Integrability conditions of an $F(K, K-2)$-structure satisfying $F^K+F^{K-2}=0$ (K=odd)

By V. C. GUPTA & C P. AWASTHI (Lucknow)

Summary: YANO, HOUH and CHEN [1] have studied the structures defined by a tensor field Φ of the type (1, 1) satisfying $\Phi^4 \pm \Phi^2=0$. GADEA and CORDERO [2] have obtained the integrability conditions of these structures. The generalised $F(K, K-2)$-structure has been defined and studied by GUPTA [3]. The purpose of the present paper is to obtain the integrability conditions of a generalised $F(K, K-2)$ structure satisfying $F^K+F^{K-2}=0$, where F is a non-null tensor field of the type (1, 1) and K is odd. Here we have also obtained the conditions of partial integrability (introducing s_K-partial integrability and t_K-partial integrability) and integrability of the generalised $F(K, K-2)$-structure in terms of its *Nijenhuis tensor* for K odd.

1. The operators s and t for an $F(K, K-2)$-structure

Let M^n be an n-dimensional differentiable manifold of class C^∞ equipped with a (1, 1) tensor field $F(F \neq 0)$ and of class C^∞ satisfying [3]

$$n = 2m, \quad F^K+F^{K-2} = 0, \quad (2\,\mathrm{rank}\, F - \mathrm{rank}\, F^{K-1}) = \dim M^n, \tag{1.1}$$

when K is odd.

The operators s and t have been defined as follows [3]:

$$s = (-1)^{1/2(K-1)} F^{K-1}, \quad t = I-(-1)^{1/2(K-1)} F^{K-1}, \tag{1.2}$$

I denoting the identity operator. Then we have

Theorem (1.1). *For a tensor field $F(F \neq 0)$ satisfying* (1.1), *the operators s and t defined by* (1.2) *and applied to the tangent space at a point of the manifold are complementary projection operators.*

PROOF. By virtue of (1.1) and (1.2), we have

$$s+t = I; \tag{1.3}$$

$$\left\{\begin{aligned} s^2 &= (-1)^{K-1} F^{2K-2} = F^K \cdot F^{K-2} \\ &= -F^{K-2} \cdot F^{K-2} = -F^K \cdot F^{K-4} = \\ &= (-1)^2 F^{K-2} \cdot F^{K-4} = (-1)^2 F^K \cdot F^{K-6} = \\ &\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots \\ &= (-1)^{1/2(K-1)} F^{K-2} \cdot F^{K-(K-1)}, = \\ &= (-1)^{1/2(K-1)} F^{K-1} = s; \end{aligned}\right. \tag{1.4}$$

(1.5) $$t^2 = I+(-1)^{K-1}F^{2K-2}-2(-1)^{1/2(K-1)}F^{K-1} = = I-(-1)^{1/2(K-1)}F^{K-1} = t;$$

(1.6) $$st = ts = (-1)^{1/2(K-1)}F^{K-1}-(-1)^{K-1}F^{2K-2} = 0.$$

This proves the theorem.

Let S and T be the complementary distributions corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r, then $\dim S=2r-n$ and $\dim T=2n-2r$. Here the dimensions of S and T are both even. Obviously, $n\leqq 2r\leqq 2n$. Such a structure has been called a generalised '$F(K, K-2)$-structure of rank r' and the manifold M^n with this structure an '$F(K, K-2)$-manifold' [3].

Theorem (1.2). *For a tensor field $F(F\neq 0)$ satisfying* (1.1) *and the operators s and t defined by* (1.2), *we have*

(1.7) $$F^{K-2}s = sF^{K-2} = F^{K-2}, \quad F^{K-2}t = tF^{K-2} = 0;$$

(1.8) $$F^{K-1}s = F^{K-1}, \qquad F^{K-1}t = 0.$$

PROOF. The proof follows by virtue of the equations (1.1) and (1.2).

Theorem (1.3). *For a tensor field $F(F\neq 0)$ satisfying* (1.1) *and the operators s and t defined by* (1.2), *we have*

(1.9) $$Fs = sF = -(-1)^{1/2(K-1)}F^{K-2}, \quad Ft = tF = F+(-1)^{1/2(K-1)}F^{K-2};$$

(1.10) $$F^2s = -s, \quad F^2t = F^2+(-1)^{1/2(K-1)}F^{K-1}.$$

PROOF. The proof is obvious.

Corollary (1.1). *An $F(K, K-2)$-structure of maximal rank is an almost complex structure.*

PROOF. If the rank of F is maximal, $r=n$. Then $\dim S=n$ and $\dim T=0$. In this case $t=0$ and $s=I$. Thus F satisfies

(1.11) $$I-(-1)^{1/2(K-1)}F^{K-1} = 0.$$

Applying F twice to (1.11) and using (1.1), we obtain

$$F^2+(-1)^{1/2(K-1)}F^{K-1} = 0,$$

which in view of (1.11) yields

$$F^2+I = 0.$$

Hence the result.

Corollary (1.2). *An $F(K, K-2)$-structure of minimal rank is an $F(K-2)$-structure.*

PROOF. If the rank of F is minimal, $2r=n$. Then $s=0$. Thus F satisfies $F^{k-1}=0$ and therefore $F^k=0$. Hence in view of (1.1), $F^{k-2}=0$. Following the general nomenclature, we call such a structure an $F(K-2)$-structure.

2. The Nijenhuis tensor of an $F(K, K-2)$-structure

Let F be an $F(K, K-2)$-structure of rank r when K is odd. Then the Nijenhuis tensor $N(X, Y)$ of F is

$$N(X, Y) = [FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y]. \tag{2.1}$$

Therefore in consequence of (1.9) and (2.1), we have

$$N(sX, sY) = [-(-1)^{1/2(K-1)}F^{K-2}X, -(-1)^{1/2(K-1)}F^{K-2}Y]- -F[-(-1)^{1/2(K-1)}F^{K-2}X, sY]-F[sX, -(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[sX, sY], \tag{2.2}$$

$$N(sX, tY) = [-(-1)^{1/2(K-1)}F^{K-2}X, FY+(-1)^{1/2(K-1)}F^{K-2}Y]- -F[-(-1)^{1/2(K-1)}F^{K-2}X, tY]-F[sX, FY+(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[sX, tY], \tag{2.3}$$

$$N(tX, sY) = [FX+(-1)^{1/2(K-1)}F^{K-2}X, -(-1)^{1/2(K-1)}F^{K-2}Y]- -F[FX+(-1)^{1/2(K-1)}F^{K-2}X, sY]-F[tX, -(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[tX, sY], \tag{2.4}$$

$$N(tX, tY) = [FX+(-1)^{1/2(K-1)}F^{K-2}X, FY+(-1)^{1/2(K-1)}F^{K-2}Y]- -F[FX+(-1)^{1/2(K-1)}F^{K-2}X, tY]-F[tX, FY+(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[tX, tY]. \tag{2.5}$$

Equations (2.2), (2.3), (2.4) and (2.5), in consequence of (1.3) and (2.1), yield

$$N(X, Y) = N(sX, sY)+N(sX, tY)+N(tX, sY)+N(tX, tY). \tag{2.6}$$

If the distribution S is integrable, $N(sX, sY)$ is exactly the Nijenhuis tensor of $F/S \overset{\text{def}}{=} F_S$. If the distribution T is integrable, $N(tX, tY)$ is exactly the Nijenhuis tensor of $F/T \overset{\text{def}}{=} F_T$.

Let $\mathscr{L}_Y F$ be the Lie derivative of the tensor field F with respect to a vector field Y. Then we have [2]

$$(\mathscr{L}_Y F)X = F[X, Y]-[FX, Y], \tag{2.7}$$

where $\mathscr{L}_Y F$ is a tensor field of the same type as F.

Now in view of (2.1) and (2.7), we get

$$N(sX, tY) = F(\mathscr{L}_{tY} F)\, sX-(\mathscr{L}_{FtY} F)\, sX \tag{2.8}$$

and

$$N(tX, sY) = F(\mathscr{L}_{sY} F)\, tX-(\mathscr{L}_{FsY} F)\, tX. \tag{2.9}$$

3. Integrability conditions

In this section we shall obtain the partial integrability conditions of the $F(K, K=2)$-structure when K is odd.

Theorem (3.1). *For any two vector fields X and Y, the following hold:*

(i) *the distribution S is integrable if and only if $t\cdot N(sX, sY)=0$;*
(ii) *the distribution T is integrable if and only if $s\cdot N(tX, tY)=0$.*

PROOF. We know that for any two vector fields X and Y, the distributions S and T are integrable if and only if $t[sX, sY]=0$ and $s[tX, tY]=0$ respectively [2]. Thus in view of (1.6), (1.7), (1.9) and (2.1), the theorem follows.

Theorem (3.2). *For any two vector fields X and Y, the distributions S and T are both integrable if and only if*

$$N(X, Y) = s\cdot N(sX, sY)+N(sX, tY)+N(tX, sY)+t\cdot N(tX, tY). \tag{3.1}$$

PROOF. In consequence of (1.3), equation (2.6) can be expressed as

$$\begin{aligned} N(X, Y) = s\cdot N(sX, sY)+t\cdot N(sX, sY)+N(sX, tY)+ \\ +N(tX, sY)+s\cdot N(tX, tY)+t\cdot N(tX, tY). \end{aligned} \tag{3.2}$$

Now the result follows from equation (3.2) and theorem (3.1).

Theorem (3.3). *If the distribution S is integrable, a necessary and sufficient condition for the almost complex structure defined by $F/S=F_S$ on each integral manifold of S to be integrable is that for any two vector fields X and Y*

$$N(sX, sY) = 0, \tag{3.3}$$

which is equivalent to

$$s\cdot N(sX, sY) = 0. \tag{3.4}$$

PROOF. Suppose that the distribution S is integrable, then F induces on each integral manifold of S an almost complex structure. The induced structure is integrable if and only if its Nijenhuis tensor vanishes identically. Thus the theorem follows.

Definition (3.1). We say that an $F(K, K-2)$-structure is "s_K-partially integrable" if the distribution S is integrable and the almost complex structure F_S induced by F on each integral manifold of S is also integrable.

Theorem (3.4). *A necessary and sufficient condition for an $F(K, K-2)$-structure to be s_K-partially integrable is that for any two vector fields X and Y,*

$$N(sX, sY) = 0. \tag{3.5}$$

PROOF. Follows from theoresm (3.1) (i) and (3.3).

Theorem (3.5). *If the distribution T is integrable, a necessary and sufficient condition for the $F(K-2)$-structure defined by $F/T=F_T$ on each integral manifold of T to be integrable is that, for any two vector fields X and Y*

$$N(tX, tY) = 0, \tag{3.6}$$

which is equivalent to

$$t\cdot N(tX, tY) = 0. \tag{3.7}$$

PROOF. The proof follows from the pattern of the proof of theorem (3.3).

Definition (3.2). We say that an $F(K, K-2)$-structure is "t_K-partially integrable" if the distribution T is integrable and the $F(K-2)$-structure F_T induced by F on each integral manifold of T is also integrable.

Theorem (3.6). *A necessary and suffcient condition for the $F(K, K-2)$-structure to be t_K-partially integrable is that for any two vector fields X and Y,*

$$N(tX, tY) = 0. \tag{3.8}$$

PROOF. Follows from theorems (3.1) (ii) and (3.5).

Definition (3.3). We say that an $F(K, K-2)$-structure is "partially integrable" if and only if it is s_K-partially integrable and t_K-partially integrable, simultaneously.

Theorem (3.7). *A necessary and sufficient condition for the $F(K, K-2)$-structure to be partially integrable is that for any two vector fields X and Y,*

$$N(X, Y) = N(sX, tY)+N(tX, sY). \tag{3.9}$$

PROOF. The theorem follows by virtue of the equations (2.6), (3.5) and (3.8).

4. Conditions $N(sX, tY)=0$ and $N(tX, sY)=0$

In this section, we shall obtain the integrability conditions of an $F(K, K-2)$-structure by means of the conditions $N(sX, tY)=0$ and $N(tX, sY)=0$ when K is odd.

Theorem (4.1). *The tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically if and only if for any vector fields X and Y,*

$$N(sX, tY) = 0. \tag{4.1}$$

PROOF. In consequence of (2.8), we have $N(sX, tY)=0$ if and only if $F(\mathcal{L}_{tY}F)sX=(\mathcal{L}_{FtY}F)sX$. Thus, if $N(sX, tY)=0$, we get

$$\begin{aligned}(-1)^{1/2(K-1)}F^{K-1}(\mathcal{L}_{tY}F)sX &= (-1)^{1/2(K-1)}F^{K-2}(\mathcal{L}_{FtY}F)sX,\\ &= (-1)^{1/2(K-1)}F^{K-3}(\mathcal{L}_{F^2tY}F)sX,\\ &\cdots\\ &\cdots\\ &= (-1)^{1/2(K-1)}F^{K-K}(\mathcal{L}_{F^{K-1}tY}F)sX,\\ &= 0,\end{aligned}$$

in consequence of (1.8).

That is, in view of (1.2), the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y.

Theorem (4.2). *The tensor field $t(\mathcal{L}_{sY}F)t$ vanishes identically if and only if for any vector fields X and Y,*

$$N(tX, sY) = 0. \tag{4.2}$$

PROOF. The proof follows from the pattern of the proof of theorem (4.1).

When the distributions S and T are both integrable, we can choose a local coordinate system such that all S are represented by putting $(2n-2r)$ local coordinates constant and all T by putting the other $(2r-n)$ coordinates constant. Let us call such a coordinate system an "adapted coordinate system".

It can be supposed that in an adapted coordinate system, the projection operators s and t have components of the form

$$s = \begin{bmatrix} I_{2r-n} & 0 \\ 0 & 0 \end{bmatrix}, \quad t = \begin{bmatrix} 0 & 0 \\ 0 & I_{2n-2r} \end{bmatrix} \tag{4.3}$$

respectively, where I_{2r-n} is a unit matrix of order $(2r-n)$ and I_{2n-2r} is that of order $(2n-2r)$.

Since the distributions S and T are integrable, $FS \subset S$ and $FT \subset T$. Therefore, the tensor F has components of the form

$$F = \begin{bmatrix} F_{2r-n} & 0 \\ 0 & F_{2n-2r} \end{bmatrix} \tag{4.4}$$

in an adapted coordinate system, where F_{2r-n} and F_{2n-2r} are square matrices of order $(2r-n)\times(2r-n)$ and $(2n-2r)\times(2n-2r)$ respectively.

Thus, the Lie derivative $\mathscr{L}_{tY}F$ has components of the form

$$\mathscr{L}_{tY}F = \begin{bmatrix} L' & 0 \\ 0 & L'' \end{bmatrix}, \tag{4.5}$$

for any vector field tY on T.

Theorem (4.3). *Suppose that the two distributions S and T are both integrable and that an adapted coordinate system has been chosen. A necessary and sufficient condition for the local components F_{2r-n} of the $F(K, K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of S is that*

$$N(sX, tY) = 0 \tag{4.6}$$

for any two vector fields X and Y.

PROOF. Let us suppose that $N(sX, tY)=0$ for any two vector fields X and Y. Therefore by theorem (4.1), the tensor field $s(\mathscr{L}_{tY}F)s$ vanishes identically for any vetor field Y. Hence $L'=0$. It follows that the components F_{2r-n} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of the distribution S in an adapted coordinate system.

Conversely, if the components F_{2r-n} of the $F(K, K-2)$-structure are independent of these coordinates, then $L'=0$. Therefore the tensor field $s(\mathscr{L}_{tY}F)s$ vanishes identically for any vector field Y. Hence $N(sX, tY)=0$ for any two vector fields X and Y.

Theorem (4.4). *Under the assumptions of Theorem (4.3), a necessary and sufficient condition for the local components F_{2r-2n} of the $F(K, K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of T*

is that

$$N(tX, sY) = 0, \tag{4.7}$$

for any two vector fields X and Y.

PROOF. The proof is similar to that of theorem (4.3).

Definition (4.1). We say that an $F(K, K-2)$-structure is 'integrable' if

(i) the $F(K, K-2)$-structure is partially integrable;
(ii) the components F_{2r-n} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system;
(iii) the components F_{2n-2r} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of T in an adapted coordinate system.

Theorem (4.5). *In order that the $F(K, K-2)$-structure be integrable, it is necessary and sufficient that*

$$N(X, Y) = 0, \tag{4.8}$$

for any two vector fields X and Y.

PROOF. The theorem follows from theorems (3.7), (4.3) and (4.4).

Acknowledgement. We are thankful to Prof. M. D. UPADHYAY, D. Sc., for his valuable suggestions during the preparation of this paper. The first author is also thankful to the University Grants Commission, New Delhi for financial assistance in the form of a Research Associateship.

References

[1] K. YANO, C. S. HOUH and B. Y. CHEN, Structures defined by a tensor field Φ of type (1, 1) satisfying $\Phi^4 \pm \Phi^2 = 0$, *Tensor, N. S.*, **23** (1972), 81—87.
[2] P. M. GADEA and LUIS A. CORDERO, Integrability conditions of a structure Φ satisfying $\Phi^4 \pm \Phi^2 = 0$, *Tensor, N. S.*, **28** (1974), 78—82.
[3] V. C. GUPTA, On a structure define by a (1, 1) tensor field ξ $(\xi \neq 0)$ satisfying $\xi^K + {}^{K-2} = 0$, *Mathematica Balkanica* **5** (1975), 127—133.

DEPARTMENT OF MATHEMATICS
AND ASTRONOMY,
LUCKNOW UNIVERSITY,
LUCKNOW (INDIA)

DEPARTMENT OF MATHEMATICS,
SRI JAI NARAIN DEGREE COLLEGE
LUCKNOW (INDIA)

(Received August 2, 1985.)