{k=1}^{k_m} c_k^m L_{y_k^m} M_{t_k^m} f_0,$$

the sum being absolutely convergent in $S_0(G)$, since

$$\sum_{m=1}^{\infty} \sum_{k=1}^{k_m} |c_k^m| \leq \sum_{m=1}^{\infty} 2 \|f_m\|_{S_0} \leq 3 \|f\|_{S_0}.$$

This implies that $f \in S_{f_0}$, and the last estimate shows that the norms $\|\cdot\|_{(f_0)}$ and $\|\cdot\|_{S_0}$ are equivalent on $S_0(G)$.

Remark 1. In view of the identity $L_y M_t = \langle y, t \rangle M_t L_y$ for all $y \in G$, $t \in \hat{G}$ it is clear that one may change the order of M_t and L_y in the atomic representation given above.

Remark 2. It is tempting to try to derive the inclusion $S_{f_0} \supseteq S_0$ from the minimality of S_0 mentioned at the beginning. Although it is clear that S_{f_0} is strongly translation invariant as well as strongly character invariant (and therefore dense in $L^1(G)$) it turns out to be a non-trivial problem to show that translation is continuous in S_{f_0} , at least for general $f_0 \in S_0(G)$.

Corollary 2. *Let G be a lca. group. Then $S_0(G)$ is the minimal strongly translation and strongly character invariant BF-space $(B, \|\cdot\|_B)$ satisfying $B \cap S_0(G) \neq \{0\}$.*

Remark 3. Since $S_0(G)$ contains the spaces $\{f | f \in L^1(G), \text{supp } \hat{f} \text{ compact}\}$, $\mathfrak{R}(G) * \mathfrak{R}(G)$ or the Schwartz—Bruhat space $\mathfrak{S}(G)$ (cf. [7]) it follows that $S_0(G)$ is contained in any such invariant space BF-space B having non-trivial intersection with one of these three spaces. That the additional condition $B \cap S_0(G) \neq \{0\}$ cannot

be avoided can be shown by way of an example. In fact, H. C. WANG (cf. [20], p. 36—39) has included an (over-) detailed description of a “semihomogenous” solid BF -space on R (discovered by the present author) which does not contain any nonzero continuous function.

Theorem 1 allows to derive a convenient characterization of $S'_0(G)$ as a subspace of the space $Q(G)$ of quasimeasures on G .

Corollary 3. *A quasimeasure $\sigma \in Q(G)$ defines an element of $S'_0(G)$ if and only if one has*

$$\sup_{y \in G} \sup_{t \in \hat{G}} |\sigma(L_y M_t f_0)| < \infty$$

for some (any) $f_0 \in S_0(G)$, $f_0 \neq 0$.

PROOF. One has $Q(G) = (A(G) \cap \mathfrak{R}(G))'$, the space $A(G) \cap \mathfrak{R}(G)$ being endowed with its natural inductive limit topology (cf. [14], [3]). Since this space is densely and continuously embedded in $S_0(G)$ it is clear that $S'_0(G)$ may be considered as a subspace of $Q(G)$. Since $S_0(G)$ is strongly translation and character invariant it is clear that the above condition is necessary for $\sigma \in Q(G)$ to act continuously on $(A(G) \cap \mathfrak{R}(G), \|\cdot\|_{S_0})$. Conversely, one has

$$|\sigma(f)| \leq \sum_{n=1}^{\infty} |a_n| |\sigma(L_{y_n} M_{t_n} f_0)| \leq 2 \|f\|_{S_0} \sup_{y \in G, t \in \hat{G}} |\sigma(L_y M_t f_0)|$$

for any $f \in A(G) \cap \mathfrak{R}(G) \subseteq S_0(G)$, and therefore σ extends to a bounded linear functional on $S_0(G)$, if the supremum is finite.

Another useful consequence is the following one:

Corollary 4. *Let B^1, B^2 be a pair of strongly character invariant BF -spaces, with B^2 strongly translation invariant. Then the following conditions are equivalent:*

- i) *There exists $f_0 \in \mathfrak{R}(G) * \mathfrak{R}(G)$, $f_0 \neq 0$, such that $f_0 * B^1 \subseteq B^2$ (for $G = \mathbb{R}^n$ one may assume $f_0 \in \mathfrak{R}(\mathbb{R}^n)$ as well).*
- ii) *Any $f \in S_0(G)$ defines a (bounded) multiplier from B^1 to B^2 .*

PROOF. By the closed graph theorem $g \rightarrow f_0 * g$ defines a bounded linear map from B^1 to B^2 . The result thus immediately from the f estimate:

$$\begin{aligned} \|(L_y M_t f_0) * f\|_{B^2} &= \|L_y M_t (f_0 * M_{-t} f)\|_{B^2} = \|f_0 * M_{-t} f\|_{B^2} \leq \\ &\leq C(f_0) \|M_{-t} f\|_{B^1} = C(f_0) \|f\|_{B^1} \quad \text{for } f \in B^1, y \in G, t \in \hat{G}. \end{aligned}$$

Remark 4. For $G = \mathbb{R}^n$ there is a particular natural choice for $f_0 \in S_0(\mathbb{R}^n)$ is the Gauß function, given by $f_0(x) := \exp(-\pi|x|^2)$ for $x \in \mathbb{R}^n$, which allows to check immediately (i.e. by direct calculation, without abstract theory) certain functorial properties of S_0 with respect to products or subgroups $H \cong \mathbb{R}^k$ for $k < n$ (cf. [7], Theorem 7). In fact, f_0 coincides not only with a multiple of its own Fourier transform, but the restriction $R_H f_0$ of f_0 to H coincides with the Gauß function on H , and the product of two Gauß functions on \mathbb{R}^k and \mathbb{R}^n is just the Gauß function on \mathbb{R}^{k+n} . One easily derives therefrom the next result which has several important implications concerning the dual space $S'_0(\mathbb{R}^*)$ (cf. [8], [9]), among them a kernel theorem for

bounded linear operators from $S_0(\mathbf{R}^n)$ to $S'_0(\mathbf{R}^k)$ (e.g. for operators from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^k)$):

Corollary 5.

- i) $\mathcal{F}[S_0(\mathbf{R}^n)] = S_0(\mathbf{R}^n)$;
- ii) $R_H S_0(\mathbf{R}^n) = S_0(H)$;
- iii) $S_0(\mathbf{R}^k) \hat{\otimes} S_0(\mathbf{R}^n) = S_0(\mathbf{R}^{k+n})$.

3. Towards an atomic decomposition of $\mathcal{S}(\mathbf{R}^n)$

The Segal algebra S_0 coincides with the Wiener-type space $W(A, L^1)$ (cf. [6], [7], [11]; see [10] for the general definition of Wiener-type spaces and their characterization using uniform partitions of unity as constructed in [6]). In this setting the more general spaces $W(A_v, L^1_w)$ can be described as follows: Given any open set Q with compact closure one has

$$W(A_v, L^1_w)(G) := \{f | f = \sum_{n=1}^{\infty} a_n L_{y_n} f_n, \text{ supp } f_n \subseteq Q, f_n \in A_v(G),$$

$$\|f\|_{A_v} \leq 1 \text{ for } n \geq 1, \sum_{n=1}^{\infty} |a_n| w(y_n) < \infty\},$$

with the according definition of the corresponding norm. Parallel to Theorem 1. one has then:

Proposition 6. *Let w, v be weight functions on G and \hat{G} respectively, both satisfying the Beurling—Domar condition. Then one has:*

- i) $W(A_v, L^1_w)$ is the minimal (nontrivial) BF-space $(B, \| \cdot \|_B)$ which is a pointwise $A_v(G)$ -module as well as translation invariant satisfying $\|L_y\|_B \leq Cw(y)$ for all $y \in G$ (or being a Banach-convolution module over $L^1_w(G)$).
- ii) Given $f_0 \in W(A_v, L^1_w)$, $f_0 \neq 0$, e.g. $f_0 \in L^1_w(G)$ with $\text{supp } \hat{f}_0$ compact in \hat{G} , one has

$$W(A_v, L^1_w) = \{f | f = \sum_{n=1}^{\infty} f_n * M_{t_n} f_0, t_n \in \hat{G},$$

$$g_n \in L^1_w(G), \text{ with } \sum_{n=1}^{\infty} \|g_n\|_{1,w} v(t_n) < \infty\},$$

the norm being again equivalent to the infimum over all such expressions.

- iii) For any $f_0 \in W(A_v, L^1_w)$, $f_0 \neq 0$, one also has the following ‘discrete atomic representations’:

$$W(A_v, L^1_w) = \{f | f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{t_n} f_0, a_n \in G, t_n \in \hat{G},$$

$$\sum_{n=1}^{\infty} |a_n| w(y_n) v(t_n) < \infty\}.$$

PROOF. i) First of all we mention that our definition of $W(A_v, L_w^1)$ is consistent with the general definition of Wiener-type spaces (cf. [10], and [6], Theorem for related questions). Consequently (or by direct inspection) it is a BF -space with A_v -module structure and satisfying $\|L_y\|_W \equiv w(y)$. Since one has for any translation invariant BF -space $(B, \|\cdot\|_B)$ with A_v -module structure $\{f|f \in A_v(G), \text{supp } f \subseteq Q\} \subseteq B$ and $\|f\|_{A_v} \equiv C_Q \|f\|_B$ it is not difficult to verify the inclusion $W(A_v, L_w^1) \subseteq B$ (cf. [6] Theorem 3).

ii) Since one has for $W(A_v, L_w^1)$ the estimate $\|M_t\|_W \equiv v(t)$ and an $L_w^1(G)$ -module structure (by vector-valued integration or by [10], Theorem 3) it is clear that the space W_{f_0} described in ii) is continuously embedded in $W(A_v, L_w^1)$, for any f_0 .

In order to show the converse one observes that W_{f_0} is an essential Banach ideal in $L_w^1(G)$. The formula $M_t(g_n * M_{t_n} f_0) = M_t g_n * M_{t+t_n} f_0$ and the submultiplicativity of v imply that W_{f_0} is character invariant (and therefore dense in $L_w^1(G)$ by Wiener's theorem, cf. [17], VI. 3.1 and II. 2.4), and satisfies $\|M_t\|_{W_{f_0}} \equiv v(t)$. It follows that $L_w^K(G)$ is dense in W_{f_0} . Since the mapping $t \rightarrow M_t f$ from \hat{G} to $L_w^1(G)$ is continuous for $f \in L_w^K(G) \subseteq L_w^1(G)$ one derives therefrom that one has $\|M_t f - M_{t_0} f\|_{W_{f_0}} \rightarrow 0$ for $t \rightarrow t_0$ in \hat{G} , for any $f \in W_{f_0}$. (cf. [5], Lemma 3.7 for a special case of this argument). By means of vector-valued integration one obtains an (essential) A_v -module structure on W_{f_0} . In view of i) W_{f_0} must now coincide with $W(A_v, L_w^1)$.

iii) The discrete representation for $W(A_v, L_w^1)$ follows from ii) by means of suitable modifications of the proof of Theorem 1. Since the operators M_t and L_y only change their roles under the Fourier transform (and since their order is not relevant, as mentioned in remark 1) the following result follows immediately from Proposition 4 (cf. [11], Proposition 3.3, for a different).

Corollary 7. *The Fourier transformation $\mathcal{F}: L^1(G) \rightarrow A(\hat{G})$ defines an isomorphism between $W(A_v, L_w^1)(G)$ and $W(A_w, L_v^1)(\hat{G})$.*

PROOF. In view of the 'discrete' representation and the preceding remarks it is sufficient to observe that $\mathcal{F} f_0 \in A_w \cap \mathfrak{R}(\hat{G}) \subseteq W(A_w, L_v^1)(\hat{G})$ for some (even any) $f_0 \neq 0$, $f_0 \in L_w^K(G) \subseteq W(A_v, L_w^1)(G)$.

Remark 5. By transposition the Fourier transform extends to an isomorphism between the dual spaces which are (rather large) Banach spaces of (ultra-) distributions (cf. [11] for further details).

We shall now turn to a characterization of the Schwartz space by atomic representations of the above kind.

Theorem 8. *Given $f_0 \in \mathfrak{S}(\mathbf{R}^m)$, $f_0 \neq 0$ (e.g. any indefinitely differentiable function with compact support) one has $\mathfrak{S}(\mathbf{R}^m) = \{f | \text{for all } r, s \in \mathbf{N} \text{ there exists a } L^1\text{-representation } f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{t_n} f_0, \text{ with } \sum_{n=1}^{\infty} |a_n| (1 + |y_n|)^r (1 + |t_n|)^s < \infty\}$.*

PROOF. In view of Proposition 6 it will be sufficient to verify that

$$\mathfrak{S}(\mathbf{R}^m) = \bigcap_{r \in \mathbf{N}} W(A_{w_r}, L_{w_r}^1) = \bigcap_{r,s} W(A_{w_{r,s}}, L_{w_{r,s}}^1), \text{ where } w_t(x) : (1 + |x|)^t$$

is a weight function on \mathbf{R}^m (satisfying of course the BD-condition).

The inclusion “ \supseteq ” follows from the fact that $A_{w_r}(\mathbf{R}^m) \subseteq C^{(r)}(\mathbf{R}^m)$ for any $r \in \mathbf{N}$ (since $p(t)\hat{f}(t)$ belongs to $L_1(\mathbf{R}^n)$ for any polynomial of degree less than r), implying

$$W(A_{w_r}, L_{w_r}^1) \subseteq W(C^{(r)}, L_{w_r}^\infty) \subseteq \{f|f^{(\alpha)}(x)(1+|x|)^r \text{ bounded for } |\alpha| \leq r\}.$$

In order to show the converse inclusion one has to verify that $\mathfrak{S}(\mathbf{R}^m) \subseteq W(A_{w_r}, L_{w_r}^1)$ for any $r \in \mathbf{N}$. As the proof of this inclusion requires only more or less technical modifications of the proof for the inclusion $\mathfrak{S}(\mathbf{R}^m) \subseteq W(A, L_{w_r}^1)$ given explicitly elsewhere (cf. [11], Lemma 6.2, the case $r=0$ has been discussed in [16]) it is left to the interested reader.

Remark 6. The above result also suggest a convenient method of verifying that a given BF-space on \mathbf{R}^n contains $\mathfrak{S}(\mathbf{R}^n)$ as a subspace. In fact, if there exists $f_0 \in B \cap \mathfrak{S}$, $f_0 \neq 0$, and for some pair (r, s) of positive integers the estimates $\|L_y\|_B \leq C(1+|y|)^r$ and $\|M_t\|_B \leq C'(1+|t|)^s$ hold true the inclusion follows. Another application (to be discussed elsewhere) concerns the equivalence of various moduli of smoothness obtained by means of different Schwartz functions.

4. Further related spaces

Without discussing details we mention that similar arguments can be used to show that the homogeneous Besov space $\dot{B}_{1,1}^0(\mathbf{R}^m)$ (cf. [1], § 6.3 [15] or [19]) is the smallest among all strongly translation invariant BF-space in $L_0^1(\mathbf{R}^m) = \{f|f \in L^1(\mathbf{R}^m), \int_{\mathbf{R}^m} f(x) dx = 0\}$, which are additionally isometrically invariant under the group of normalized dilations, given by $M_\varrho f(x) = \varrho^{-m} f(x/\varrho)$, $\varrho > 0$, and contain $\mathfrak{S}_0(\mathbf{R}^m) = \{f|f \in \mathfrak{S}(\mathbf{R}^m), \int_{\mathbf{R}^m} f(x) dx = 0\}$ as a dense subspace. One can show in analogy to Theorem 1:

Theorem 9. Given $f_0 \in \mathfrak{S}(\mathbf{R}^m)$ with $\int_{\mathbf{R}^m} f_0(x) dx = 0$ one has:

$$\dot{B}_{1,1}^0(\mathbf{R}^m) := \{f|f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{\varrho_n} f_0, \sum_{n=1}^{\infty} |a_n| < \infty\},$$

if and only if f_0 satisfies the following ‘Tauberian condition’: for $t \in \mathbf{R}^n$, $t \neq 0$, there exists $\lambda > 0$ such that $\hat{f}_0(\lambda t) \neq 0$. Furthermore, the expression $\inf \{ \sum_{n=1}^{\infty} |a_n|, \dots \}$ defines an equivalent norm on $\dot{B}_{1,1}^0$ in that case.

Remark 7. Among other applications this characterizations can be used in order to show that the expression

$$\left[\int_0^\infty \|M_\varrho k_{k_0} * g\|_p^q dt/t \right]^{1/q}$$

defines an equivalent norm on $\dot{B}_{p,q}^0$ for $1 \leq p, q \leq \infty$, for any $k_0 \in \dot{B}_{1,1}^0(\mathbf{R}^m)$, $k_0 \neq 0$ (cf. e.g. [18]). Detailed results in this direction as well as sufficient conditions for $f \in L_0^1(\mathbf{R}^m)$ to belong to $\dot{B}_{1,1}^0(\mathbf{R}^m)$ are to be given elsewhere (cf. [13] for connections to the space $A_1(\mathbf{R}^m)$ of ‘good vectors’ as treated in [4], which therefore has also an atomic representation).

Remark 8. Again for $f \in \mathfrak{S}(\mathbf{R}^m)$, with vanishing integral, better informations concerning the summability of a suitable sequence of coefficients (with weights) can be obtained.

Remark 9. It is worth being mentioned here that the well known atomic characterization of the Hardy space $H^1(\mathbf{R}^m)$ (cf. [2], p. 591) can be described by the equality

$$H^1(\mathbf{R}^m) = \left\{ f \mid f = \sum_{n=1}^{\infty} a_n L_{y_n} M_{\varrho_n} f_n, \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

where f_n has to be taken from the set of ‘atoms’ $At = \{f \mid f \in L^\infty(\mathbf{R}^m), \|f_n\|_\infty \leq 1, \text{supp } f \subseteq Q_1, \int_{\mathbf{R}^m} f(x) dx = 0\}$, which is bounded in $L_0^1(\mathbf{R}^m)$. In contrast to the spaces considered above it cannot be expected that it might be possible to replace the set At above by some finite set of atoms (since $\dot{B}_{1,1}^0(\mathbf{R}^m) \subseteq H^1(\mathbf{R}^m)$ as a proper subspace it is certainly impossible to replace At by some finite subset of $At \cap \mathfrak{S}(\mathbf{R}^m)$).

As another consequence of Theorem one obtains a result concerning the (modified) Hardy-spaces $H_\varphi^1(\mathbf{R})$ (cf. [21]), which may be considered as an improvement of the main results of [22]):

COROLLARY 10: *If $H_\varphi^1(\mathbf{R})$ contains any nonzero, real-valued $f_0 \neq 0$, then the Besov space $\dot{B}_{1,1}^0(\mathbf{R})$ is continuously embedded into $H_\varphi^1(\mathbf{R})$. In particular, $H_\varphi^1(\mathbf{R})$ contains any step function with vanishing integral, as well as any Schwartz function with this property. Consequently $H^1 \cap H_\varphi^1$ is dense in H^1 if and only if H_φ^1 contains any nonzero real-valued L^1 -function.*

PROOF. $H_\varphi^1(\mathbf{R})$ is always isometrically invariant under (normalized) dilations $(M_\varrho)_{\varrho > 0}$ as well as isometrically invariant under translations (cf. [22]). The minimality of $\dot{B}_{1,1}^0$ among BF-spaces with this property (observe that any real-valued L^1 -function on \mathbf{R} satisfies the ‘Tauberian condition’!) implies the required embedding.

That any Schwartz function with vanishing integral as well as any simple function (i.e. any linear combination of indicator functions of intervals) with this property, and in particular the function $a(x)$ treated in [22], belongs to the Besov space $\dot{B}_{1,1}^0$ can be verified separately (cf. [13], see also [1], § 6.3). In fact, it is a simple special case of the results to be given in the expanded version of [13] that any function on \mathbf{R} with vanishing integral belongs to $\dot{B}_{1,1}^0(\mathbf{R})$ if it is Lipschitz-continuous with the exception of a finite number of jumps and satisfies $|f(x)| \leq C(1+|x|)^\alpha$ for some $\alpha > 1$. The density of $\{f \mid f \in \mathfrak{S}(\mathbf{R}), \hat{f}(0) = 0\}$ in $H^1(\mathbf{R})$ implies the last assertion.

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Added in proof: Recently the author has discovered constructive and linear methods of finding the appropriate coefficients of atomic decompositions (to appear in Rocky Mountain J. Math).

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