Invariants of special semi-symmetric Finsler connection transformations

By PETRE STAVRE and FRANCISC C. KLEPP (Timisoara)

§ 1. Introduction

Let M be an n-dimensional differentiable manifold of class C^∞ and let (x^i, y^i) be the canonical coordinates of a point $y \in T(M)$, where $T(M)$ is the tangent bundle of M [3] and let π be the canonical projection. The natural basis of $T(M)$ with respect to canonical coordinates is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ and the mapping N: $y \in T(M) \to N_y \in T(M)_y$ is a regular distribution on $T(M)$, such that: $T(M)_y = N_y \oplus T(M)^v_y$. Let $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$ be a local basis of the n dimensional local distribution N, where $N^i_k(x, y)$ are called the coefficients of the non-linear connection defined by N. The notions and notations of M. MATSUMOTO [2] and R. MIRON [3] are used.

Let $F\Gamma = (N, F, C)$ be a Finsler connection with the coefficients (N^i_j, F^i_k, C^i_{jk}). $\mathscr{T}$ the group of general Finsler connection transformations t: $(N, F, C) \to (\bar N, \bar F, \bar C)$, and $\mathscr{T}_N = \{t \mid t \in \mathscr{T};\ t = t(0, B, D)\}$ the subgroup of $\mathscr{T}$, formed by the transformations t: $(N, F, C) \to (\bar N = N, \bar F, \bar C)$, which preserve the non-linear connection N. The transformations from $\mathscr{T}_N$ have the form:

$$\bar N^i_j = N^i_j;\ \bar F^i_{jk} = F^i_{jk} - B^i_{jk};\ \bar C^i_{jk} = C^i_{jk} - D^i_{jk} \tag{1.1}$$

where $B, D \in Z^1_2(M)$ are arbitrary Finsler tensor fields [3]. In the following we denote by $|,|$ and $\|,\|$ the h- and v-covariant derivatives relative to (N, F, C) and to $(\bar N, \bar F, \bar C)$ respectively.

For a Finsler connection $F\Gamma = (N, F, C)$ we can define the notion of semi-symmetric Finsler connection, analogously with the definition of semi-symmetric linear-connection. We consider the h- and v-Finsler torsion tensors:

$$T^i_{jk} = F^i_{jk} - F^i_{kj};\ S^i_{jk} = C^i_{jk} - C^i_{kj} \tag{1.2}$$

and the associated tensors:

$$\underset{1}{I}{}^i_{jk} = \underset{1}{T}{}^i_{jk} - \frac{1}{(n-1)}(\delta^i_j T_k - \delta^i_k T_j);\ \underset{2}{I}{}^i_{jk} = S^i_{jk} - \frac{1}{(n-1)}(\delta^i_j S_k - \delta^i_k S_j) \tag{1.3}$$

where: $T_k = T^i_{ik}$ and $S_k = S^i_{ik}$ are the h- and v-torsion Finsler covectors respectively. A study of these tensors relative to its projective transformations are given in [6].

A Finsler connection $F\Gamma=(N, F, C)$ is called a semi-symmetric Finsler connection, if $\underset{1}{I}{}^i_{jk}=0$ and $\underset{2}{I}{}^i_{jk}=0$.

This condition is equivalent with the existence of an 1-form on $T(M)$:

$$\tau = \tau_j\, dx^j + \omega_k \delta y^k \tag{1.4}$$

such that:

$$T_k = (1-n)\tau_k; \quad S_k = (1-n)\omega_k \tag{1.5}$$

For a fixed semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ the h-torsion tensor and the v-torsion tensor have the form:

$$T^i_{jk} = \tau_j \delta^i_k - \tau_k \delta^i_j; \quad S^i_{jk} = \omega_j \delta^i_k - \omega_k \delta^i_j \tag{1.6}$$

In general, for a Finsler connection $F\Gamma=(N, F, C)$ we have the following three curvature tensors:

$$\begin{aligned} S^i_{jkh} &= \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk} \\ R^i_{jkh} &= \frac{\delta F^i_{jk}}{\delta x^h} - \frac{\delta F^i_{jh}}{\delta x^k} + F^r_{jk} F^i_{rh} - F^r_{jh} F^i_{rk} + C^i_{jr} R^r_{kh} \\ P^i_{jkh} &= \frac{\partial F^i_{jk}}{\partial y^h} - C^i_{jh|k} + C^i_{jr} P^r_{kh} \end{aligned} \tag{1.7}$$

where:

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}; \quad P^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - F^i_{kj} \tag{1.8}$$

We define also the Finsler tensor:

$$K^i_{jkh} \overset{\text{def}}{=} R^i_{jkh} - C^i_{jr} R^r_{kh} \tag{1.9}$$

where R^r_{kh} is the curvature tensor of the non-linear connection N^i_j, given by the first relation of (1.8).

If the non-linear connection N is integrable, then: $K^i_{jkh}=R^i_{jkh}$. We denote by:

$$s_{kh} = S^i_{ikh}; \quad r_{kh} = R^i_{ikh}; \quad B_{kh} = K^i_{ikh} \tag{1.10}$$

the tensors of Bianchi type and by:

$$S_{jk} = S^i_{jki}; \quad R_{jk} = R^i_{jki}; \quad K_{jk} = K^i_{jki} \tag{1.11}$$

the tensors of Ricci type, and use also the notations:

$$(h \operatorname{div} T)_{ij} \overset{\text{def}}{=} T^k_{ij|k}; \quad (v \operatorname{div} S)_{ij} \overset{\text{def}}{=} S^k_{ij}|_k \tag{1.12}$$

If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then between the Finsler tensors of Ricci type and of Bianchi type we have the relations [1]:

$$K_{jk} - K_{kj} = B_{kj} + (n-2)(h \operatorname{div} T)_{kj} \tag{1.13}$$

$$S_{jk} - S_{kj} = s_{kj} + (n-2)(v \operatorname{div} S)_{kj} \tag{1.14}$$

In the present paper we study invariants of those Finsler connection transformations, which transform semi-symmetric Finsler connections in semi-symmetric Finsler connections again. First we study invariants obtained by the use of S and R curvature tensors, and afterwords invariants obtained using the third curvature tensor P. Finally the properties of the Finsler 1-form $\omega\in\Lambda\ (T(M))$ associate to the Finsler connection transformation are given.

§ 2. Invariants of S and R type

In this section the invariants obtained by the use of the curvature tensors S and R (or K) are established.

The most general Finsler connection transformation $t\in\mathcal{T}_N$, which has the invariants $\underset{1}{I}$ and $\underset{2}{I}$, has been studied in [8] obtaining the following result:

Lemma 2.1. *The set $\mathcal{T}_{NI}$ of all Finsler connections, which preserves the linear connection N and has the invariants I_1 and I_2, is given by:*

$$\bar{N}^i_j = N^i_j;\ \ \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\beta_k-\delta^i_k\beta_j-U^i_{jk};\ \ \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\alpha_k-\delta^i_k\alpha_j-V^i_{jk} \tag{2.1}$$

where $F\Gamma=(N, F, C)$ is a fixed Finsler connection; α_k and β_k are arbitrary Finsler covectors, and $U, V\in Z^1_2(M)$ are arbitrary Finsler tensor fields of type (1, 2), *with the properties $U^i_{jk}=U^i_{kj}$; $V^i_{jk}=V^i_{kj}$.*

The most general case of the Finsler connection transformations $t\in\mathcal{T}=\{t|t\colon (N, F, C)\to(\bar{N}, \bar{F}, \bar{C})\}$ with the invariant tensors $\underset{1}{I}$ and $\underset{2}{I}$ has been studied in [7].

If we have $\underset{1}{I}=0$ and $\underset{2}{I}=0$, then from (2.1) it follows $\underset{1}{\bar{I}}=0$, $\underset{2}{\bar{I}}=0$, and reciprocally. Thus $\mathcal{T}_{NI}$ contains also the set of the general semi-symmetric connection transformations.

Afterwards we consider the special case of the Finsler connection transformations (2.1) with $U=0$ and $V=0$. In this case we obtain:

$$\bar{N}^i_j = N^i_j;\ \ \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\beta_k-\delta^i_k\beta_j;\ \ \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\alpha_k-\delta^i_k\alpha_j \tag{2.2}$$

where $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection. Then also $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ is a semi-symmetric Finsler connection.

Between the curvature tensor K^i_{jkh}, S^i_{jkh} and $\bar{K}^i_{jkh}$, $\bar{S}^i_{jkh}$ respectively we obtain the relations:

$$\bar{K}^i_{jkh} = K^i_{jkh}+\delta^i_j(\varphi_{kh}-\varphi_{hk})-\delta^i_k\varphi_{jh}+\delta^i_h\varphi_{jk} \tag{2.3}$$

$$\bar{S}^i_{jkh} = S^i_{jkh}+\delta^i_j(\Phi_{kh}-\Phi_{hk})-\delta^i_k\Phi_{jh}+\delta^i_h\Phi_{jk} \tag{2.4}$$

where we have:

$$\Phi_{kh} = \alpha_k|_h+\alpha_k\alpha_h-\alpha_k\omega_h;\ \ \varphi_{kh} = \beta_{k|h}+\beta_k\beta_h-\beta_k\tau_h \tag{2.5}$$

From (2.3) and (2.4) follows the relations:

$$\bar{K}_{jk} = K_{jk}+(n-2)\,\varphi_{jk}+\varphi_{kj};\ \ B_{kh} = \bar{B}_{kh}+(n-1)(\varphi_{kh}-\varphi_{hk}) \tag{2.6}$$

$$\bar{S}_{jk} = S_{jk}+(n-2)\,\Phi_{jk}+\Phi_{kj};\ \ s_{kh} = \bar{s}_{kh}+(n-1)(\Phi_{kh}-\Phi_{hk}) \tag{2.7}$$

From (2.6) and (2.7) we have:

$$\overline{K}_{jk}-\overline{K}_{kj} = K_{jk}-K_{kj}+(n-3)(\varphi_{jk}-\varphi_{kj}); \quad \overline{B}_{kh}+\overline{B}_{hk} = B_{kh}+B_{hk} \tag{2.8}$$

$$\overline{S}_{jk}-\overline{S}_{kj} = S_{jk}-S_{kj}+(n-3)(\Phi_{jk}-\Phi_{kj}); \quad \bar{s}_{kh}-\bar{s}_{hk} = s_{kh}-s_{hk} \tag{2.9}$$

Thus it follows the:

Proposition 2.1. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then the transformation* (2.2) *has the invariants:*

$$\overline{K}_{jk}-\overline{K}_{kj}-\frac{n-3}{n-1}\,\overline{B}_{jk} = K_{jk}-K_{kj}-\frac{n-3}{n-1}\,B_{jk}; \quad \overline{B}_{kh}+\overline{B}_{hk} = B_{kh}+B_{hk} \tag{2.10}$$

$$\overline{S}_{jk}-\overline{S}_{kj}-\frac{n-3}{n-1}\,\bar{s}_{jk} = S_{jk}-S_{kj}-\frac{n-3}{n-1}\,s_{jk}; \quad \bar{s}_{kh}+\bar{s}_{hk} = s_{kh}+s_{hk} \tag{2.11}$$

From (2.6) and (2.7) we have:

$$\varphi_{jk} = \frac{1}{n-1}(\overline{K}_{jk}-K_{jk})+\frac{1}{(n-1)^2}(\overline{B}_{jk}-B_{jk}) \tag{2.12}$$

$$\Phi_{jk} = \frac{1}{n-1}(\overline{S}_{jk}-S_{jk})+\frac{1}{(n-1)^2}(\bar{s}_{jk}-s_{jk}) \tag{2.13}$$

From (2.1), (2.2), (2.4), (2.5), (2.8) and (2.9) follow the relations:

$$\underset{1}{\overline{A}}{}^i_{jkh} \stackrel{\text{def}}{=} \overline{K}^i_{jkh}+\frac{1}{n-1}\,\delta^i_j\,\overline{B}_{hk}+\delta^i_k\left(\frac{\overline{K}_{jh}}{n-1}+\frac{\overline{B}_{jh}}{(n-1)^2}\right)-\delta^i_h\left(\frac{\overline{K}_{jk}}{n-1}+\frac{\overline{B}_{jk}}{(n-1)^2}\right)= \tag{2.14}$$

$$= K^i_{jkh}+\frac{1}{n-1}\,\delta^i_j\,B_{hk}+\delta^i_k\left(\frac{K_{jh}}{n-1}+\frac{B_{jh}}{(n-1)^2}\right)-\delta^i_h\left(\frac{K_{jk}}{n-1}+\frac{B_{jk}}{(n-1)^2}\right) \stackrel{\text{def}}{=} \underset{1}{A}{}^i_{jkh}$$

$$\underset{2}{\overline{A}}{}^i_{jkh} \stackrel{\text{def}}{=} \overline{S}^i_{jkh}+\frac{1}{n-1}\,\delta^i_j\,\bar{s}_{hk}+\delta^i_k\left(\frac{\overline{S}_{jh}}{n-1}-\frac{\bar{s}_{jh}}{(n-1)^2}\right)-\delta^i_h\left(\frac{\overline{S}_{jk}}{n-1}+\frac{\bar{s}_{jk}}{(n-1)^2}\right)= \tag{2.15}$$

$$= S^i_{jkh}+\frac{1}{n-1}\,\delta^i_j\,s_{hk}+\delta^i_k\left(\frac{S_{jh}}{n-1}+\frac{s_{jh}}{(n-1)^2}\right)-\delta^i_h\left(\frac{S_{jk}}{n-1}+\frac{s_{jk}}{(n-1)^2}\right) \stackrel{\text{def}}{=} \underset{2}{A}{}^i_{jkh}$$

Thus we have the following:

Theorem 2.1. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then the transformation* (2.2) *has the invariants $\underset{1}{A}$ and $\underset{2}{A}$ given by* (2.14) *and* (2.15) *respectively.*

From (2.5) it follows:

$$\Phi_{kh}-\Phi_{hk} = \frac{\partial\alpha_k}{\partial y^h}-\frac{\partial\alpha_h}{\partial y^k}; \quad \varphi_{kh}-\varphi_{hk} = \frac{\delta\beta_k}{\delta x^h}-\frac{\delta\beta_h}{\delta x^k} \tag{2.16}$$

Thus we have the:

Lemma 2.1. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then:*

$$\Phi_{kh}-\Phi_{hk}=0 \quad \text{iff} \quad \frac{\partial \alpha_k}{\partial y^h}=\frac{\partial \alpha_h}{\partial y^k};\ \varphi_{kh}-\varphi_{hk}=0 \quad \text{iff} \quad \frac{\delta \beta_k}{\delta x^h}-\frac{\delta \beta_h}{\delta x^k}=0, \tag{2.17}$$

From this Lemma and the relations (2.6), (2.7), (2.8), (2.9) it follows the:

Proposition 2.2. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then*
(a) *the Finsler—Bianchi tensors B_{kh} and s_{kh} are invariants of the transformation* (2.2) *if and only if:*

$$\frac{\partial \alpha_h}{\partial y^k}=\frac{\partial \alpha_k}{\partial y^h};\quad \frac{\partial \beta_h}{\delta x^k}=\frac{\delta \beta_k}{\delta x^h} \tag{2.18}$$

(b) *the Finsler tensors $K_{jk}-K_{kj}$ and $S_{jk}-S_{kj}$ are invariants of the transformation* (2.2) *if and only if the relations* (2.18) *hold.*

From (2.12) and (2.18) it follows:

$$\varphi_{jk}=\frac{1}{n-1}(\bar{K}_{jk}-K_{jk});\quad \Phi_{jk}=\frac{1}{n-1}(\bar{S}_{jk}-S_{jk}) \tag{2.19}$$

From (2.3), (2.4) and (2.19) we have the relations:

$$\underset{1}{\bar{B}}{}^i_{jkh} \stackrel{\text{def}}{=} \bar{K}^i_{jkh}-\frac{1}{n-1}(\delta^i_h\bar{K}_{jk}-\delta^i_k\bar{K}_{jh})= \tag{2.20}$$

$$=K^i_{jkh}-\frac{1}{n-1}(\delta^i_hK_{jk}-\delta^i_kK_{jh}) \stackrel{\text{def}}{=} \underset{1}{B}{}^i_{jkh}$$

$$\underset{2}{\bar{B}}{}^i_{jkh} \stackrel{\text{def}}{=} \bar{S}^i_{jkh}-\frac{1}{n-1}(\delta^i_h\bar{S}_{jk}-\delta^i_k\bar{S}_{jh})= \tag{2.21}$$

$$=S^i_{jkh}-\frac{1}{n-1}(\delta^i_hS_{jk}-\delta^i_kS_{jh}) \stackrel{\text{def}}{=} \underset{2}{B}{}^i_{jkh}$$

Reciprocally: From (2.20) and (2.21), using (2.3)—(2.4) and (2.6)—(2.7) we have:

$$\varphi_{kh}-\varphi_{hk}=0;\quad \Phi_{kh}-\Phi_{hk}=0 \tag{2.22}$$

Theorem 2.2. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then the necessary and sufficient condition that the transformation* (2.2) *has the invariants of Weyl-type* (2.20)—(2.21) *is given by* (2.18).

We associate to the connection $F\Gamma=(N, F, C)$ the Finsler tensors of projective Weyl-type:

$$\underset{1}{W}{}^i_{jkh}=K^i_{jkh}+\frac{1}{n+1}\delta^i_jB_{hk}+\delta^i_k\left(\frac{K_{jh}}{n-1}+\frac{B_{jh}}{n^2-1}\right)-\delta^i_h\left(\frac{K_{jk}}{n-1}+\frac{B_{j}}{n^2-1}\right) \tag{2.23}$$

$$\underset{2}{W}{}^i_{jkh}=S^i_{jkh}+\frac{1}{n+1}\delta^i_js_{hk}+\delta^i_k\left(\frac{S_{jh}}{n-1}+\frac{s_{jh}}{n^2-1}\right)-\delta^i_h\left(\frac{S_{jk}}{n-1}+\frac{s_{jk}}{n^2-\ .}\right) \tag{2.24}$$

analogously with the projective tensors of Weyl from the linear connection theory.

The projective transformations of the semi-symmetric connections, characterized by the invariants $\underset{1}{W}$ and $\underset{2}{W}$ are studied in [4] and [5]. In the case of the transformations (2.2) we have:

(2.25)

$$\underset{1}{\overline{W}}{}^i_{jkh} = \underset{1}{W}{}^i_{jkh} + \frac{2}{n+1}\,\delta^i_j(\varphi_{hk} - \varphi_{kh}) + \frac{2}{n^2-1}\,\delta^i_k(\varphi_{jh} - \varphi_{hj}) - \frac{2}{n^2-1}\,\delta^i_h(\varphi_{jk} - \varphi_{kj})$$

(2.26)

$$\underset{2}{\overline{W}}{}^i_{jkh} = \underset{2}{W}{}^i_{jkh} + \frac{2}{n+1}\,\delta^i_j(\Phi_{hk} - \Phi_{kh}) + \frac{2}{n^2-1}\,\delta^i_k(\Phi_{jh} - \Phi_{hj}) - \frac{2}{n^2-1}\,\delta^i_h(\Phi_{jk} - \Phi_{kj})$$

From Lemma 2.1 and the relations (2.23)—(2.24) it follows the:

Theorem 2.3. *If* $F\Gamma = (N, F, C)$ *is a semi-symmetric Finsler connection, then the necessary and sufficient condition that the transformation* (2.2) *has the invariants* $\underset{1}{W}$ *and* $\underset{2}{W}$ *is given by the relations* (2.18).

Corollary 2.1. *If* $\underset{1}{\overline{B}} = \underset{1}{B}$, $\underset{2}{\overline{B}} = \underset{2}{B}$ *then and only then we have:*

$$\underset{1}{\overline{W}} = \underset{1}{W} \quad \text{and} \quad \underset{2}{\overline{W}} = \underset{2}{W}.$$

From (1.13)—(1.14) and (2.23)—(2.24) we have:

$$\underset{1}{W}{}^i_{ikh} = \frac{n-2}{n-1}(h\,\mathrm{div}\,T)_{kh}; \quad \underset{2}{W}{}^i_{ikh} = \frac{n-2}{n-1}(v\,\mathrm{div}\,S)_{kh} \tag{2.27}$$

Thus it follows the:

Proposition 2.3. *The necessary and sufficient condition that for a semi-symmetric Finsler connection* $F\Gamma = (N, F, C)$ *the Finsler tensors* $\underset{1}{W}{}^i_{ikh}$, $\underset{2}{W}{}^i_{ikh}$ *wanish, is that the torsional divergence of* $F\Gamma = (N, F, C)$ *wanishes.*

From (2.25)—(2.26) it follows:

$$\underset{1}{\overline{W}}{}^i_{ikh} = \underset{1}{W}{}^i_{ikh} + 2(n-2)(\varphi_{kh} - \varphi_{hk}); \quad \underset{2}{\overline{W}}{}^i_{ikh} = \underset{2}{W}{}^i_{ikh} + 2(n-2)(\Phi_{kh} - \Phi_{hk}) \tag{2.28}$$

From the Proposition 2.3 and the relation (2.22), (2.28) it follows the:

Proposition 2.4. *The necessary and sufficient condition that a semi-symmetric Finsler connection* $F\Gamma = (N, F, C)$ *with null torsional divergence be transformed by the transformation* (2.2) *in a semi-symmetric Finsler connection* $F\overline{\Gamma} = (N, \overline{F}, \overline{C})$ *with the same property, is that the transformation has the property* (2.18).

Next it follows the:

Proposition 2.5. *If $F\Gamma=(N, F, C)$ and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ are semi-symmetric Finsler connections with a null torsional divergence, then we have:*

$$(2.29)\qquad K_{jk}-K_{kj}=B_{kj};\quad S_{jk}-S_{kj}=s_{kj};\quad \bar{K}_{jk}-\bar{K}_{jk}=\bar{B}_{kj};\quad \bar{S}_{jk}-\bar{S}_{kj}=\bar{s}_{kj}$$

and it follows that $\underset{1}{B}, \underset{2}{B}, \underset{1}{A}, \underset{2}{A}, \underset{1}{W}, \underset{2}{W}$ are invariants for the transformation (2.2).

Definition 2.1. The semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ is equiaffine, if $B_{kj}=0$, $s_{kj}=0$.

Proposition 2.6. *If $F\Gamma=(N, F, C)$ is an equiaffine, semi-symmetric Finsler connection, with null torsional divergence, then the Finsler connection $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ obtained by the transformation* (2.2) *has the same properties. In this case the invariants $\underset{1}{A}, \underset{1}{B}, \underset{1}{W}$ coincide and the invariants $\underset{2}{A}, \underset{2}{B}, \underset{2}{W}$ coincide also.*

Proposition 2.7. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, $\bar{K}^i_{jkh}=0$, $\bar{S}^i_{jkh}=0$ and α_n, β_n satisfies the condition* (2.18), *then $F\Gamma$ is equiaffine Finsler connection with null torsional divergence.*

In this case we have:

$$(2.30)\qquad S^i_{jkh}=\frac{1}{n-1}(\delta^i_h S_{jk}-\delta^i_k S_{jh});\quad K^i_{jkh}=\frac{1}{n-1}(\delta^i_h K_{jk}-\delta^i_k K_{jh})$$

$$(2.31)\qquad S_{jk}=S_{kj};\quad K_{jk}=K_{kj}$$

From (2.30) it follows:

$$(2.32)\qquad S^i_{jkh}|_r=\frac{1}{n-1}(\delta^i_h S_{jk}|_r-\delta^i_k S_{jh}|_r);\quad S^i_{jkh}|_i=\frac{1}{n-1}(S_{jk}|_h-S_{jh}|_k)$$

$$(2.33)\qquad K^i_{jkh|r}=\frac{1}{n-1}(\delta^i_h K_{jk|r}-\delta^k_i K_{jh|r});\quad K^i_{jkh|i}=\frac{1}{n-1}(K_{jk|h}-K_{jh|k})$$

and we obtain the Bianchi identities:

$$(2.34)\qquad S^i_{jkh}+S^i_{khj}+S^i_{hjk}=0;\quad K^i_{jkh}+K^i_{khj}+K^i_{hjk}=0$$

Thus it follows the:

Proposition 2.8. *For a semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ the Bianchi identities* (2.34) *holds if and only if $F\Gamma$ is with null torsional divergence.*

The second group of Bianchi identities for a semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ is given in [5]:

$$(2.35)\qquad \sum_{(jkh)} K^r_{ijk|h}=2\sum_{(jkh)} K^r_{ijk}\tau_h-\frac{\partial F^r_{ij}}{\partial y^s}R^s_{kh};\quad \sum_{(jkh)} S^r_{ijk}|_h=2\sum_{(jkh)} S^r_{ijk}\omega_h$$

From (2.32) and (2.35) it follows:

$$(2.36)\qquad S_{ij}|_k-S_{ik}|_j=2S_{ir}S^r_{kj}$$

If N is integrable we obtain analogously:

$$K_{ij|k}-K_{ik|j} = 2K_{ir}T^r_{kj} \tag{2.37}$$

Proposition 2.9. *If $F\Gamma=(N, F, C)$ is an equiaffin semi-symmetric Finsler connection with null torsional divergence, then the relations* (2.30)—(2.36) *hold, and if moreover N is integrable, then the relations* (2.30)—(2.35) *and* (2.37) *hold however.*

Consequently if the conditions of Propositions 2.7 and 2.9 are satisfied, then we have the following Propositions:

Proposition 2.10. *If $F\Gamma=(N, F, C)$ is a symmetric Finsler connection and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ is a projectively flate Finsler connection ($\underset{1}{\bar{W}}=0$, $\underset{2}{\bar{W}}=0$), then $F\Gamma$ is an equiaffine and projectively flate Finsler connection.*

Definition 2.1. A Finsler connection $F\Gamma=(N, F, C)$ is called a Finsler F-connection, if:

$$\frac{\partial F^i_{jk}}{\partial y^r} R^r_{hi}+\frac{\partial F^i_{jh}}{\partial y^r} R^r_{ik}+\frac{\partial F^i_{ji}}{\partial y^s} R^s_{kh} = 0 \tag{2.38}$$

Proposition 2.11. *If $F\Gamma=(N, F, C)$ is a symmetric Finsler connection and we have $\bar{K}^i_{jkh}=0$, $\bar{S}^i_{jkh}=0$, then $F\Gamma$ is equiaffine, projectively flate and the tensor $S_{ijk} \overset{\text{def}}{=} S_{ij}|_k-S_{ik}|_j$ is symmetric. The tensor $K_{ijk} \overset{\text{def}}{=} K_{j|k}-K_{k|j}$ is symmetric if and only if $F\Gamma$ is a Finsler F-connection.*

If $\underset{1}{\bar{W}}{}^i_{jkh|r}=0; \underset{2}{\bar{W}}{}^i_{jkh}|_r=0$, it follows $\underset{1}{\bar{W}}{}^i_{ikh|r}=0$, $\underset{2}{\bar{W}}{}^i_{ikh}|_r=0$ and from (2.28) we have:

$$(\varphi_{kh}-\varphi_{hk})_{|r} = 0; \quad (\Phi_{kh}-\Phi_{hk})|_r = 0 \tag{2.39}$$

From (2.25), (2.26) and (2.39) we have:

$$\underset{1}{W}{}^i_{jkh|r} = 0; \quad \underset{2}{W}{}^i_{jkh}|_r = 0 \tag{2.40}$$

Thus it follows the:

Proposition 2.12. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection and $\underset{1}{\bar{W}}{}^i_{jkh|r}=0$, $\underset{2}{\bar{W}}{}^i_{jkh}|_r=0$, then also $\underset{1}{W}{}^i_{jkh|r}=0$, $\underset{2}{W}{}^i_{jkh}|_r=0$.*

Particularly we have the:

Proposition 2.13. *If $F\Gamma=(N, F, C)$ is a symmetric Finsler connection and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ is a projective symmetric Finsler connection ($\underset{1}{\bar{W}}{}^i_{jkh|r}=0$, $\underset{2}{\bar{W}}{}^i_{jkh}|_r=0$), then also $F\Gamma$ is projective symmetric ($\underset{1}{W}{}^i_{jkh|r}=0$, $\underset{2}{W}{}^i_{jkh|r}=0$).*

§ 3. Invariants of $\mathscr{P}$ type

In this section the invariants of the transformations (2.2) obtained from the third curvature tensor P, or from the tensor $\mathscr{P}$ are studied.

The third curvature tensor P^i_{jkh} and the tensor:

$$\mathscr{P}^i_{jkh} \overset{\text{def}}{=} \left(P^i_{jkh} - C^i_{jr}\frac{\partial N^r_k}{\partial y^h}\right) - \left(P^i_{jkh} - C^i_{jr}\frac{\partial N^r_h}{\partial y^k}\right) \tag{3.1}$$

are given in [3].

By a Finsler connection transformation (2.2) we obtain:

$$\bar{\mathscr{P}}^i_{jkh} = \mathscr{P}^i_{jkh} + \delta^i_j(\Theta_{kh} - \Theta_{hk}) - \delta^i_k\Theta_{jh} + \delta^i_h\Theta_{jk} \tag{3.2}$$

where:

$$\Theta_{kh} = \beta_{k|h} + \alpha_k|_h + \beta_k\alpha_h + \alpha_k\beta_h - \beta_k\omega_h - \alpha_k\tau_h \tag{3.3}$$

From (3.2) it follows:

$$\bar{\mathscr{P}}_{jk} = \mathscr{P}_{jk} + (n-1)\Theta_{jk} - (\Theta_{jk} - \Theta_{kj}); \quad \bar{\pi}_{kh} = \pi_{kh} + (n-1)(\Theta_{kh} - \Theta_{hk}) \tag{3.4}$$

where:

$$\mathscr{P}_{jk} = \mathscr{P}^i_{jki}; \quad \pi_{kh} = \mathscr{P}^i_{ikh}; \quad \bar{\mathscr{P}}_{jk} = \bar{\mathscr{P}}^i_{jki}; \quad \bar{\pi}_{kh} = \bar{\mathscr{P}}^i_{ikh}$$

and it follows also:

$$\bar{\mathscr{P}}_{jk} - \bar{\mathscr{P}}_{kj} = \mathscr{P}_{jk} - \mathscr{P}_{kj} + (n-3)(\Theta_{jk} - \Theta_{kj}); \quad \bar{\pi}_{kh} - \pi_{kh} = -(\bar{\pi}_{hk} - \pi_{hk}) \tag{3.5}$$

From (3.2)—(3.5) we obtain the invariant $\underset{3}{\bar{A}}{}^i_{jkh} = \underset{3}{A}{}^i_{jkh}$, given by:

$$\underset{3}{\bar{A}}{}^i_{jkh} \overset{\text{def}}{=} \bar{\mathscr{P}}^i_{jkh} + \frac{1}{n-1}\delta^i_j\bar{\pi}_{hk} + \delta^i_k\left(\frac{\bar{\mathscr{P}}_{jh}}{n-1} + \frac{\bar{\pi}_{jh}}{(n-1)^2}\right) - \delta^i_h\left(\frac{\bar{\mathscr{P}}_{jk}}{n-1} + \frac{\bar{\pi}_{jk}}{(n-1)^2}\right) = \tag{3.6}$$

$$= \mathscr{P}^i_{jkh} + \frac{1}{n-1}\delta^i_j\pi_{hk} + \delta^i_k\left(\frac{\mathscr{P}_{jh}}{n-1} + \frac{\pi_{jh}}{(n-1)^2}\right) - \delta^i_h\left(\frac{\mathscr{P}_{jk}}{n-1} + \frac{\pi_{jk}}{(n-1)^2}\right) \overset{\text{def}}{=} \underset{3}{A}{}^i_{jkh}$$

It follows the:

Theorem 3.1. *The Finsler connection transformation* (2.2) *between the semi-symmetric Finsler connections* $F\Gamma = (N, F, C)$ *and* $F\bar{\Gamma} = (\bar{N}, \bar{F}, \bar{C})$ *has the invariant* $\underset{3}{A}{}^i_{jkh}$.

For $n \geqq 3$ the relations are equivalents:

$$\text{(a) } \Theta_{kh} = \Theta_{hk}; \quad \text{(b) } \bar{\pi}_{jk} = \pi_{jk}; \quad \text{(c) } \bar{\mathscr{P}}_{jk} - \bar{\mathscr{P}}_{kj} = \mathscr{P}_{jk} - \mathscr{P}_{kj}.$$

It follows the:

Theorem 3.2. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, then a necessary and sufficient condition that the transformation* (2.2) *has the invariant:*

$$(3.7)\qquad \underset{3}{\bar{B}}{}^i_{jkh} \overset{\text{def}}{=} \bar{\mathscr{P}}^i_{jkh}-\frac{1}{n-1}(\delta^i_h\bar{\mathscr{P}}_{jk}-\delta^i_k\bar{\mathscr{P}}_{jh}) = \mathscr{P}^i_{jkh}-\frac{1}{n-1}(\delta^i_h\mathscr{P}_{jk}-\delta^i_k\mathscr{P}_{jh}) \overset{\text{def}}{=} \underset{3}{B}{}^i_{jkh}$$

is given by: $\Theta_{kh}=\Theta_{kh}$.

We associate to $\mathscr{P}^i_{jkh}$ a Finsler tensor $\underset{3}{W}{}^i_{jkh}$ analogous with the Weyl projective curvature tensor where:

$$(3.8)\qquad \underset{3}{W}{}^i_{jkh} \overset{\text{def}}{=} \mathscr{P}^i_{jkh}+\frac{1}{n+1}\delta^i_j\pi_{hk}+\delta^i_k\left(\frac{\mathscr{P}_{jh}}{n-1}+\frac{\pi_{jh}}{n^2-1}\right)-\delta^i_h\left(\frac{\mathscr{P}_{jk}}{n-1}+\frac{\pi_{jk}}{n^2-1}\right)$$

We obtain:

$$(3.9)\qquad \underset{3}{\bar{W}}{}^i_{jkh} = \underset{3}{W}{}^i_{jkh}+\frac{2}{n+1}\delta^i_j(\Theta_{kh}-\Theta_{hk})-\frac{2}{n^2-1}\left[\delta^i_k(\Theta_{jh}-\Theta_{hj})-\delta^i_h(\Theta_{jk}-\Theta_{kj})\right]$$

and:

$$(3.10)\qquad \underset{3}{\bar{W}}{}^i_{ikh} = \underset{3}{W}{}^i_{ikh}+2\frac{n-2}{n-1}(\Theta_{kh}-\Theta_{hk});\quad \underset{3}{\bar{W}}{}^i_{jki} = \underset{3}{W}{}^i_{jki}-0.$$

It follows the:

Theorem 3.3. *The necessary and sufficient condition that the Finsler connection transformation* (2.2) *between the semi-symmetric Finsler connections $F\Gamma=(N, F, C)$ and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ has the invariant $\underset{3}{\bar{W}}{}^i_{jkh}=\underset{3}{W}{}^i_{jkh}$, is given by: $\Theta_{kh}=\Theta_{hk}$.*

In this case theinvar iant $\underset{3}{\bar{B}}{}^i_{jkh}=\underset{3}{B}{}^i_{jkh}$ is an outcome of the invariant $\underset{3}{W}{}^i_{jkh}$.

§ 4. Properties of the Finsler 1-form $\omega\in\Lambda(T(M))$, associate to the Finsler connection transformation

Let $\omega\in\lambda\ (T(M))$ be the Finsler 1-form

$$(4.1)\qquad \omega = \beta_k\,dx^k+\alpha_i\delta y^i$$

associate to the Finsler connection transformation (2.2). We denote:

$$(4.2)\qquad \Theta_k = \beta_k+N^i_k\alpha_i$$

$$(4.3)\qquad \alpha_{kh} = \frac{\delta\beta_k}{\delta x^h}-\frac{\delta\beta_h}{\delta x^k}+R^i_{kh}\alpha_i$$

$$(4.4)\qquad \beta_{kh} = \frac{\partial\Theta_k}{\delta y^h}-\frac{\partial\Theta_h}{\partial y^k}$$

$$(4.5)\qquad \gamma_{kh} = \frac{\partial\alpha_k}{\partial y^h}-\frac{\partial\alpha_h}{\partial y^k}$$

Thus it follows the:

Theorem 4.1. *The necessary and sufficient condition that the Finsler 1-from* ω *be closed* $(d\omega=0)$ *is given by:* $\alpha_{kh}=0,\ \beta_{kh}=0,\ \gamma_{kh}=0.$

PROOF. The exterior differential of ω is given by:

$$d\omega=\frac{1}{2}\underset{1}{\alpha}_{kh}\,dx^h\wedge dx^k+\underset{2}{\alpha}_{kh}\,\delta y^h\wedge dx^k+\frac{1}{2}\underset{3}{\alpha}_{kh}\,\delta y^h\wedge\delta y^k \tag{4.6}$$

where:

$$\underset{1}{\alpha}_{kh}=\beta_{k|h}-\beta_{h|k}+T^s_{kh}\beta_s+R^i_{kh}\alpha_i \tag{4.7}$$

$$\underset{2}{\alpha}_{kh}=\beta_{k|h}-\alpha_h|_k+C^s_{kh}\beta_s+P^s_{kh}\alpha_s \tag{4.8}$$

$$\underset{3}{\alpha}_{kh}=\alpha_k|_h-\alpha_h|_k+S^s_{kh}\alpha_s \tag{4.9}$$

It follows:

$$\underset{1}{\alpha}_{kh}=\alpha_{kh},\ \ \underset{2}{\alpha}_{kh}=\beta_{kh}-N^s_k\gamma_{sh};\ \ \underset{3}{\alpha}_{kh}=\gamma_{kh} \tag{4.10}$$

Then $d\omega=0$ if and only if $\alpha_{kh}=0,\ \beta_{kh}=0,\ \gamma_{kh}=0.$
We have also the:

Theorem 4.2. *Let* $F\Gamma=(N,F,C)$ *be a semi-symmetric Finsler connection. The necessary and sufficient condition that* B_{kh} *and* $K_{kh}-K_{hk}$ *be invariants of the transformation* (2.2), *is given by:*

$$d\omega=\frac{1}{2}R^s_{kh}\,dx^h\wedge dx^k+\underset{2}{\alpha}_{kh}\,\delta y^h\wedge dx^k+\frac{1}{2}\underset{3}{\alpha}_{kh}\,\delta y^h\wedge\delta y^k \tag{4.11}$$

or by:

$$d\omega=\underset{2}{\alpha}_{kh}\,\delta y^h\wedge dx^k+\frac{1}{2}\underset{3}{\alpha}_{kh}\,\delta y^h\wedge\delta y^k \tag{4.12}$$

in case if N is integrable.

From Theorem 2.2 and 1.2 it follows:

Theorem 4.3. *The necessary and sufficient condition that a Finsler connection transformation* (2.2), *where* $F\Gamma=(N,F,C)$ *is a semi-symmetric Finsler connection, has the invariant* $\underset{1}{B}{}^i_{jk}$ *is given by* (4.11) *or by* (4.12) *if N is integrable.*

From (4.5), (2.1) and (2.5) it follows the:

Theorem 4.4. *If* $F\Gamma=(N,F,C)$ *is a semi-symmetric Finsler connection then the necessary and sufficient condition that* $\underset{2}{B}{}^i_{jk}$ *be an invariant of the transformation* (2.2) *is given by:*

$$d\omega=\frac{1}{2}\underset{1}{\alpha}_{kh}\,dx^h\wedge dx^k+\underset{2}{\alpha}_{kh}\,\delta y^h\wedge dx^k \tag{4.13}$$

or by

$$d\omega = \frac{1}{2} \underset{1}{\alpha}_{kh} dx^h \wedge dx^k + \beta_{kl} \delta y^l \wedge dx^k \tag{4.14}$$

if N is integrable.

From the two above Theorems it follows the:

Theorem 4.5. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, the necessary and sufficient condition that $\underset{1}{B}$ and $\underset{2}{B}$ are invariants by a transformation* (2.2) *is given by:*

$$d\omega = \frac{1}{2} R^s_{kh} dx^h \wedge dx^k + \beta_{kh} \delta y^h \wedge dx^k \tag{4.15}$$

or by:

$$d\omega = \beta_{kh} \delta y^h \wedge dx^k \tag{4.16}$$

if N is integrable.

From (3.3) and (4.4) it follows the relation:

$$\beta_{kh} - \beta_{hk} - (N^s_k \gamma_{sh} - N^s_h \gamma_{sk}) - \left(\frac{\partial N^i_k}{\partial y^h} - \frac{\partial N^i_h}{\partial y^k}\right) \alpha_i = \Theta_{kh} - \Theta_{hk} \tag{4.17}$$

Thus we have the:

Theorem 4.6. *If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, the necessary and sufficient condition that $\underset{1}{B}$, $\underset{2}{B}$ and $\underset{3}{B}$ are invariants by a transformation* (2.2), *is that $d\omega$ is of the form* (4.14) *and β_{kh} satisfies the relation:*

$$\beta_{kh} - \beta_{hk} = \left(\frac{\partial N^i_k}{\partial y^h} - \frac{\partial N^i_h}{\partial y^k}\right) \alpha_i \tag{4.18}$$

or that $d\omega$ be of the form (4.15), *and β_{kh} satisfies the realtion* (4.17), *if N integrable.*

If ω is closed ($d\omega=0$), then the Finsler covectors α_k and β_k have the form $\alpha_k=\varrho|_k$, $\beta_k=\varrho_{|k}$, where ϱ is a Finsler function, while ω is locally exact ($\omega=d\varrho$). In this case from the above theorems it follows:

Theorem 4.7. *If ω is a closed Finsler* 1*-form, then the necessary and sufficient condition that $\underset{1}{\bar{B}}=\underset{1}{B}$, $\underset{2}{\bar{B}}=\underset{2}{B}$, $\underset{3}{\bar{B}}=\underset{3}{B}$ is that $\alpha_k=\varrho|_k$ satisfies the conditions:*

$$R^s_{kh}\alpha_s = 0; \ \left(\frac{\partial N^i_k}{\partial y^h} - \frac{\partial N^i_h}{\partial y^k}\right) \alpha_i = 0.$$

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