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On generalized pseudo symmetric manifolds

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Abstract. In this paper it is shown that out of four 1-forms in terms of which a generalized pseudo symmetric manifold $G(PS)_n$ is defined, only two are independent. Further, introducing a new notion of hyper quasi-constant curvature of a Riemannian manifold it is proved that a conformally flat $G(PS)_n$ $(n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

1. Introduction

The notion of a generalized pseudo symmetric manifold was introduced by one of the authors (Chaki) in a recent paper [1]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ was called by him a generalized pseudo symmetric manifold if its curvature tensor R satisfies the condition:

$$(1)\qquad \begin{gathered}(\nabla_X R)(Y,Z,W) = 2A(X)\,R(Y,Z,W) + B(Y)\,R(X,Z,W)\\ +C(Z)\,R(Y,X,W) + D(W)\;R(Y,Z,X) + g(R(Y,Z,W),X)P,\\ X,Y,Z,W,P \in \mathfrak{X}(M),\end{gathered}$$

where A, B, C, D are 1-forms, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and P is defined by

$$(2)\qquad g(X,P) = A(X) \quad \forall X.$$

The 1-forms A, B, C, D are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $G(PS)_n$. It may be mentioned in this connection that although the definition of a $G(PS)_n$ is similar to that of a weakly symmetric Riemannian manifold studied by Tamássy and Bihn [2], the defining condition of the latter is a little weaker than that of a $G(PS)_n$. Let

$$g(X,Q)=B(X),\quad g(X,T)=C(X) \\ g(X,U)=D(X)\quad \forall X. \tag{3}$$

Then P,Q,T,U are called the basic vector fields corresponding to the associated 1-forms A,B,C,D respectively.

Let L be the symmetric endomorphism of the tangent space corresponding at each point to the Ricci tensor S of type $(0,2)$. Then

$$g(LX,Y)=S(X,Y)\quad \forall X,Y. \tag{4}$$

We write

$$A(LX)=\bar{A}(X),\quad B(LX)=\bar{B}(X), \\ C(LX)=\bar{C}(X),\quad D(LX)=\bar{D}(X). \tag{5}$$

Then $\bar{A},\bar{B},\bar{C},\bar{D}$ are called the auxiliary 1-forms corresponding to the associated 1-forms A,B,C,D respectively.

In this paper it is shown that the defining condition of a $G(PS)_n$ can always be expressed in the following form:

$$(\nabla_X R)(Y,Z,W)=2A(X)\,R(Y,Z,W)+B(Y)\,R(X,Z,W) \\ +B(Z)\,R(Y,X,W)+A(W)\,R(Y,Z,X)+g(R(Y,Z,W),X)P. \tag{6}$$

Thus out of the four 1-forms A,B,C,D only two are different, namely A and B and out of the four basic vector fields P,Q,T,U only two, namely P and Q are different. For a conformally flat $G(PS)_n$ $(n>3)$, it is proved that if it is of non-zero constant scalar curvature then the Ricci tensor S has the following form:

$$S(X,Y)=a_1 g(X,Y)+b_1 A(X)\,B(Y)+b_2 A(X)\,\bar{B}(Y) \\ +b_3 B(X)\,B(Y)+b_4 B(X)\,\bar{B}(Y)+b_5 \bar{A}(X)\,B(Y)+b_6 \bar{B}(X)\,B(Y)$$

where $a_1,b_1,b_2,\dots,b_6$ are suitable scalars.

Finally, introducing the notion of hyper quasi-constant curvature in a Riemannian manifold, it is shown that a conformally flat $G(PS)_n$ $(n>3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

2. Preliminaries

From (1) we get

$$(\nabla_X {}'R)(Y,Z,W,V) = 2A(X)\,{}'R(Y,Z,W,V) + B(Y)\,{}'R(X,Z,W,V) + C(Z)\,{}'R(Y,X,W,V) + D(W)\,{}'R(Y,Z,X,V) + A(V)\,{}'R(Y,Z,W,X), \tag{2.1}$$

where

$${}'R(Y,Z,W,V) = g(R(Y,Z,W),V). \tag{2.2}$$

Now, contracting (2.1) over Z and W we get

$$(\nabla_X S)(Y,V) = 2A(X)\,S(Y,V) + B(Y)\,S(X,V) + C(R(X,Y,V)) + D(R(X,V,Y)) + A(V)\,S(X,Y). \tag{2.3}$$

Next, contracting (2.3) over Y and V, by (3)–(5) we obtain

$$\begin{aligned} dr(X) &= 2A(X)r + S(X,Q) + S(X,T) + S(X,U) + S(X,P) \\ &= 2A(X)r + \bar{B}(X) + \bar{C}(X) + \bar{D}(X) + \bar{A}(X), \end{aligned} \tag{2.4}$$

where r is the scalar curvature. These formulas will be used in the sequel.

3. Associated 1-forms of a $G(PS)_n$

In this section it will be shown that the four associated 1-forms A, B, C, D of a $G(PS)_n$ cannot be all different.

Interchanging Y and Z in (2.1) we get

$$(\nabla_X {}'R)(Z,Y,W,V) = 2A(X)\,{}'R(Z,Y,W,V) + B(Z)\,{}'R(X,Y,W,V) + C(Y)\,{}'R(Z,X,W,V) + D(W)\,{}'R(Z,Y,X,V) + A(V)\,{}'R(Z,Y,W,X). \tag{3.1}$$

Now, adding (2.1) and (3.1) we obtain,

$$\begin{aligned} 0 &= B(Y)\,{}'R(X,Z,W,V) + B(Z)\,{}'R(X,Y,W,V) \\ &\quad + C(Z)\,{}'R(Y,X,W,V) + C(Y)\,{}'R(Z,X,W,V) \end{aligned}$$

or,

$$[B(Y) - C(Y)]\,{}'R(X,Z,W,V) + [B(Z) - C(Z)]\,{}'R(X,Y,W,V) = 0. \tag{3.2}$$

Now, contracting (3.2) over Y and Z we get

$$'R(W,V,X,Q) - {'R}(W,V,X,T) = 0. \tag{3.3}$$

From (3.3) it follows that $Q = T$. Hence

$$B(X) = C(X) \quad \forall X. \tag{3.4}$$

Similarly, interchanging W and V in (2.1) and proceeding as before we get

$$D(X) = A(X) \quad \forall X. \tag{3.5}$$

From (3.4) and (3.5) we see that the associated 1-forms A, B, C, D are not all different, because $A = D$ and $B = C$. In virtue of this we can state the following

Theorem 1. *The defining equation of a $G(PS)_n$ can always be expressed in the following form:*

$$\begin{aligned}&(\nabla_X {'R})(Y,Z,W,V) = 2A(X)\,{'R}(Y,Z,W,V)\\ &+B(Y)\,{'R}(X,Z,W,V) + B(Z)\,{'R}(Y,X,W,V)\\ &+A(W)\,{'R}(Y,Z,X,V) + A(V)\,{'R}(Y,Z,W,X).\end{aligned}$$

In virtue of this theorem the formula (2.4) takes the following form:

$$dr(X) = 2[A(X)r + \bar{A}(X) + \bar{B}(X)]. \tag{3.6}$$

If a $G(PS)_n$ is of non-zero constant scalar curvature, then from (3.6) it follows that $A(X)r + \bar{A}(X) + \bar{B}(X) = 0$.

This leads to the following

Theorem 2. *If in a $G(PS)_n$ the scalar curvature r is a non-zero constant, then*

$$A(X)r + \bar{A}(X) + \bar{B}(X) = 0 \quad \forall X.$$

4. Conformally flat $G(PS)_n$ $(n \geq 3)$

It is known [(4), p. 41 Theorem 4.1] that in a conformally flat (M^n, g) $(n \geq 3)$

$$\begin{aligned}&(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X)\\ &= \frac{1}{2(n-1)}[dr(X)\,g(Y,Z) - dr(Z)\,g(X,Y)].\end{aligned} \tag{4.1}$$

In virtue of (3.6) and Theorem 1 the formula (2.3) takes the following form:

$$(4.2)\qquad \begin{aligned}(\nabla_X S)(Y,V) &= 2A(X)\,S(Y,V) + B(Y)\,S(X,V)\\ &+B[R(X,Y,V)] + A[R(X,V,Y)] + A(V)\,S(X,Y).\end{aligned}$$

Hence, with the help of the Ricci identity we obtain

$$(4.3)\qquad \begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) &= A(X)\,S(Y,Z) - A(Z)\,S(Y,X)\\ &- {}'R(Z,X,Y,Q) - 2\,{}'R(Z,X,Y,P).\end{aligned}$$

Using (4.3) and (3.6) we can express (4.1) as follows:

$$(4.4)\qquad \begin{aligned}&(n-1)A(X)\,S(Y,Z) - (n-1)A(Z)\,S(X,Y)\\ &+(n-1)\,{}'R(X,Z,Y,Q) + 2(n-1)\,{}'R(X,Z,Y,P)\\ &= g(Y,Z)(A(X)r + \bar{A}(X) + \bar{B}(X)) - g(Y,X)(A(Z)r + \bar{A}(Z) + \bar{B}(Z)).\end{aligned}$$

Putting $Z = Q$ in (4.4) we get

$$(4.5)\qquad \begin{aligned}&(n-1)A(X)\,\bar{B}(Y) - (n-1)A(Q)\,S(X,Y)\\ &+(n-1)\,{}'R(X,Q,Y,Q) + 2(n-1)\,{}'R(X,Q,Y,P)\\ &= g(Y,Q)(A(X)r + \bar{A}(X) + \bar{B}(X)) - g(X,Y)(A(Q)r + \bar{A}(Q) + \bar{B}(Q)).\end{aligned}$$

If the scalar curvature r is a non-zero constant, then in virtue of Theorem 2 the right-hand side of (4.5) vanishes and (4.5) takes the following form:

$$\begin{aligned}&(n-1)A(X)\,\bar{B}(Y) - (n-1)A(Q)\,S(X,Y)\\ &+(n-1)\,{}'R(X,Q,Y,Q) + 2(n-1)\,{}'R(X,Q,Y,P) = 0\end{aligned}$$

or

$$(4.6)\qquad A(Q)\,S(X,Y) = A(X)\bar{B}(Y) + {}'R(X,Q,Y,Q) + 2\,{}'R(X,Q,Y,P).$$

In a conformally flat (M^n, g) $(n \geq 3)$ we have

$$(4.7)\qquad \begin{aligned}{}'R(X,Y,Z,W) &= \frac{1}{n-2}[S(Y,Z)\,g(X,W) - S(X,Z)\,g(Y,W)\\ &+S(X,W)\,g(Y,Z) - S(Y,W)\,g(X,Z)]\\ &+\frac{r}{(n-1)(n-2)}[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)].\end{aligned}$$

Using this formula we can write (4.7) as follows:

$$(4.8)\quad {}'R(X,Q,Y,Q) + 2\,{}'R(X,Q,Y,P)$$
$$= \frac{1}{n-2}\{-S(X,Y)[2A(Q)+B(Q)] - g(X,Y)[2\bar{A}(Q)+\bar{B}(Q)]$$
$$+\bar{B}(X)\,B(Y) + B(X)\,\bar{B}(Y) + 2[\bar{A}(X)\,B(Y) + A(X)\,\bar{B}(Y)]\}$$
$$+\frac{r}{(n-1)(n-2)}\{g(X,Y)[2A(Q)+B(Q)] - B(Y)[2A(X)+B(X)]\}.$$

In virtue of (4.8) we can express (4.6) as follows:

$$S(X,Y)\left\{A(Q) + \frac{1}{n-2}[2A(Q)+B(Q)]\right\}$$
$$= A(X)\,\bar{B}(Y) - \frac{1}{n-2}g(X,Y)[2\bar{A}(Q)+\bar{B}(Q)]$$
$$+\frac{1}{n-2}[\bar{B}(X)\,B(Y) + B(X)\,\bar{B}(Y)]$$
$$+\frac{2}{n-2}[\bar{A}(X)\,B(Y) + A(X)\,\bar{B}(Y)]$$
$$+\frac{r}{(n-1)(n-2)}[2A(Q)+B(Q)\,g(X,Y)]$$
$$-\frac{rB(Y)}{(n-1)(n-2)}[2A(X)+B(X)]$$

or

$$(4.9)\quad \begin{aligned} S(X,Y) &= a_1 g(X,Y) + b_1 A(X)\,B(Y) + b_2 A(X)\,\bar{B}(Y) \\ &+ b_3 B(X)\,B(Y) + b_4 B(X)\,\bar{B}(Y) + b_5 \bar{A}(X)\,B(Y) + b_6 \bar{B}(X)\,B(Y), \end{aligned}$$

where $a_1, b_1, \dots, b_6$ are scalars in terms of r, $A(Q)$ and $B(Q)$. This leads to the following

Theorem 3. *In a conformally flat $G(PS)_n$ of non-zero constant scalar curvature, the Ricci tensor S has the form* (4.9).

5. Hyper quasi-constant curvature

In this section we shall introduce in a Riemannian manifold the notion of hyper quasi-constant curvature as a generalization of the notion of quasi-constant curvature introduced by Chen and Yano [5]. According to these authors a Riemannian manifold (M^n, g) $(n > 3)$ is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor ${}'R$ of type (0,4) satisfies the condition

$$
\begin{aligned}
'R(X,Y,Z,W) &= a[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)] \\
&\quad + b[g(X,W)\,A(Y)\,A(Z) - g(Y,W)\,A(X)\,A(Z) \\
&\quad + g(Y,Z)\,A(X)\,A(W) - g(X,Z)\,A(Y)\,A(W)],
\end{aligned} \tag{5.1}
$$

where A is a 1-form and a, b are scalars.

We shall generalize this notion as follows: A Riemannian manifold (M^n, g) $(n > 3)$ will be said to be of *hyper quasi-constant curvature* if it is conformally flat and its curvature tensor $'R$ of type $(0,4)$ satisfies the condition

$$
\begin{aligned}
'R(X,Y,Z,W) &= a[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)] \\
&\quad + g(X,W)\{bAB\}(Y,Z) - g(Y,W)\{bAB\}(X,Z) \\
&\quad + g(Y,Z)\{bAB\}(X,W) - g(X,Z)\{bAB\}(Y,W),
\end{aligned} \tag{5.2}
$$

where $\{bAB\} \equiv b_1 AB + b_2 A\bar{B} + b_3 BB + b_4 B\bar{B} + b_5 \bar{A}B + b_6 \bar{B}B$ and $b_1, b_2, \dots, b_6$ are scalars.

If $B = A$, then the $G(PS)_n$ reduces to a pseudo-symmetric manifold $(PS)_n$ introduced by Chaki in [3]. In that paper it has been proved that in a conformally flat $(PS)_n$ the auxiliary 1-form $\bar{A}$ is proportional to the associated 1-form A [(3), p. 57 (6.7)]. Taking the factor of proportionality as k the relation (5.2) takes the form:

$$
\begin{aligned}
'R(X,Y,Z,W) &= a[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)] \\
&\quad + b_1[g(X,W)\,A(Y)\,A(Z) - g(Y,W)\,A(X)\,A(Z) \\
&\quad + g(Y,Z)\,A(X)\,A(W) - g(X,Z)\,A(Y)\,A(W)]
\end{aligned} \tag{5.3}
$$

where a and b_1 are appropriate scalars and b_1 is related to k in a simple way. In view of (5.1) it follows from (5.3) that in this case the manifold $(PS)_n$ is of quasi-constant curvature. This is the reason why a conformally flat $G(PS)_n$ $(n > 3)$ satisfying the condition (5.2) has been called a manifold of hyper quasi-constant curvature. The word 'hyper' has been used because the name hypercomplex number has been given to a quaternion as a generalization of a complex number.

The question now arises whether there exists a conformally flat Riemannian manifold whose curvature tensor $'R$ satisfies a condition of the form (5.2). The remaining part of this paper provides an answer to this question.

We consider a $G(PS)_n$ $(n > 3)$ which is conformally flat and of non-zero constant scalar curvature. We can therefore use the expression (4.9) for $S(X,Y)$. Substituting this expression for $S(X,Y)$ in (4.7), it can be expressed, after simplification, in the following form:

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= a_1[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)] \\
&+ g(X,W)\{b'AB\}(Y,Z) - g(Y,W)\{b'AB\}(X,Z) \\
&+ g(Y,Z)\{b'AB\}(X,W) - g(X,Z)\{b'AB\}(Y,W),
\end{aligned} \tag{5.4}
$$

where $\{b'AB\} \equiv b'_1AB + b'_2A\bar{B} + b'_3BB + b'_4B\bar{B} + b'_5\bar{A}B + b'_6\bar{B}B$ and $b'_1, b'_2, \dots, b'_6$ are scalars. Comparing (5.4) and (5.2) we conclude that the manifold under consideration is of hyper quasi-constant curvature. We can therefore state the following theorem which provides an answer to the question raised above:

Theorem 4. *A conformally flat $G(PS)_n$ $(n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.*

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