Extensions of cuspidal characters of $GL_m(q)$

By WALTER, FEIT[1] (Nev Haven, Con.)

§ 1. Introduction

Let q be a power of an odd prime r. Let $m \geqq 3$ be an integer. If $x \in GL_m(q)$ let x' denote the transpose of x. If $q = q_0^2$ for an integer q_0 let * denote the automorphism of order 2 of $\mathbf{F}_q$. Define the following groups

$$H_m^1(q_0^2) = GL_m(q_0^2)\langle\tau\rangle \quad \text{where} \quad \tau^2 = 1,\ x^\tau = x^{*'-1} \quad \text{for} \quad x \in GL_m(q_0^2).$$

$$H_m^2(q) = GL_m(q)\langle\tau\rangle \quad \text{where} \quad \tau^2 = 1,\ x^\tau = x'^{-1} \quad \text{for} \quad x \in GL_m(q_0^2).$$

Let $i=1$ or 2. Let χ be a cuspidal character of $GL_m(q)$ such that $\chi(x^\tau) = \chi(x)$ for all $x \in GL_m(q)$. Then χ extends to an irreducible character $\tilde{\chi}$ of $H_m^i(q)$. It is known (see Section 7) that if such a χ exists then m is odd for $i=1$ and m is even for $i=2$. The main object of this paper is to evaluate $\tilde{\chi}$. The precise results are stated in Theorems 9B and 10J.

The same question can be asked for an arbitrary character and an arbitrary automorphism of $GL_m(q)$. If χ is the Steinberg character the questions can be answered in a much more general context, the details will appear elsewhere. However for other characters it seems to be quite difficult.

The question studied here is related to [9], [10]. Our results are much more special but also much more explicit. In case $i=1$ Theorem 9B verifies the suggestions in the last paragraph of [9] for the cuspidal character χ.

If q is even the question remains open, even for cuspidal characters. The arguments used here don't apply.

The methods used here depend on Brauer's work in the theory of modular representations of finite groups, especially the second main theorem on blocks. The relevant material is summarized in Section 4. The case $i=1$ (Section 9) is considerably simpler than the case $i=2$ (Section 10). One reason for this is that in case $i=1$ it is essentially sufficient to study 2-blocks of $GL_m(q)$ of defect 0, while in case $i=2$ it is necessary to consider 2-blocks of $GL_m(q)$ with a cyclic defect group. In [4] p-blocks for $p \nmid q$ of $GL_m(q)$ have been investigated. However the deeper results of that paper are not needed here as from this point of view the cuspidal characters are the simplest characters of $GL_m(q)$.

Theorem 5C is a strengthening of a classical result of Zsigmondy which may be of independent interest. It is very convenient for parts of the argument.

[1] This work was partly supported by NSF Grant MCS—8201333.

The question discussed above is natural in its own right but the work in this paper was motivated by some questions of A. Moy and R. Howe which arose in their study of Hecke algebras [7]. Some of these can be a answered as consequences of Theorems 9B and 10J. For instance the following is proved in Corollaries 9D and 10L.

Theorem 1A. *Let* $i=1$ *or* 2. *Let* χ *be a cuspidal character of* $GL_m(q)$ *with* $\chi(x)=\chi(x^{\tau})$ *for all* $x\in GL_m(q)$. *Let* A *be a complex representation which affords* χ.

(i) *Let* $i=1$. *Then* m *is odd,* $q=q_0^2$ *and*

$$\Big\{\sum_{x^{*\prime-1}=x} A(x)\Big\}^2 = q_0^{m(m-1)}(q_0^m-1)^2 I.$$

(ii) *Let* $i=2$. *Then* $m=2n$ *and*

$$\Big\{\sum_{x'=x} A(x)\Big\}^2 = q^{2n^2}(q^n-1)^2 I.$$

(iii) *Let* $i=2$. *Then* $m=2n$. *Define* $S=\{x+cv'v|x=-x',\ c\in F_q^{\times},\ v$ *is a column vector of size* $2n\}\subseteqq GL_{2n}(q)$. *Then*

$$\Big\{\sum_{x\in S} A(x)\Big\}^2 = q^{2(n^2-n)}(q^{2n}-1)^2 I.$$

All the work in this paper depends very essentially on [2], [6]. The relevant results are summarized in Section 6. However the proof of Theorem 1A (iii) goes even deeper and requires the evaluation of an appropriate Green function which is done in [11].

I wish to thank Alan Moy and Roger Howe for many interesting and illuminating discussions on the topics in this paper. I also wish to acknowledge a great debt to Paul Fong who answered many of my questions concerning irreducible characters of various classical groups. My first proof of Theorem 10J was considerably more complicated as I did not fully appreciate the power of the available explicit formulas such as (6.6) and (6.7). Consequently many of the original questions I asked are not relevant to the present proof. However Fong's responses provided an education on the subject without which is would have been much more difficult for me to prove Theorem 10J.

Circulation of a preprint of this paper elicited the information that N. Kawanaka had independently proved Theorem 9B in unpublished work by using different methods. His result also holds for even q. I was also informed that Theorem 10B could be proved by using the methods of [2] directly although this had apparently not been done previously.

Added Later. I have just realized that the proof of Theorem 10J is not complete in that it does not cover the group $GL_6(5)$. The problem occurs in Lemma 10H and is due to the fact that there is no large Zsigmondy prime for (5, 6). The proof can be completed (if the result is true) by an inspection of the character tables of $O_6^-(5)$ and $Sp_6(5)$ once these character tables become available. By inspection $U_3(5)$ has the required properties.

§ 2. Notation

In general the notation used in this paper is standard. Here we list a few items to avoid confusion.

If x is a matrix then x' denotes its transpose.

Let a be natural number and let P be a collection of primes.

P' denotes the set of all primes not in P.

a_P is the largest divisor of a which is a product of primes in P.

If p is a prime and $P=\{p\}$ then $a_p=a_P$, $a_{p'}=a_{P'}$.

a is a P-number if $a_P=a$. If $\{p\}=P$ then a P-number, P'-number is a p-number, p'-number respectively.

Let G be a finite group and let $x\in G$. Then $x=x_P x_{P'}=x_{P'}x_P$, where $|\langle x_P\rangle|$, $|\langle x_{P'}\rangle|$ is a *P-number, P'-number* respectively. Furthermore x_P, $x_{P'}$ are uniquely determined by x and are called the *P-part, P'-part* of x respectively. If $|\langle\lambda\rangle|$ is a P-number then x is a *P-element*.

As above we will identify p with $\{p\}$ and p' with $\{p\}'$. If x is not a p'-element then x is said to be *p-singular*.

If S is a nonempty subset of G then $N(S)=N_G(S)$ denotes the normalizer of S and $Z(S)=Z_G(S)$ denotes the centralizer of S. This latter is the notation used in the theory of algebraic groups rather than in the theory of finite groups.

If $S\subseteq G$ let $\hat{S}=\sum_{x\in S} x$ in the group algebra of G over any domain.

Let A be an absolutely irreducible representation of G over a field F of characteristic 0. Let χ be the character afforded by A. If C is a conjugacy class of G then $\sum_{x\in C} A(x)=A(\hat{C})=\omega(C)I$ is a scalar by Schur's lemma. By taking traces it follows that for $x\in C$

$$\omega(\hat{C})=\omega_\chi(\hat{C})=\frac{|C|\chi(x)}{\chi(1)}=\frac{|G|}{|Z(x)|}\frac{|\chi(x)}{\chi(1)}.$$

We will also write $\omega(x)=\omega(C)=\omega(\hat{C})$. If ω is extended to the center of $F[G]$ by linearity then it is easily seen to be a homomorphism of the center of $F[G]$ into F. It is called the central character of $F[G]$ corresponding to χ or to A.

§ 3. τ-conjugacy

If a, b are in the finite group H let $a^b=b^{-1}ab$.

Let $\tau\in H$. The elements x and y are *τ-conjugate* in H if $x=a^{-\tau}ya$ for some $a\in H$.

It is easily seen that *τ-conjugacy* is an equivalence relation. Clearly 1-conjugacy is the same as conjugacy.

A similar definition can be made if the inner automorphism corresponding to τ is replaced by an arbitrary automorphism. However we won't need this more general comcept.

Lemma 3A. *Let $x, y, a\in H$ and let $S\subseteq H$.*

(i) *$\tau x=a^{-1}\tau ya$ if and only if $x=a^{-\tau}ya$.*

(ii) a *commutes with τx if and only if $a^{-\tau}xa=x$.*

(iii) *S is a conjugacy class in H if and only if $\tau^{-1}S$ is a τ-conjugacy class in H.*

PROOF. (i) is clear. This implies (ii) and (iii). □

Lemma 3B. *Suppose that* $\tau^2=1$. *Let* $x, z\in H$. *Then*

$$(\tau x)^2 = z \quad \text{if and only if} \quad x^{-\tau} = xz^{-1}.$$

PROOF. $(\tau x)^2=x^\tau x$. The result follows. □

Lemma 3C. *Suppose that* $\tau^2=1$. *Let* A *be a complex irreducible representation of* H, *let* χ *be the character afforded by* A *and let* ω *be the central character corresponding to* χ. *Let* $C_1, \dots, C_k$ *be pairwise distinct conjugacy classes of* H *and let* $S=\bigcup_{i=1}^{k} C_i$ *Then*

$$A(\tau\hat{S})^2 = \{\sum_{x\in S} A(\tau x)\}^2 = \omega(\hat{S})^2 I = \{\sum_{i=1}^{k} \omega(C_i)\}^2 I.$$

PROOF.

$$A(\tau\hat{S})^2 = A((\tau\hat{S})^2) = A(\hat{S}^2) = A(\hat{S})^2 = \omega(\hat{S})^2 I. \quad \square$$

Lemma 3D. *Suppose that* $\tau^2=1$, $|H:G|=2$ *and* $H=G\langle\tau\rangle$. *Then* G *acts transitively by conjugation on any conjugacy class of* H *which lies in the coset* $G\tau$. *Furthermore if* $x, y\in G$ *then the following are equivalent.*

(i) τx *is conjugate to* τy *in* H.
(ii) $x=a^{-\tau}ya$ *for some* $a\in H$.
(iii) $x=a^{-\tau}ya$ *for some* $a\in G$.

PROOF. Let C be a conjugacy class of H with $C\subseteq G\tau$. Let $x\tau\in C$ with $x\in G$. Thus $x\tau\in Z_H(x\tau)$ and so $|Z_H(x\tau)\colon Z_G(x\tau)|=2$. Thus G acts transitively on C.

(i) is equivalent to (ii) by Lemma 3A. By the first part of the Lemma, (ii) is equivalent to (iii). □

Let $H_m^i(q)$ for $i=1, 2$ and τ be defined as in the introduction. In the rest of this section the results above will be applied to these two groups.

Let $x, y\in GL_m(q)$. Lemma 3D implies the following

(3.1) (i) τx is conjugate to τy in $H_m^1(q)$ if and only if $x=a^{*\prime}ya$ for some $a\in GL_m(q)$.

(3.1) (ii) τx is conjugate to τy in $H_m^2(q)$ if and only if $x=a'ya$ for some $a\in GL_m(q)$.

Let $\varepsilon=\pm 1\in GL_m(q)$. Lemma 3B implies the following

$$\text{(3.2)} \qquad \text{(i) } (\tau x)^2 = \varepsilon \quad \text{in} \quad H_m^1(q) \quad \text{if and only if} \quad x^{*\prime} = \varepsilon x.$$

$$\text{(3.2)} \qquad \text{(ii) } (\tau x)^2 = \varepsilon \quad \text{in} \quad H_m^2(q) \quad \text{if and only if} \quad x' = \varepsilon x.$$

Lemma 3E (i) *Define*

$$C^+ = \{\tau x | x\in GL_m(q), x^{*\prime} = x\} \subseteq H_m^1(q),$$
$$C^- = \{\tau x | x\in GL_m(q), x^{*\prime} = -x\} \subseteq H_m^1(q).$$

Then C^+ *and* C^- *are conjugacy classes of* $H_m^1(q)$. *Furthermore* C^+ *is the set of all involutions in the coset* $GL_m(q)\tau$, *and* C^- *is the set of all elements of order* 4 *in* $GL_m(q)\tau$.

(ii) *Define*

$C^{++} = \{\tau x | x\in GL_{2n}(q),\ x=x',\ x \text{ has maximum Witt index}\}\subseteq H_{2n}^2(q)$,
$C^{+}\ = \{\tau x | x\in GL_{2n}(q),\ x=x',\ x \text{ does not have maximum Witt index}\}\subseteq H_{2n}^2(q)$,
$C^{-}\ = \{\tau x | x\in GL_{2n}(q),\ x=-x'\}\subseteq H_{2n}^2(q)$.

Then C^{++}, C^{+}, C^{-} are conjugacy classes of $H_{2n}^2(q)$. Furthermore $C^{++}\cup C^{+}$ is the set of all involutions in the coset $GL_{2n}(q)\tau$, and C^{-} is the set of all elements of order 4 in $GL_{2n}(q)\tau$.

PROOF. There is one congruence class of hermitian and skew-hermitian matrices in $GL_m(q)$ for $q=q_0^2$. There are two congruence classes of symmetric matrices and one congruence class of skew matrices in $GL_{2n}(q)$. The results follow from (3.1) (i), (3.1) (ii), (3.2) (i), (3.2) (ii). □

Lemma 3F. *Define S by*
$S=\{x+cv'v|x=-x'\in GL_{2n}(q),\ c\in\mathbf{F}_q^\times,\ v \text{ is a column vector of size } 2n\}$.
Then $S\subseteq GL_{2n}(q)$ and τS is a conjugacy class of $H_{2n}^2(q)$. An element in τS is of the form $\tau xu=u\tau x$ where $x\in GL_{2n}(q)$, τx has order 4 and u is a unipotent element in $GL_{2n}(q)$ such that $u-1$ has a null space of dimension $2n-1$.

PROOF. Let $J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ and let J_k denote the $2k$ by $2k$ matrix which is the direct sum of k copies of J. Every element in S is congruent in $GL_{2n}(q)$ to $J_{2k}+cv'v$ for some v. The group $Sp_{2n}(q)$ acts transitively on the underlying vector space. Thus if $v_0=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$, there exists $y\in Sp_{2n}(q)$ with $y'v'vy=v_0'v_0$. As $y'J_{2n}y=J_{2n}$ it follows that every element of S is congruent to $x_0=\begin{pmatrix} c & 1 & \\ -1 & 0 & \\ & & J_{2n-2}\end{pmatrix}$. Thus $S\subseteq GL_{2n}(q)$. By Lemma 3.A (i) and (3.1) (ii) τS is a conjugacy class of $H_{2n}^2(q)$. Also $x_0'^{-1}=$ $=\begin{pmatrix} 0 & 1 & \\ -1 & c & \\ & & J_{2n-2}\end{pmatrix}$. Hence $(\tau x_0)^2=x_0^\tau x_0=\begin{pmatrix} -1 & & \\ -c & -1 & \\ & & -I_{2n-2}\end{pmatrix}$. The result follows. □

Lemma 3G. *Let $i=1$ or 2. Let A be a complex irreducible representation of $H_m^i(q)$, let χ be the character afforded by A and let ω be the corresponding central character.*

(i) *Let $i=1$. Then*

$$\Big\{\sum_{x^{*\prime}=-x\in GL_m(q)} A(x)\Big\}^2=\omega(C^-)^2 I.$$

(ii) *Let $i=2$ and $m=2n$. Then*

$$\Big\{\sum_{x'=x\in GL_m(q)} A(x)\Big\}^2=\{\omega(C^{++})+\omega(C^+)\}^2 I.$$

(iii) *Let $i=2$ and $m=2n$. Let S be defined as in Lemma 3F. Then*

$$\Big\{\sum_{x\in S} A(x)\Big\}^2=\omega(\tau S)^2 I.$$

PROOF. These formulas follow directly from Lemmas 3C, 3E and 3F. □

§ 4. Results from modular representation theory

We will assume familiarity with the basic concepts and results in the theory of modular representations. See [3] for a general reference. However as a matter of convenience we will here state some results from that theory which are pertinent to this paper. These results will be used as a tool for getting information about ordinary characters of finite groups. Almost all the results in this section are due to R. BRAUER. The only exceptions are the results about blocks with a cyclic defect group whic are due to E. C. DADE and J. G. THOMPSON and generalize earlier results of Brauer on blocks of defect 1.

The following notation will be used.

G is a finite group and p is a prime.

F is a finite extension of the p-adic numbers Q_p which is a splitting field for G and all its subgroups.

R is the ring of integers in F and π is a prime in R. If $\alpha \in R$ then $\bar{\alpha}$ denotes its image in $\bar{R} = R/\pi R$.

If χ is a character of G then as usual we may consider $\chi(x) \in R$ for $x \in G$ by choosing monomorphism from the algebraic numbers in F to the complex numbers.

If x is a p'-element in G then a S_p-groups P of $Z_G(x)$ is a *defect group of* x. If $|P| = p^{d(x)}$ then $d(x)$ is the *defect of* x.

Let $\{\chi_u\}$ be the set of all irreducible characters of G and let ω_u be the central character corresponding to χ_u.

For the next four results see [3] Chapter IV, Section 4.

(4A) *χ_u and χ_v are in the same p-block if and only if $\bar{\omega}_u = \bar{\omega}_v$.*

For the next three statements let B be a p-block of G with defect group D and defect d.

(4B) *If $\chi_u \in B$ then $\chi_u(x) = 0$ unless the p-part of x is conjugate to an element of D. If P is a p-groups with $P \lhd G$ then $P \subseteq D$.*

(4C) *If $\chi_u \in B$ and x is a p'-element with $d(x) < d$ then $\overline{\omega_u(x)} = 0$*

(4D) *Let $\chi_u \in B$. There exists a p'-element $x \in G$ with $d(x) = d$ and $\overline{\omega_u(x)} \neq 0$. In that case D is conjugate to a defect group of x.*

(4C) and (4D) could be used to define a defect group of a block. In fact this was essentially Brauers' original definition.

Let K be a subgroup of G. Let b be a p-block of K and let λ be the central character of $\bar{R}[K]$ corresponding to b. If C is a conjugate class of G define

$$\lambda^G(C) = \lambda(\widehat{H \cap C}).$$

Extend λ^G to $\bar{R}[G]$ by linearity. If λ^G is a central character of $\bar{R}[G]$ then it determines a p-block b^G of G. In this case we say that *b^G is defined.* The map from b to b^G is called the *Brauer correspondence.* Much of the theory of modular representations is concerned with studying this correspondence.

(4E) ([3] III. 9.2). *Let $K \subseteq A \subseteq G$ for groups K, A. Let b be a p-block of K. If b^A is defined and either one of b^G or $(b^A)^G$ is defined then so is the other and $b^G = (b^A)^G$.*

(4F) ([3] III. 9.4.). *Let P be a p-subgroup of G and let K be a subgroup with $PZ_G(P) \subseteq K \subseteq N_G(P)$. If b is a block of K then b^G is defined.*

(4G) ([3] III. 9.6). *Let K be a subgroup of G and let b be a p-block of K such that b^G is defined. Then a defect group of b is contained in a defect group of b^G.*

Suppose that $G \lhd \tilde{G}$. A p-block $\tilde{B}$ of $\tilde{G}$ covers the p-block B of G if there exist irreducible characters $\tilde{\chi} \in \tilde{B}$, $\chi \in B$ such that χ is a constituent of $\tilde{\chi}_G$.

(4H) ([3] V.3.7 and V.3.9). *Suppose that $G \lhd \tilde{G}$. Let $\tilde{B}$ be a p-block of $\tilde{G}$ with defect group D. Suppose that $Z_{\tilde{G}}(D) \subseteq G$. Let B be a p-block of G. Then $\tilde{B}$ covers B if and only if $B^{\tilde{G}} = \tilde{B}$.*

If y is a p-element in G let $\{\varphi_i^y\}$ be the set of all irreducible Brauer characters of $Z_G(y)$. Let (c_{ij}^y) be the Cartan matrix of $Z_G(y)$. For the next three results see [3] Chapter IV Section 6.

(4I) *Let y be a p-element in G. There exist algebraic integers d_{ui}^y such that if x is a p'-element in $Z_G(y)$ then for all u*

$$\chi_u(yx) = \sum_i d_{ui}^y \varphi_i^y(x).$$

The algebraic integers d_{ui}^y are the *higher decomposition numbers*. We will also write $d^y(\chi, \varphi^y) = d_{ui}^y$ if $\chi = \chi_u$, $\varphi^y = \varphi_i^y$.

(4J) *Let y, y' be p-elements in G. Let * denote complex conjugation. Then*

$$\sum_u d_{ui}^y (d_{uj}^{y'})^* = \begin{cases} 0 & \text{if } y \text{ is not conjugate } y', \\ c_{ij}^y & \text{if } y = y'. \end{cases}$$

(4K) (Second Main Theorem on Blocks) *Let y be a p-element in G. Suppose that $d_{ui}^y \neq 0$ for some u, i. Let b be the p-block of $Z_G(y)$ which contains φ_i^y. Then $\chi_u \in b^G$.*

If B is a p-block of G and y is a p-element let $l(y, B)$ denote the number of irreducible Brauer characters of $Z_G(y)$ which lie in blocks b with $b^G = B$. As an immediate consequence of (4B) and (4G) we get.

(4L) *If y is a p-element which is not conjugate to an element of a defect group of the p-block B then $l(y, B) = 0$.*

(4M) ([3] IV.6.5) *Let B be a p-block. Let $\{y_i\}$ be a complete set of representations of all conjugacy classes of G consisting of p-elements. Then $\sum_i l(y_i, B)$ is the number of irreducible characters in B.*

In case B has a cyclic defect group a great deal is known. See [3] Chapter VII. We will here only state one special result.

(4N) *Let B be a 2-block with a cyclic defect group of order 2^d. Then B contains exactly 2^d irreducible characters χ_u, $u = 1, \dots, 2^d$ and B contains a unique irreducible Brauer character φ. Furthermore if x is a $2'$-element in G then $\chi_u(x) = \varphi(x)$ for $u = 1, \dots, 2^d$.*

(4O) ([3] IV.4.16 (i)) *Let B be a p-block of defect d with contains a unique irreducible Brauer character. The corresponding Cartan invariant is p^d.*

We will also quote some technical results occasionally as in the proof of Lemm 6A.

§ 5. Zsigmondy primes

Let a, m be integers with $a>1$, $m>2$. There exists a prime l such that $l|a^m-1$ but $l\nmid a^i-1$ for $1\leqq i\leqq m-1$ unless $(a, m)=(2, 6)$. See [1] Corollary 2. At Paul Fong's suggestion we will call such a prime a *Zsigmondy prime for* (a, m). In this section a refinement of this result is proved.

If l is a Zsigmondy prime then $l\equiv 1 \pmod m$ and so $l\geqq m+1$. A prime l is a *large Zsigmondy prime* for (a, m) if l is a Zsigmondy prime for (a, m) and $(a^m-1)_l>m+1$.

Let $\Phi_m(x)$ denote the m^{th} cyclotomic polynomial. Let $P(m)$ denote the product of all the distinct primes dividing m.

Lemma 5A. *Suppose that* $m\geqq 3$ *is an integer such that* $m\neq 2n$ *with n odd. Let a be complex number with* $|a|\geqq 3$. *Then*

$$|\Phi_m(a)| > 4(m+1)\,P(m) \tag{5.1}$$

unless one of the following occurs.

$$m = 12,\ |a| < 5;\ m = 5,\ |a| < 4;\quad m = 4,\ |a| < 7;\quad m = 3,\ |a| < 8.$$

Proof. Let $m=\prod_{i=1}^{k} p_i^{b_i}$ for primes $p_1<\ldots<p_k$. The proof is by induction on k. Suppose that $k=1$. Then $m=p^b$. If (5.1) does not hold then

(5.2)
$$4(p^b+1)p \geqq \left|\frac{a^{p^b}-1}{a^{p^{b-1}}-1}\right| \geqq \frac{|a|^{p^b}-1}{|a|^{p^{b-1}}+1} \geqq \frac{1}{2}\,\frac{|a|^{p^b}-1}{|a|^{p^{b-1}}-1} > \frac{1}{2}|a|^{p^b-p^{b-1}} \geqq \frac{1}{2}\,3^{p^b-p^{b-1}}.$$

Assume first that $b=1$. Then (5.2) implies that $4p(p+1)\geqq\frac{1}{2}3^{p-1}$. Hence $p<7$. If $p=5$ then (5.2) implies that $120\geqq\frac{|a|^5-1}{|a|+1}$. Thus $|a|^5\leqq 120\,|a|+121$ and so $|a|<4$. If $p=3$ then (5.2) implies that $48\geqq\frac{|a|^3-1}{|a|+1}$. Thus $|a|^3\leqq 48|a|=49$. Hence $|a|<8$.

Suppose that $b>1$. Let $2x=p^b$. Then (5.2) implies that $8x^2>4(p^b+1)p>\frac{1}{2}3^x$. Hence $16x^2>3^x$ and so $x<5$. Thus $p^b<12$ and so $p^b=4$ or 9. If $p^b=9$ then (5.2) yields that $120\geqq\frac{1}{2}|a|^6$ and so $|a|<3$ contrary to assumption. If $p^b=4$ then $40\geqq\frac{|a|^4-1}{|a|^2+1}=|a|^2-1$. Thus $|a|<7$.

Let $m=p^b n$ for a prime p with $p\nmid n$, $b>0$, $n>1$. For a suitable p^b-th root ε of 1 we get

$$|\Phi_m(a)| \geqq |\Phi_n(\varepsilon a)|^{p^b-p^{b-1}}. \tag{5.3}$$

Suppose that $n=q^c>1$ for a prime q and (5.1) does not hold. Then (5.2) and (5.3) imply that

$$4(p^b q^c+1)\,pq \geqq \left(\frac{|a|^{p^b}-1}{|a|^{p^{b-1}}+1}\right)^{q^c-q^{c-1}}. \tag{5.4}$$

Assume that $p=2$, $b\geqq 2$. By (5.4)

$$8q(2^b q^c+1) \geqq (|a|^{2^{b-1}}-1)^{q^c-q^{c-1}} \geqq (3^{2^{b-1}}-1)^{q^c-q^{c-1}}. \tag{5.5}$$

If $q^c\geqq 5$ this yields that $40(5\cdot 2^b+1)\geqq(3^{2^{b-1}}-1)^4$ which is impossible for $b\geqq 2$. If $q^c\geqq 3$ then $24(3\cdot 2^b+1)\geqq(3^{2^{b-1}}-1)^2$ and so $b<3$. Thus $b=2$ and $m=12$. In this case (5.5) implies that $24\cdot 13\geqq(|a|^2-1)^2$ and so $|a|<5$.

Suppose that $m=15$. By (5.4) $960\geqq\left(\frac{|a|^5-1}{|a|+1}\right)^2\geqq\left(\frac{3^5-1}{4}\right)^2$, wihch is not the case.

Now proceed by induction on m. It may be assumed that (5.1) holds for n and n is not a power of 2. Thus $P(n)\geqq 3$. If (5.1) does not hold for m then by (5.3)

$$4(np^b+1)\,pP(n) \geqq |\Phi_m(a)| \geqq |\Phi_n(\varepsilon a)|^{p^b-p^{b-1}} > \{4(n+1)\}^{p^b-p^{b-1}}P(n)\cdot 3^{p^b-p^{b-1}-1}.$$

Hence

$$12(np^b+1)\,p > \{12(n+1)\}^{p^b-p^{b-1}}. \tag{5.6}$$

Let $2x=p^b$. Thus

$$24x(2xn+1) > \{12(n+1)\}^x.$$

This implies that $x<2$ and so $p^b<4$. Thus $p^b=3$. Hence (5.6) implies that $36(3n+1)>144(n+1)^2$ which is not the case. $\square$

Lemma 5B. *Suppose that* $m\geqq 3$ *and* $a\geqq 3$ *are integers. Then*

$$|\Phi_m(a) > (m+1)\,P(m)$$

unless one of the following cases occurs. $m=12$, $a<5$; $m=5$ *or* 10, $a<4$; $m=4$; $a<7$; $m=3$ *or* 6, $a<8$.

PROOF. If $m\neq 2n$ with n odd the result follows from Lemma 5A. If $m=2n$ with n odd then $\Phi_m(a)=\Phi_n(-a)$ and $P(m)=2P(n)$. Thus the result follows from Lemma 5A. $\square$

Theorem 5C. *Let* $a\geqq 3$, $m\geqq 3$ *be integers. Then there exists a large Zsigmondy prime for* (a, m) *unless* $(a, m)=(3, 4)$, $(3, 6)$ *or* $(5, 6)$.

PROOF. If there are at least two Zsigmondy primes $l>l_0$ for (a, m) then $l>m+1$. If $l^2|a^m-1$ for some Zsigmondy prime then $l^2>m+1$. Thus if there is no large Zsigmondy prime for (a, m) then $l=m+1$ is the unique Zsigmondy prime for (a, m) and $l^2\nmid a^m-1$. Thus m is even and $lP(m)=\Phi_m(a)$. See [1] Section 1. Hence the inequality in Lemma 5B does not hold and so $m=12$, $a<5$; $m=10$, $a<4$; $m=4$,

$a<7$; or $m=6$, $a<8$. In each of the following cases we list a large Zsigmondy prime.

$$m = 12 \quad \begin{array}{lll} a & 4, & 4 \\ l & 73, & 241. \end{array}$$

$$m = 10, \ a = 3, \ l = 61$$

$$m = 6 \quad \begin{array}{llll} a & 4 & 6 & 7 \\ l & 13 & 31 & 43 \end{array}$$

$$m = 4 \quad \begin{array}{lll} a & 5 & 6 \\ l & 13 & 37. \end{array}$$

This leaves the cases listed in the Theorem. □

§ 6. Cuspidal characters

Throughout this section q is a power of an odd prime r and G is a classical group over $\mathbf{F}_q$.

T is a Coxeter torus of G in the sense of [11]. Thus T and $N_G(T)/T$ are cyclic groups. Let $N=N_G(T)=\langle T, w\rangle$.

We will only be concerned with the four cases listed in table I.

TABLE I

	G	$\lvert T\rvert$
(A)	$GL_m(q)$, $m>2$	q^m-1
(B)	$U_m(q_0)$, $q=q_0^2$, m is odd, $m>2$	q_0^m+1
(C)	$Sp_{2n}(q)$, $m=2n>2$	q^n+1
(D)	$O_{2n}^-(q)$, $m=2n>2$	$q+1.$

In each of the cases (A)—(D) of table I, T is determined up to conjugacy by $|T|$ and the following hold.

(6.1) $$|N:T| = m.$$

(6.2) $$x^w = x^q \quad \text{for all} \quad x \in T.$$

If K is the group of $\mathbf{F}_q$-rational points of an lagebraic group, let K° denote the group of $\mathbf{F}_q$-rational points of the connected component of the identity.

Observe that $G=G^\circ$ in cases (A)—(C) of table I but in case (D) $G^\circ=SO_{2n}^-(q)$.

We will freely use basic facts about algebraic groups. For instance if H is a torus of G then $Z_G(H)$ and $N_G(H)$ are $\mathbf{F}^q$-rational points of an algebraic group.

Let l be a Zsigmondy prime for (q, m) is cases (A), (C), (D). Let l be a Zsigmondy prime for $(q_0, 2m)$ in case (B). Thus $l \mid |T|$ in cases (A)—(D) in table I. Every l-singular element in T is regular, i.e., if x is l-singular then $Z_G(x)=T$. Thus Sylow's theorem implies

(6.3) $$T=Z(T)=Z_G(T)=Z_{G^\circ}(T)=Z^\circ(T).$$

(6.4) If $T\subseteqq K\subseteqq G$ with K algebraic and $x\in K$, then there exists $x^\circ\in K^\circ$ with $T^x=T^{x^\circ}$. Hence also

(6.5) $$|N_K(T):N_{K^\circ}(T)|=|K:K^\circ|.$$

A character θ of T is *regular* if θ is irreducible and θ^N is an irreducible character of N. There is a bijection $\theta\leftrightarrow\chi_\theta$ between the set of all regular characters of T and the set of all cuspidal characters (corresponding to T) of G. See [2] for the following formulas.

Let $\chi=\chi_\theta$ for the regular character θ of T.

Let $su=us\in G$ with s semi-simple and u unipotent.

(6.6) If $s\notin T^g$ for any $g\in G$ then $\chi(su)=0$.

Suppose that $s\in T$. Let $Q_T^{Z^\circ(s)}$ be the Green function. Then

(6.7) $$\chi(su)=\frac{1}{|Z^\circ(s)|}\sum_{\substack{g\in G\\ T^g\subset Z^\circ(s)}}\theta^{g^{-1}}(s)Q_{T^g}^{Z^\circ(s)}(u)=Q_T^{Z^\circ(s)}(u)\frac{1}{|N\cap Z^\circ(s):T|}\theta^N(s).$$

It is known that

(6.8) $$Q_T^{Z^\circ(s)}(1)=|Z^\circ(s):T|_{r'}.$$

Thus by (6.7) and (6.8)

(6.9) $$\chi(s)=\frac{|Z^\circ(s)|_{r'}}{|N\cap Z^\circ(s)|}\theta^N(s).$$

Hence by (6.9)

(6.10) $$\chi(1)=\frac{|G^\circ|_{r'}}{|N\cap G^\circ|}\theta^N(1)=\frac{|G^\circ|_{r'}|N:N\cap G^\circ||N:T|}{|N|}=\frac{|G^\circ|_{r'}|N:N\cap G^\circ|}{|T|}.$$

In cases (A)—(D) of table I $|G:G^\circ|\leqq 2$. As r is odd (6.5) implies that

$$|G|_{r'}=|G^\circ|_{r'}|G:G^\circ|=|G^\circ|_{r'}|N:N\cap G^\circ|.$$

Therefore (6.10) becomes

(6.11) $$\chi(1)=\frac{|G|_{r'}}{|T|}.$$

Also (6.9) implies that

(6.12) If s is regular in T then $\chi(s)=\theta^N(s)$.

Now (6.5), (6.9) and (6.11) yield that if ω is the central character corresponding to χ then

(6.13) $$\omega(s)=\frac{|G|}{|Z(s)|}\frac{\chi(s)}{\chi(1)}=\frac{|G|_r}{|Z(s)|}\frac{|Z^\circ(s)|_{r'}}{|N\cap Z^\circ(s):T|}\theta^N(s)=$$

$$=\frac{|G|_r}{|Z(s)|_r}\frac{1}{|Z(s):Z^\circ(s)|}\frac{1}{|N\cap Z^\circ(s):T|}\theta^N(s)=\frac{|G|_r}{|Z(s)|_r}\frac{1}{|N\cap Z(s):T|}\theta^N(s).$$

Lemma 6A. *Let χ be an irreducible character of G in case* (A)—(D) *of table I which is not cuspidal with respect to T. Assume that* $(q, m) \neq (3, 4)$ *or* $(3, 6)$. *Then there exists a large Zsigmondy prime l for* (q, m). *Furthermore χ is constant on the set of l-elements in T.*

Proof. By Theorem 5C such an l exists. Let L be a S_l-group of T. Then $L \cap L^g = \langle 1 \rangle$ for $g \notin N$. Thus the number of irreducible characters of G which are not constant on $L - \{1\}$ (the exceptional characters for some l-block) is equal to the number of regular characters θ of T such that θ^N is not constant on $L - \{1\}$. See [3] Chapter VII. Thus the cuspidal characters include all these. □

Let $p=2$ and let F, R, π etc. be defined as in section 4.

Lemma 6B. *Suppose that G is a group in one of the cases* (B)—(D) *of table I. Assume that* $(q, m) \neq (3, 4), (3, 6)$ *or* $(5, 6)$. *There exists a large Zsigmondy prime l for* (q, m). *Let L be a S_l-group of T. If θ is a regular character of T then there exists a regular character θ_0 of T such that the following hold.*

(i) *$\overline{\theta_0^N}$ is not constant on $L - \{1\}$.*

(ii) *If $s \in T$ but s is not l-singular then $\theta^N(s) = \theta_0^N(s)$.*

Proof. The existence of l follows from Theorem 5C. If $\overline{\theta^N}$ is not constant $L - \{1\}$ let $\theta = \theta_0$. Suppose that $\overline{\theta^N}$ is constant on $L - \{1\}$.

Let $T = L \times M$ and let α be a character of T with M in its kernel such that α_L is faithful. Define $\theta_0 = \theta\alpha$. Then (ii) holds.

Let β be a character of T with M in its kernel such that β_L is faithful. Thus $(\beta^N)_L$ is multiplicity free. If $(\overline{\beta^N})_L$ is constant on $L - \{1\}$ then $\beta = a_0 1 + a_1(\varrho - 1)$, where ϱ is afforded by the regular representation of L as $|L|$ is odd. Since $|L| > m+1$, $a_1 = 0$ contrary to the fact that β_L is faithful. Therefore θ_L is not faithful and so $(\theta_0)_L$ is faithful. Hence $\overline{\theta_0^N}$ is not constant on $L - \{1\}$ and (i) holds. □

Lemma 6C. *Suppose that G is the group in one of the cases* (A)—(D) *of table I. Let $T = X \times Y$ where X is a $S_{2'}$-group of T and Y is a S_2-group of T. Let α be a regular character of T which has Y in its kernel. Let $\{\beta_i \mid 1 \leqq i \leqq |Y|\}$ be the set of all irreducible characters of $T/X \cong Y$. Let B_α be the 2-block which contains χ_α.*

(i) *Y is a defect group of B_α.*

(ii) *$\{\chi_{\beta_i\alpha} \mid 1 \leqq i \leqq |Y|\}$ is the set of all irreducible characters in B_α.*

(iii) *If $\theta \in B_\alpha$ and $s \in T$ then*

$$\omega_\theta(s) \equiv \frac{1}{|N \cap Z(s) : T|} \theta^N(s) \pmod{\pi}.$$

Proof. (iii) follows from (6.13).

As $\chi_{\beta_i\alpha}(x) = \chi_\alpha(x)$ for every element of odd order in G by (6.6) and (6.7) it follows that $\chi_{\beta_i\alpha} \in B_\alpha$ for all i. Hence by (4N), (ii) will follow once (i) is proved.

We next show that

(6.14) $$\overline{\alpha^N(s)} \neq 0 \quad \text{for some regular element } s \text{ in } T.$$

Let l be a Zsigmondy prime for (q, m) and let L be a S_l-subgroup of X. Suppose first that L is in the kernel of α.

If $\overline{\alpha^N(s)}=0$ for all $s\in X$ then every constituent of $(\alpha^N)_X$ occurs with even multiplicity as $|X|$ is odd. Since α is regular this is not the case and so there exists $s\in T$ with $\overline{\alpha^N(s)}\neq 0$. If s is regular (6.14) is proved. If s is not regular then s is an l'-element. Thus if $z\in L-\{1\}$ then zs is l-singular and hence regular. This implies (6.14) since $\alpha^N(sz)=\alpha^N(s)$.

Assume next L is not in the kernel of α. Choose $z\in L$ such that z is not in the kernel of α but z^l is in the kernel of α. Then $\alpha^N(z)$ is a Gaussian sum of l^{th} roots of 1. If Tr denotes the trace from $Q(\alpha^N(z))$ to Q, then $\mathrm{Tr}\,(\alpha^N(z)) = -1$. Hence there exists an i with $1\leqq i\leqq l-1$ such that $\overline{\alpha^N(z^i)}\neq 0$. As z^i is regular (6.14) is proved in all cases.

By (iii) and (6.14) $\overline{\omega_\alpha(x)}\neq 0$. Let $Y=\langle y\rangle$. By (6.12) and (6.14) $\chi_\alpha(ys)\neq 0$. Since Y is a S_2-group of $Z_G(s)$, (i) follows from (4B) and (4C). □

Corollary 6D. *Let G be a group in one of the cases* (A)—(D) *of table I. Let α, α' be regular characters of T of odd order with $\alpha^N\neq\alpha'^N$. Then there exists $s\in T$ with s of odd order such that $\overline{\omega_\alpha(s)}\neq\overline{\omega_{\alpha'}(s)}$.*

PROOF. Suppose the result is false. By Lemma 6C (iii) $\overline{\alpha^N(s)}=\overline{(\alpha')^N(s)}$ for all $s\in X$, where X is a $S_{2'}$ group of T. As X has odd order and $(\alpha^N)_X$, $(\alpha'^N)_X$ are multiplicity free it follows that $(\alpha^N)_X=(\alpha'^N)_X$ and so $\alpha^N=\alpha'^N$ as α, α' have odd order, contrary to assumption. □

§ 7. Cuspidal characters of $GL_m(q)$

The results in this section are known but are included for convenience. They were first brought to my attention by A. MOY.

It will be helpful to use the next result which was proved by the author and D. PASSMAN, [8] Proposition 4.3.

Lemma 7A. *Let $G=GL_m(q)$. Then it is possible to replace T by a conjugate such that every element of T is a symmetric matrix.*

Throughout the rest of this section $G=GL_m(q)$ with $m>2$, q odd and T is chosen so that it consists of symmetric matrices.

Lemma 7B. *Let $q=q_0^2$ and let * denote the automorphism of order 2 of $\mathbf{F}_q$. Let $x^\tau=x^{*'-1}$ for $x\in G=GL_m(q)$ with $m>2$. Let $\chi=\chi_\theta$ be a cuspidal character of G. Then $\chi=\chi^\tau$ if and only if m is odd and $\theta^{q_0^m+1}=1$.*

PROOF. Let $T=\langle\tau\rangle$. The characteristic values of t are $\{\alpha^{q^i}|0\leqq i\leqq m-1\}$, where α is some generator of the cyclic group $\mathbf{F}_{q^m}^\times$. Hence the characteristic values of $t^\tau=t^{*-1}$ are $\{\alpha^{-q_0q^i}|0\leqq i\leqq m-1\}$. Thus t^τ is conjugate to $t^{-q_0q^i}$ for some i with $0\leqq i\leqq m-1$. As t is conjugate to t^{q^i} in G for all i, there exists $x\in G$ such that $t^\tau=x^{-1}t^{-q_0}x$. Hence by (6.6) and (6.7) each of the following statements is equivalent to the next.

$$\chi = \chi^\tau.$$

$$\chi(t) = \chi(t^{-q_0})$$

$$\theta(t^{q^i}) = \theta(t^{-q_0}) \quad \text{for some } i \text{ with } 0 \leqq i \leqq m-1.$$

$$\theta^{q_0 q^i+1} = 1 \quad \text{for some } i \text{ with } 0 \leqq i \leqq m-1. \tag{7.1}$$

Suppose that $m=2i+1$ is odd and $\theta^{q_0^m+1}=1$. Then (7.1) holds and so $\chi=\chi^\tau$.

Suppose that $\chi=\chi^\tau$. Then (7.1) holds. Raising both sides of (7.1) to the $(q_0q^i-1)^{\text{th}}$ power yields that $\theta^{q^{2i+1}}=\theta=\theta^{q^m}$. As θ is regular and $0<2i+1<2m$ it follows that $m=2i+1$ is odd. Hence (7.1) becomes $\theta^{q_0^m+1}=1$. □

Lemma 7C. *Let $m>2$ and et $x^\tau=x'^{-1}$ for $x\in G=GL_m(q)$. Let $\chi=\chi_\theta$ be a cuspida character of G. Then $\chi=\chi^\tau$ if and only if $m=2n$ is even and $\theta^{q^n+1}=1$.*

Proof. Let $T=\langle\tau\rangle$. Thus $t^\tau=t^{-1}$. By (6.6) and (6.7) each of the following statements is equivalent to the next.

$$\chi = \chi^\tau.$$

$$\chi(t) = \chi(t^{-1}).$$

$$\theta(t^{q^i}) = \theta(t^{-1}) \quad \text{for some } i \text{ with } 0 \leqq i \leqq m-1.$$

$$\theta^{q^i+1} = 1 \quad \text{for some } i \text{ with } 0 \leqq i \leqq m-1. \tag{7.2}$$

Suppose that $m=2n$ and $\theta^{q^n+1}=1$. Then (7.2) holds and so $\chi=\chi^\tau$.

Suppose that $\chi=\chi^\tau$. Then (7.2) holds. Thus $i\neq 0$ and $\theta^{q^{2i}}=\theta=\theta^{q^m}$. As θ is regular and $0<2i<2m$ it follows that $2i=m$. Hence $m=2n$ is even and (7.2) becomes $\theta^{q^n+1}=1$. □

§ 8. A property of 2-groups

In this section we state a result about 2-groups without proof. See [5] Theorem 5.4.4.

Theorem 8A. *Let D be a 2-group of order 2^{k+1} which contains a cyc ic subgroup $\langle y\rangle$ of index 2. Then $y^{2^{k-1}}=-1\in Z(D)$. Assume that $k\geqq 3$. Then $D=\langle\sigma, y\rangle$ and σ can be chosen so that one of the following occurs.*

(i) *D is abelian.*

(ii) *$y^\sigma=y^{-1}$, D is a quaternion group and -1 is the only involution in D.*

(iii) *$y^\sigma=y^{-1}$, D is dihedral. Every element in $D-\langle y\rangle$ is an involution. $D-\langle y\rangle$ contains exactly 2 conjugate classes: σ and σy are representatives of these.*

(iv) *$y^\sigma=-y^{-1}$, D is quasi-dihedral. σ is an involution, σy has order 4 and every element in $D-\langle y\rangle$ is conjugate to one of these.*

(v) *$y^\sigma=-y$, D is semi-dihedral. Every element in $D-\langle y\rangle$ is conjugate to exactly one of σy^i with $1\leqq i\leqq 2^{k-1}$. Furthermore if $k=2^s a$ with a odd then σy^i has order 2^{k-s}.*

Lemma 8B. *Let $D=\langle\sigma, y\rangle$ where $\langle y\rangle$ is a 2-group and $\sigma^2\in\langle y\rangle$. Then $(\sigma y^k)^2\in Z(D)$ for all k.*

Proof. $(\sigma y^k)^2\in\langle y\rangle$. Thus $D=\langle y, \sigma y^k\rangle\subseteq Z_D((\sigma y^k)^2)$. □

9. The group $H^1_m(q_0^2)$

Let $q=q_0^2$ where q_0 is a power of the odd prime r. Let $m>2$. Let $x^\tau=x^{*'-1}$ for $x\in G=GL_m(q)$, where * denotes the automorphism of order 2 of $\mathbf{F}_q$. Thus $H=H^1_m(q)=G\langle\tau\rangle$. Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau=\chi_\theta$. By Lemma 7B m is odd and $\theta^{q_0^m+1}=1$. Then χ_θ extends in two ways to a character of H. Let $\tilde{\chi}_\theta$ denote one of these extensions. Then the other is $\varepsilon\tilde{\chi}_\theta$, where ε is the character of order 2 of H/G. By definition

$$Z_H(\tau)=\langle\tau\rangle\times Z_G(\tau)=\langle\tau\rangle\times U,$$

where $U=U_n(q_0)$. If K is a subgroup of G let $K_U=K\cap U$.

Let l be a Zsigmondy prime for $(q_0, 2m)$. Let L be a S_l-group of G. By Sylow's theorem L can be chosen so that $L^\tau=L$ and hence $Z_G(L)^\tau=Z_G(L)$. Let $T=Z_G(L)$. Then T is a torus of G with $|T|=q^{m-1}$. Hence $T_U=Z_G(L)\cap U$ is a torus of U with $|T_U|=q_0^m+1$.

Let Y be a S_2-group of T and let X be a $S_{2'}$-group of T. Then $T=Y\times X$. As m is odd $|Y|\,|\,q-1$. Hence Y consists of scalar matrices and so $Y\subseteq Z(G)$. Let $Y=\langle y\rangle$. Then $y^\tau=y^{-q_0}$. Hence $Y_U=\langle y_U\rangle$ with $y_U=y^{q_0-1}$ and $|Y_U|=q_0+1$. Thus the element w in N may be chosen of odd order. Then $N=Y\times\langle X, w\rangle$ and $x^w=x^q$ for $x\in X$. Furthermore θ is a regular character of T if and only if $\theta^N_{\langle X,w\rangle}$ is irreducible.

Lemma 9A. *Let θ be a regular character of T which has odd order such that $\theta^{q_0^m+1}=1$. Then θ restricted to T_U is a regular character of T_U of odd order. Conversely if θ is a regular character of T_U of odd order then θ extends uniquely to a regular character of odd order of T such that $\theta^{q_0^m+1}=1$.*

PROOF. Clear by the remarks above. □

If θ is a regular character of T_U let ζ_θ denote the corresponding cuspidal character of U.

The object of this section is to prove the next result.

Theorem 9B. *Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau=\chi_\theta$. Let $\tilde{\chi}_\theta$ denote an extension of χ_θ to H. Then $Z_G(\tau y^k)=U\langle\tau y^k\rangle$.*

If $\sigma\in H-G$ and the 2-part of σ is not conjugate to any τy^k then $\tilde{\chi}_\theta(\sigma)=0$.

Suppose that $\sigma\in H-G$ and the 2-part of σ is τy^k. Thus $\sigma=\tau y^k x$ with x of odd order in U. Then

$$\tilde{\chi}_\theta(\tau y^k x)=\pm\theta(y^k)\zeta_\theta(x).$$

The sign only depends on the choice of extension of χ_θ to H.

Before proving this we note some consequences.

Corollary 9C. *Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau=\chi_\theta$. Let $\tilde{\chi}_\theta$ denote an extension of χ_θ to H. Then*

$$\left|\tilde{\chi}_\theta\big(\tau(\pm 1)\big)\right|=\Big|\prod_{i=1}^{m-1}\big((-q_0)^i-1\big)\Big|.$$

If $\tilde{\omega}$ is the central character corresponding to $\tilde{\chi}$ then

$$\left|\tilde{\omega}_\theta(\tau(\pm 1))\right| = q_0^{m(m-1)/2}(q_0^m-1).$$

PROOF. By Theorem 9B $\left|\tilde{\chi}_\theta(\tau(\pm 1))\right| = \zeta_\theta(1)$. By (6.11) this has the required value. By Theorem 9B and (6.11)

$$\left|\tilde{\omega}_\theta(\tau(\pm 1))\right| = \frac{|H|}{2|U|}\frac{|T|}{|G|_{r'}}\frac{|U|_{r'}}{|T_U|}\frac{|G|_r}{|U|_r}(q_0^m-1).$$

As $|G|_r = q^{\frac{m(m-1)}{2}}$ and $|U|_r = q_0^{\frac{m(m-1)}{2}}$ the result follows.

Corollary 9D. *Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau = \chi_\theta$. Let A be a complex representation which affords $\tilde{\chi}_\theta$. Then*

$$\Big\{\sum_{x^{*\prime}=-x\in GL_m(q)} A(x)\Big\}^2 = q_0^{m(m-1)}(q_0^m-1)^2 I$$

where $q=q_0^2$.

PROOF. By Lemma 3E (i) every element of order 4 in $H-G$ is conjugate to $\tau(-1)$. Thus $\tau(-1)\in C^-$, where C^- is defined in Lemma 3E. The result now follows from Lemma 3G and Corollary 9C. $\square$

Theorem 9B will be proved in a series of Lemmas.

Define

$$G_0 = \{x|x\in G,\ \det\ x \text{ has odd order in } F_q^x\},\ U_0 = G_0\cap U.$$

Since m is odd.

$$G = Y\times G_0,\quad U = Y_U\times U_0 \tag{9.1}$$

Lemma 9E (i) *Let χ be an irreducible character of G. Then χ_{G_0} is of 2-defect* 0 *if and only if χ is cuspidal.*

(ii) *Let ζ be an irreducible character of U. Then ζ_{U_0} is of 2-defect* 0 *if and only if ζ is cuspidal with respect to T_U.*

PROOF. By (6.10) $\dfrac{|G|}{\chi(1)} = |T|a$, $\dfrac{|U|}{\zeta(1)} = |T_U|a'$ for a, a' odd. As $|T\colon Y|$ and $|T_U\colon Y_U|$ are odd it follows that χ_{G_0} and ζ_{G_0} are of 2-defect 0.

Let $\varphi(x) = \prod_1^m (x^i-1)$. Let χ be an irreducible character of G and let ζ be an irreducible character of U. There exists a polynomial $g(x)$ of degree m which is a product of terms of the form x^i-1, depending on χ, ζ respectively, such that

$$\chi(1)_{r'} = \varphi(q)/g(q),\ \zeta(1)_{r'} = |\varphi(-q)/g(-q)|.$$

See [6], [4] p. 115.

As m is odd $|(\delta q)-1|_2 = |\delta q-1|_2$ where $\delta=\pm 1$. Hence

$$|G_0|_2 = \left|\frac{\varphi(q)}{q-1}\right|_2,\quad |U_0|_2 = \left|\frac{\varphi(-q)}{q+1}\right|_2.$$

For $\delta=\pm 1$ $(\delta q-1)|(\delta q)^i-1$ for all i. Thus $|g(\delta q)|_2>|\delta q-1|_2$ unless there is only one term in $g(x)$, i.e. $g(x)=x^m-1$.

Thus if χ, ζ is of 2-defect 0 for G_0, U_0 respectively. Then $\chi(1)_{r'}=\dfrac{\varphi(q)}{q^m-1}$ and $\zeta(1)_{r'}=\left|\dfrac{\varphi(-q)}{q^m+1}\right|$. This implies that χ, ζ respectively is cuspidal. See [6], [4], p. 115. □

Corollary 9F (i) *Let B be a 2-block of G with defect group contained in Y. Then B contains a cuspidal character.*

(ii) *Let b be a 2-block of U with defect group contained in Y_U. Then b contains a cuspidal character for T_U.*

PROOF. As $Y\lhd G$ and $Y_U\lhd G$ this follows from (4B) and Lemma 9E. □

Let $\{\beta_i|1\leqq i\leqq |Y|\}$ be the set of all irreducible characters of $T/X\cong Y$. Then $\beta_i^\tau=\beta_i$ if and only if $\beta_i^{q_0+1}=1$. An irreducible character θ of T is of the form $\beta_i\alpha$ for some i, where α is an irreducible character of $T/Y\cong X$. Since m is odd θ is regular if and only if α is regular.

Suppose that α is regular with $\alpha^\tau=\alpha$. Let B_α denote the 2-block of G with $\chi_\alpha\in B_\alpha$. Then Lemma 6C applies to B_α. Let $\tilde{B}_\alpha$ be the 2-block of H which contains an extension of χ_α to H.

Let $p=2$ and let F, R, π etc. be defined as in Section 4.

Lemma 9G. *Let α be a regular character of T with $\alpha^{q_0^m+1}=1$ and α of odd order.*

(i) *$\langle Y, \tau\rangle$ is a defect group of $\tilde{B}_\alpha$.*

(ii) *Let ε be the character of order 2 of H/G. If χ_θ extends to H let $\tilde{\chi}_\theta$ denote one of the extensions. Then $\beta_i\chi_\alpha=\chi_{\beta_i\alpha}$ on T for all i and*

$$\{\tilde{\chi}_{\beta_i\alpha}|\beta_i^{q_0+1}=1\}\cup\{\varepsilon\tilde{\chi}_{\beta_i\alpha}|\beta_i^{q_0+1}=1\}\cup\{\chi^H_{\beta_i\alpha}|\beta_i^{q_0+1}\neq 1\}$$

is the set of all irreducible characters in $\tilde{B}_\alpha$.

(iii) *Suppose $s\in X$ with $s^{q_0^m+1}=1$. Let $\theta=\beta_i\alpha$, $\chi=\chi_\theta$. If $\beta_i^{q_0+1}\neq 1$ then $\overline{\omega_{\chi^H}(s)}=0$. If $\beta_i^{q_0+1}=1$ and $\tilde{\omega}_\theta$ is the central character corresponding to $\tilde{\chi}_\theta$ then*

$$\tilde{\omega}_\theta(s)\equiv\frac{1}{|N\cup Z_G(s):T|}\,\theta^N(s)\equiv\frac{1}{|N\cap Z_G(s):T|}\,\alpha^N(s)\,(\mathrm{mod}\;\pi)$$

PROOF. (ii) and (iii) are direct consequences of Lemma 6C.

(i) Let $\langle Y, \tau x\rangle$ be a defect group of $\tilde{B}_\alpha$. Then $y^{\tau x}=y^{-q_0}\neq y^{-1}$. Hence $\langle Y, \tau x\rangle$ is not a quaternion group and so the coset $Y_{\tau x}$ contains an involution τx_0. By Lemma 3E τx_0 is conjugate to τ in H. □

If α is a regular character of odd order of T and $\alpha^{q_0^m+1}=1$ then by Lemma 9A α may be identified with a regular character of T_U, also denoted by α.

Let $\tau y^k\in\langle Y, \tau\rangle$. Then $(\tau y^k)^2\in Y_U$ by Lemma 8B. Hence

$$Z_H(\tau y^k)=U\langle\tau y^k\rangle\quad\text{and}\quad(\tau y^k)^2\in Y_U\subseteqq U. \tag{9.2}$$

Thus every irreducible character of U extends to one of $Z_H(\tau y^k)$.

Let b_α be the 2-block of U which contains ζ_α. Let $\tilde{b}_\alpha = \tilde{b}_{\alpha,k}$ denote the extension of b_α to $Z_H(\tau y^k)$.

Lemma 9H. *Let α be a regular character of T of odd order with $\alpha^{q_0^m+1}=1$. Choose k.*

(i) *$\langle Y_U, \tau y^k\rangle$ is a defect group of $\tilde{b}_\alpha$.*

(ii) *If ζ is a character of U let $\tilde{\zeta}$ denote an extension of ζ to $Z_H(\tau y^k)$. Let ε denote the character of order 2 of $Z_H(\tau y^k)/U$. Let $\{\gamma_i | 1 \leq i \leq |Y_U|\}$ be the set of all irreducible characters of $T_U/X_U \cong Y_U$. Then $\gamma_i \zeta_\alpha = \zeta_{\gamma_i \alpha}$ on T_U for all i and*

$$\{\tilde{\zeta}_{\gamma_i\alpha}\} \cup \{\varepsilon \tilde{\zeta}_{\gamma_i \alpha}\}$$

is the set of all irreducible characters in $\tilde{b}_\alpha$.

(iii) *Let $s \in X_U$. If $\theta = \gamma_i \alpha$ and $\tilde{\omega}_\theta$ is the central character corresponding to $\tilde{\zeta}_\theta$ then*

$$\tilde{\omega}_\theta(s) \equiv \frac{1}{|N \cap Z_U(s) : T_U|} \theta^{N \cap U}(s) \equiv \frac{1}{|N \cup Z_U(s) : T_U|} \alpha^N(s) \equiv$$

$$\equiv \frac{1}{|N \cup Z(s) : T|} \alpha^N(s) \pmod{\pi}.$$

Proof. Since $|N \cap Z(s) \colon T| = |N \cap Z_U(s) \colon T_U|$ the result follows directly from Lemma 6C. □

Lemma 9I. *Let α be a regular character of T which has Y in its kernel and satisfies $\alpha^{q_0^m+1}=1$. Fix k and let $\tilde{b}$ be a 2-block of $Z_H(\tau y^k)$. Then $\tilde{b}^H = \tilde{B}_\alpha$ if and only if $\tilde{b} = \tilde{b}_\alpha$.*

Proof. If $\tilde{b}^H = \tilde{B}_\alpha$ then the defect group of $\tilde{b}$ is contained in $\langle Y, \tau\rangle$ by (4G). Thus by Corollary 9F $\tilde{b} = \tilde{b}_{\alpha'}$ for some regular character α' of T_U of odd order. By Corollary 6D $(\alpha')^N = \alpha^N$ and so $\tilde{b}_{\alpha'} = \tilde{b}_\alpha$. It remains to show that $\tilde{B}_\alpha = \tilde{b}^H$ for some block $\tilde{b}$ of $Z_H(\tau y^k)$.

χ_α is in a block of H/Y with defect group $\langle \tau, Y\rangle/Y$. Thus the second main theorem on blocks (4K) implies that $\tilde{B}_\alpha = \tilde{b}^H$ for some block $\tilde{b}$ of $Z_H(\tau y^k)$. □

Proof of theorem 9B. Let $\tilde{\chi}_\theta \in \tilde{B}_\alpha$. By (4N) b_α, and hence $\tilde{b}_\alpha$, has a unique irreducible Brauer character, call it $\varphi_\alpha = \tilde{\varphi}_\alpha$. Furthermore if x is an element of odd order in U then $\zeta(x) = \varphi_\alpha(x)$ for any $\zeta \in \tilde{b}_\alpha$.

Y is in the kernel of $\tilde{\chi}_\alpha$. Thus $\tilde{\chi}_\alpha$ lies in a block of H/Y of defect 1. As $(\tau y^k)^2 \in Y$, $\tilde{\chi}_\alpha(\tau y^k)$ is rational. Hence the corresponding higher decomposition number is ± 1 by (4J). Now Lemma 9I and (4K) imply

$$\tilde{\chi}_\alpha(\tau y^k x) = \pm \zeta_\alpha(x) \tag{9.3}$$

for x of odd order in U.

Let $\theta = \beta\alpha$ for $\beta = \beta_i$ for some i with $\beta^{q_0+1} = 1$. Then

$$\tilde{\chi}_\theta(\tau y^k x) = \tilde{\chi}_\theta(\tau x)\, \theta(y^k) = \tilde{\chi}_\alpha(\tau x)\, \theta(y^k) = \tilde{\chi}_\alpha(\tau y^k x)\, \theta(y^k)$$

and $\zeta_\theta(x) = \zeta_\alpha(x)$. Substituting these equations in (9.3) completes the proof. □

§ 10. The group $H_m^2(q)$

Let $m>2$. Let $x^\tau=x'^{-1}$ for $x\in G=GL_m(q)$ and q a power of the odd prime r. Thus $H=H_m^2(q)=G\langle\tau\rangle$. Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau=\chi_\theta$. By Lemma 7C $m=2n$ is even and $\theta^{q^n+1}=1$. Then χ_θ extends in two ways to a character of H. Let $\tilde{\chi}_\theta$ denote one of these extensions. Then the other is $\varepsilon\tilde{\chi}_\theta$, where ε is the character of order 2 of H/G.

Let X be a $S_{2'}$-group of T and let Y be a S_2-group of T. Then $T=Y\times X$. Let $Y=\langle y\rangle$, let $T=\langle t\rangle$. It may be assumed that $t'=t$. Thus $t^\tau=t^{-q^n}$ and so $y^\tau=y^{-q^n}$. Let $N=\langle T, w\rangle$ where $t^w=t^q$. Let $t^{w_0}=t^{q^n}$. It may be assumed that w_0 is a 2-element. Hence $\langle y, \tau, w_0\rangle$ is a 2-group.

Let $\{\beta_i|1\leqq i\leqq |Y|\}$ be the set of all irreducible characters of $T/X\cong Y$.

Lemma 10A. *Let $\theta=\beta\alpha$ where $\beta=\beta_i$ for some i and α is an irreducible character of $T/Y\cong X$. Suppose that $\theta^\tau=\theta$. Then θ is regular if and only if α is regular.*

Proof. If α is regular then so is θ. Suppose that θ is regular but α is not. Aut (Y) is a 2-group. Thus $\alpha^{q^n}=\alpha^{w_0}=\alpha$. Hence $\alpha^{q^n-1}=1=\alpha^{q^n+1}$. Thus $\alpha^2=1$ and so $\alpha=1$. Therefore β is regular. Hence n is a power of 2 and $n>1$. Thus $q^n+1\equiv 2 \pmod 4$. Therefore $\beta^2=1$ and so cannot be regular. □

Let θ be a regular character of T with $\theta^{q^n+1}=1$. By Lemma 10A $\theta=\beta_i\alpha$ for some i, where α is a regular character of T of odd order. Let B_α be the 2-block of G which contains χ_θ. By Lemma 6C Y is a defect group of B_α. Let $\tilde{B}_\alpha$ denote the extension of B_α to H.

Lemma 10B. *Let $m=2^kc$ with c odd. Let $q^2-1=2^{j+2}c'$ with c' odd.*

(i) *If $0\leqq i<k$ then $Z_G(y^{2^i})\approx GL_{2^ic}(q^{2^{k-i}})$.*

(ii) *If $q\equiv 1 \pmod 4$ and $k\leqq i$ then $Z_G(y^{2^i})=G$.*

(iii) *If $q\equiv -1 \pmod 4$ and $k\leqq i\leqq k+j$ then $Z_G(y^{2^i})\cong GL_{2^{k-1}c}(q^2)$ and $Z_G(\pm 1)=G$.*

Proof. Clear. □

Lemma 10C. *Let α be a regular character of T of odd order. Then $\langle\tau w_0, y\rangle$ is a defect group of $\tilde{B}_\alpha$.*

Proof. Let D be a defect group of $\tilde{B}_\alpha$ with $Y\subseteq D$. Then $|D:Y|=2$ and so $Y\lhd D$.

By Lemma 10B $Z=Z_H(y)\cong GL_c(q^{2^k})$. Let $*$ denote the automorphism of $\mathbf{F}_{q^m}$ of order 2. Then $t^{\tau w_0}=t^{*-1}$. Thus there exists $a\in Z$ such that $x^{\tau w_0}=a^{-1}x^{*'-1}a$ for all $x\in Z$. Hence $a\in Z(t)=T$. Furthermore it may be assumed a is a 2-element and so $a\in Y$. Hence $\langle\tau w_0 a, y\rangle=\langle\tau w_0, y\rangle$. Thus if τw_0 is replaced by $\tau w_0 a$ it may be assumed that $x^{\tau w_0}=x^{*'-1}$ for all $x\in Z$.

By (6.9) $\chi_\alpha(y)\neq 0$. Thus (4K) implies the existence of a 2-block b of Z with $b^H=\tilde{B}_\alpha$. By (4B) and (4G), Y is a defect group of b. Hence by Corollary 9F $b=b_{\alpha'}$ for some regular character α' of T with $(\alpha')^{q^{\frac{2^{k-1}}{c}}+1}=(\alpha')^{q^n+1}=1$. Hence b extends to a block $\tilde{b}$ of $Z\langle\tau w_0\rangle$. If $c=1$ then $Z=T$ and $\langle y, \tau w_0\rangle$ is a S_2-group of $Z\langle\tau w_0\rangle$. Hence it is a defect group of $\tilde{b}$. If $c\geqq 3$ then $\langle y, \tau w_0\rangle$ is a defect group of $\tilde{b}$ by Lemma 9G.

By (4H) $\tilde{b}=b^{Z\langle\tau w_0\rangle}$. Hence (4E) implies that $\tilde{B}_\alpha=b^H=\tilde{b}^H$. Thus by (4G) $\langle y, \tau w_0\rangle$ is in a defect group of $\tilde{B}_\alpha$ and so is equal to a defect group of $\tilde{B}_\alpha$.

Lemma 10D. *Let* $D=\langle y, \tau w_0\rangle$ *and let* $q^n\equiv\delta \pmod 4$ *with* $\delta=\pm 1$. *Then*

$$y^{\tau w_0} = -y^{-\delta}. \tag{10.1}$$

Thus D is quasi-dihedral if $\delta=1$ *and D is semi-dihedral if* $\delta=-1$. *In any case* $D-Y$ *contains involutions and elements of order* 4. *Let* σ *be an element in* $D-Y$.

(i) *If* σ *is an involution then* $\sigma=\tau x$ *with* $x=x'$ *not of maximum Witt index and* $Z_H(\sigma)\cong O^-_{2n}(q)\langle\sigma\rangle$.

(ii) *If* σ *has order* 4 *then* $Z_H(\sigma)\cong Sp_{2n}(q)\langle\sigma\rangle$.

(iii) *If* σ *has order greater than* 4 *then* $\delta=-1$, *n is odd and* $Z_H(\sigma)\cong U_n(q)\langle\sigma\rangle$.

PROOF. By definition

$$y^{\tau w_0} = y^{-q^n} = y^{\delta-q^n}y^{-\delta}.$$

As $q^{2n}-1=(q^n-\delta)(q^n+\delta)$ and $q^n+\delta\equiv 2 \pmod 4$ it follows that $y^{\delta-q^n}=-1$. This proves (10.1). The next statement follows directly from Theorem 8A.

(i) By Lemma 3E (ii) $\sigma=\tau x$ with $x'=x$ and $\sigma\in C^+$ or C^{++}. By (3.1) (ii) $Z_G(\sigma)=\{a|a'xa=x\}$. Thus $Z_H(\sigma)\cong O_{2n}(q)\langle\sigma\rangle$, where $O_{2n}(q)$ is one of the orthogonal groups. It remains to determine which one. [1]

$t^\sigma=t^{-q^n}$ and so $(t^j)^\sigma=t^j$ if and only if $(t^j)^{q^n+1}=1$. Thus $|Z_T(\sigma)|=q^n+1$. Let l be a Zsigmondy prime for $(q, 2n)$. Then $l\nmid|O^+_{2n}(q)|$. Hence $Z_H(\sigma)\not\approx O^+_{2n}(q)\langle\sigma\rangle$ and so $Z_H(\sigma)\cong O^-_{2n}(q)\langle\sigma\rangle$.

(ii) This follows directly from (3.1) (ii) and Lemma 3E (ii).

(iii) By Theorem 8.1 D is semi-dihedral and $\delta=-1$. Thus n is odd. By (10.1) $(\tau w_0 y^i)^2=(-1)^i y^{2i}$. Thus if $\sigma=\tau w_0 y^i$ then $y^{2i}\neq\pm 1$ and

$$Z_H(\sigma) \subseteq Z_H(y^{2i}) = GL_n(q^2)\langle\sigma\rangle.$$

σ acts as an outer automorphism on $GL_n(q^2)$ and $t^\sigma=t^{*'-1}$ where $*$ denotes the automorphism of order 2 of $\mathbf{F}_{q^2}$. Since $\dfrac{q^{2n}+1}{q^2+1}$ is odd, $|Y|\big|q^2+1$ and so $Y\subseteq Z\big(GL_n(q^2)\big)$. Thus $\sigma=\sigma_0 y^i$ for an involution σ_0 and some i. By Lemma 3E (i) σ_0 may be replaced by a conjugate such that $x^{\sigma_0}=x^{*'-1}$ for $x\in GL_n(q^2)$. Hence $x^\sigma=x^{\sigma_0}=x^{*'}-1$ for $x\in GL_n(q^2)$. The result follows from Lemma 3A (i). □

Let $\sigma\in D-Y$ where $D=\langle Y, \tau w_0\rangle$. Let $Z=Z_G(\sigma)$ and let $\tilde{Z}=Z_H(\sigma)$. Then Z and $\tilde{Z}$ are described in Lemma 10D. Furthermore $|\tilde{Z}\cdot Z|=2$ and $\tilde{Z}=\tilde{Z}\langle\sigma\rangle$. The notation may be chosen so that $T_Z=T\cap Z$ is a maximal torus of Z such that Z, T_Z is one of the cases (B) (C) (D) of table I.

If α is a regular character of T of odd order with $\alpha^{q^n+1}=1$ then α defines a regular character of odd order of T_Z and this yields a bijection between the two sets of regular characters.

Let $\theta=\beta_i\alpha$ for some i. Let ζ_θ be the cuspidal character of Z corresponding to θ. Let $\tilde{\zeta}_\theta$ be an extension of ζ_θ to an irreducible character of $\tilde{Z}$. Let b_α be the 2-block of Z which contains ζ_α and let $\tilde{b}_\alpha$ be the 2-block of $\tilde{Z}$ which contains $\tilde{\zeta}_\alpha$.

1). I am indebted to **Roger Howe** for the following argument which is much simpler than the original proof.

Let $p=2$ and let F, R, π etc. be defined as in Section 4.

Lemma 10E. *Let α be a regular character of T of odd order with $\alpha^{q^n+1}=1$.*

(i) *$D=\langle Y, \tau w_0\rangle$ is a defect group of $\tilde{B}_\alpha$.*

(ii) *Let ε be the character of order 2 of H/G. If χ_θ extends to H let $\tilde{\chi}_\theta$ denote one of the extensions. Then*

$$\{\tilde{\chi}_{\beta_i\alpha} \mid \beta_i^{q^n+1}=1\} \cup \{\varepsilon\tilde{\chi}_{\beta_i\alpha} \mid \beta_i^{q^n+1}=1\} \cup \{\chi^H_{\beta_i\alpha} \mid \beta_i^{q^n+1}\neq 1\}$$

is the set of all irreducible characters in $\tilde{B}_\alpha$. Thus the number of irreducible characters in $\tilde{B}_\alpha$ equals the number of irreducible characters of D. There is a unique irreducible Brauer character φ_α in $\tilde{B}_\alpha$ and $\tilde{\chi}_\alpha(x)=\varphi_\alpha(x)$ for x of odd order in H.

(iii) *Suppose s is an element of odd order in T with $s^{q^n+1}=1$. Let $\theta=\beta_i\alpha$, $\chi=\chi_\theta$. If $\beta_i^{q^n+1}\neq 1$ then $\overline{\omega_{\chi^H}(s)}=0$.* If $\beta_i^{q^n+1}=1$ *and $\tilde{\omega}_\theta$ is the central character corresponding to $\tilde{\chi}_\theta$ then*

$$\tilde{\omega}_\theta(s) \equiv \frac{1}{|N\cap Z_G(s):T|}\,\theta^N(s) \equiv \frac{1}{|N\cap Z_G(s):T|}\,\alpha^N(s) \pmod{\pi}.$$

Proof. This is a direct consequence of Lemma 6C, (6.13) and Lemma 10C. □

Lemma 10F. *Let α be a regular character of T of odd order with $\alpha^{q^n+1}=1$. Let $\sigma\in D-Y$. Let $Z=Z_G(\sigma)$ and $\tilde{Z}=Z_G(\sigma)\langle\sigma\rangle$.*

(i) *$D\cap\tilde{Z}=\langle Y\cap Z, \sigma\rangle$ is a defect group of $\tilde{b}_\alpha$.*

(ii) *Let ε be the character of order 2 of $\tilde{Z}/Z$. Let $\{\gamma_i \mid 1\leq i\leq |Y\cap Z|\}$ be the set of all irreducible characters of $T\cap Z/X\cap Z\cong Y\cap Z$. Then $\{\tilde{\zeta}_{\gamma_i\alpha}\}\cup\{\varepsilon\tilde{\zeta}_{\gamma_i\alpha}\}$ is the set of all irreducible characters in $\tilde{b}_\alpha$. There is a unique irreducible Brauer character $\varphi_\alpha^\sigma\in\tilde{b}_\alpha$ and $\tilde{\zeta}_\alpha(x)=\varphi_\alpha^\sigma(x)$ for every element x of odd order in $\tilde{Z}$.*

(iii) *Let s be an element of odd order in $T\cap Z$. If $\theta=\gamma_i\alpha$ and $\tilde{\omega}_\theta$ is the central character corresponding to $\tilde{\zeta}_\theta$ then*

$$\tilde{\omega}_\theta(s) \equiv \frac{1}{|N\cap Z_G(s)\cap Z:T\cap Z|}\,\theta^{N\cap Z}(s) \equiv \frac{1}{|N\cap Z_G(s)\cap Z:T\cap Z|}\,\alpha^N(s) \equiv$$

$$\equiv \frac{1}{|N\cap Z_G(s):T|}\,\alpha^N(s) \pmod{\pi}.$$

Proof. Since $|N\cap U(s)\cap Z\colon T\cap Z|=|N\cap Z(s)\colon T|$ the result follows directly from Lemma 6C and (6.13). □

Lemma 10G. *Let $\sigma\in D-Y$. Let θ be a regular character of T with $\theta^{q^n+1}=1$. Then there exists an element x of odd order in $Z_H(\sigma)$ such that $\overline{\tilde{\chi}_\theta(\sigma x)}\neq 0$ and $\overline{\tilde{\omega}_\theta(x)}\neq 0$.*

Proof. By Lemma 10A $\theta=\beta_i\alpha$ for some i and some regular character α of odd order of T with $\alpha^{q^n+1}=1$. By Lemma 10E D is a defect group of $\tilde{B}_\alpha$. Thus by (4D) there exists an element x of odd order such that $\overline{\tilde{\omega}_\theta(x)}\neq 0$ and D is a S_2-group of $Z_H(x)$. Hence $|H|_2=|Z_H(x)|_2\chi_\theta(1)_2$ and so $\overline{\tilde{\chi}_\theta(x)}\neq 0$. Therefore

$$\tilde{\chi}_\theta(\sigma x) \equiv \tilde{\chi}_\theta(x) \not\equiv 0 \pmod{\pi}. \quad \square$$

Lemma 10H. *Let $\sigma \in D-Y$. Let α be a regular character of T of odd order with $\alpha^{q^n+1}=1$. Let $\tilde{b}$ be a 2-block of $\tilde{Z}=Z_H(\sigma)$. Then $\tilde{b}^H=\tilde{B}_\alpha$ if and only if $\tilde{b}=\tilde{b}_\alpha$.*

PROOF. Suppose that $(q, m)\neq(3, 4)$ or $(3, 6)$. Let l, L be defined as in Lemma 6B. Then that Lemma implies the existence of a regular character α' of T of odd order with $(\alpha')^{q^n+1}=1$ such that $\overline{(\alpha')^N}$ is not constant on $L-\{1\}$ and $(\alpha')^N(s)=\alpha^N(s)$ for $s\in T$ but s not l-singular.

Let $\tilde{\omega}_\alpha, \tilde{\omega}_{Z,\alpha}$ be the central character corresponding to $\tilde{\chi}_\alpha, \tilde{\zeta}_\alpha$ respectively. By (6.13) $\overline{\tilde{\omega}_{\alpha'}}$ and $\overline{\tilde{\omega}_{Z,\alpha'}}$ are not constant on $L-\{1\}$. Thus by Lemma 6A $\tilde{b}_{\alpha'}{}^H$ contains a cuspidal character. By Corollary 6D $\tilde{b}_{\alpha'}^H=\tilde{B}_{\alpha'}$.

If x is not a regular element of odd order in T then

$$\overline{\tilde{\omega}_\alpha(x)} = \overline{\tilde{\omega}_{\alpha'}(x)} = \overline{\tilde{\omega}_{Z,\alpha'}(x)} = \overline{\tilde{\omega}_{Z,\alpha}(x)}.$$

By Lemmas 10E (iii) and 10F (iii) $\overline{\tilde{\omega}_\alpha(s)}=\overline{\tilde{\omega}_{Z,\alpha}(s)}$ for s a regular element of odd order in T. Hence $\tilde{b}_\alpha^H=\tilde{B}_\alpha$.

The number of irreducible characters in B_α is equal to the number of conjugate classes in D by Lemma 10E (ii). By Lemmas 10E (ii) and 10F (ii), every block $\tilde{b}_\alpha$ or $\tilde{B}_\alpha$ contains a unique irreducible Brauer character. Thus (4M) implies that $l(\sigma, B)\leqq 1$. The result is proved in this case.

Suppose that $q=3$ and $m=4$ or 6. Let $s\in T$ with s of order 5 if $m=4$ and s of order 7 if $m=6$. In each case there is a unique α, and $\chi_\alpha(s)=1$. Inspection of the character tables of $O_4^-(3)$ $Sp_4(3)$, $O_6^-(3)$, $Sp_6(3)$, $U_3(3)$ shows that for each of these groups Z, b_α is the unique 2-block whose defect group is in Y which contains an irreducible character not vanishing on s. By (6.13) and Lemma 10F (iii) $\overline{\tilde{\omega}_{Z,\alpha}(s)}\neq 0$. Thus if $\tilde{B}_\alpha=\tilde{b}^H$ then $\tilde{b}=\tilde{b}_\alpha$. By Lemma 10G and (4K) $\tilde{B}_\alpha=\tilde{b}^H$ for some $\tilde{b}$. The result is proved also in this case.

Lemma 10I. *Let $\sigma\in D-Y$. Let θ be a regular character of T with $\theta^{q^n+1}=1$. Then $\theta=\beta\alpha$ for some $\beta=\beta_i$ and some regular character α of T of odd order such that $\alpha^{q^n+1}=1=\beta^{q^n+1}$, $\tilde{\chi}_\theta\in\tilde{B}_\alpha$ and there is a unique irreducible Brauer character $\varphi_\alpha\in\tilde{b}_\alpha$. Furthermore*

$$d^\sigma(\tilde{\chi}_\theta, \varphi_\alpha) = \tilde{\theta}(\sigma), \tag{10.2}$$

where $\tilde{\theta}$ is a suitably chosen extension of θ to $(T\cap Z_G(\sigma))\langle\sigma\rangle$, the choice depending on the choice of the extension $\tilde{\chi}_\theta$.

PROOF. The first two statements follow from Lemmas 10A and 10F (ii). It remains to prove (10.2).

Let $Z=Z_G(\sigma)$, $\tilde{Z}=Z_H(\sigma)=Z_G(\sigma)\langle\sigma\rangle$, $Y_Z=Y\cap Z$. Let $S=\{\beta_i|\beta_i^{q^n+1}=1\}$. Then $|S|=|Y_Z|$. By Lemma 10G $d^\sigma(\tilde{\chi}_{\beta_i\alpha}, \varphi_\alpha)\neq 0$ for $\beta_i\in S$ and if ε is the character of order 2 of $\tilde{Z}/Z$ then

$$d^\sigma(\varepsilon\chi_{\beta_i\alpha}, \varphi_\alpha) = -d^\sigma(\tilde{\chi}_{\beta_i\alpha}, \varphi_\alpha). \tag{10.3}$$

As each higher decomposition number is an algebraic integer in a cyclotomic field and the set for each σ is closed under conjugation, the arithmetic-geometric inequality, (4J) and (10.3) imply that

$$(10.4)\qquad 1 \leqq \prod_S |d^\sigma(\tilde{\chi}_{\beta_i\alpha}, \varphi_\alpha)^2|^{1/|S|} \leqq \frac{1}{|S|}\sum_S |d^\sigma(\tilde{\chi}_{\beta_i\alpha}, \varphi_\alpha)|^2 = \frac{1}{2|S|} c^\sigma_{\alpha\alpha}.$$

By (40) $c^\tau_{\alpha\alpha}=2|Y_Z|=2|S|$. Hence equality holds in (10.4). Thus

$$(10.5)\qquad |d^\sigma(\tilde{\chi}_{\beta_i\alpha}, \varphi_\alpha)| = 1.$$

Let l be a Zsigmondy prime for (q, m). Let L be a S_l-group of T and let M be a $S_{l'}$-group of T. Define an irreducible character λ of $T/M\cong L$ as follows.

$$\lambda_L = \alpha_L^{-1} \quad \text{if} \quad \alpha_L \neq 1_L$$

λ_L is a faithful character of L if $\alpha_L=1_L$.

Let $\alpha'=\alpha\lambda$. Then α' has odd order, $(\alpha')^{q^n+1}=1$ and

$$(10.6)\qquad (\alpha^N)_L \neq ((\alpha')^N)_L,$$

$$(10.7)\qquad (\alpha-\alpha')(s) = 0 \quad \text{if} \quad s\in T, \quad s \text{ is not } l\text{-singular.}$$

Let $T_Z=T\cap Z, X_Z=X\cap Z$, $\tilde{T}_Z=T_Z\langle\sigma\rangle$. Then $D\cap\tilde{Z}=Y_Z\langle\sigma\rangle$ and

$$(10.8)\qquad \tilde{T}_Z = Y_Z\langle\sigma\rangle\times X_Z = (Y_Z\times X_Z)\langle\sigma\rangle$$

Thus α, α' extend to characters of $\tilde{T}_Z$, also denoted by α, α' respectively, with $Y_Z\langle\sigma\rangle$ in their kernel. Let $d^\sigma=d^\sigma(\tilde{\chi}_\theta, \varphi_\alpha)$. Define the function η on $\tilde{T}_Z$ by

$$(10.9)\qquad \left.\begin{aligned} \eta(s) &= \theta^N(s),\\ \eta(\sigma s) &= d^\sigma\theta^N(s), \end{aligned}\right\} \quad s\in T.$$

Lemma 10F (ii) and (6.12) applied to G and Z imply that

$$(10.10)\qquad \left.\begin{aligned} \tilde{\chi}(s) &= \eta(s),\\ \tilde{\chi}(\sigma s) &= \eta(\sigma s), \end{aligned}\right\} \quad s \text{ a regular element in } T.$$

If $\beta_i\in S$ let $\tilde{\beta}_i$ denote an extension of β_i to a character of D. Since every l-singular element in T is regular (10.7) and (10.10) imply that

$$10.11)\qquad (\tilde{\chi}_{\tilde{T}_Z}, \tilde{\beta}_i\alpha-\tilde{\beta}_i\alpha') = (\eta, \tilde{\beta}_i\alpha-\tilde{\beta}_i\alpha'),$$

where (,) denotes the inner product over $\tilde{T}_Z$.

Define the function $\hat{\beta}_i$ on $D\cap\tilde{Z}$ by

$$(10.12)\qquad \left.\begin{aligned} \hat{\beta}_i(s) &= \beta_i(s),\\ \hat{\beta}_i(\sigma s) &= d^\sigma\beta_i(s), \end{aligned}\right\} \quad s\in T.$$

Hence by (10.9)

$$(10.13)\qquad \eta = \sum_{j=0}^{m-1} \widehat{\beta^{q^j}}\alpha^{q^j}.$$

Furthermore

$$(\beta^{q^j}\alpha^{q^j}, \tilde{\beta}_i(\alpha-\alpha')) = (\beta^{q^i}, \tilde{\beta}_i)_{D\cap Z}(\alpha^{q^j}, \alpha-\alpha')_{\overline{X\cap Z}}.$$

By the definition of α', $(\alpha^{q^j}, \alpha')_{X\cap Z}=0$. Since α is regular $(\alpha^{q^j}, \alpha)_{X\cap Z}=\delta_{j,0}$. Hence (10.13) implies that

$$(\eta, \tilde{\beta}_i(\alpha-\alpha')) = (\hat{\beta}, \tilde{\beta}_i)_{D\cap Z}. \tag{10.14}$$

Let $\hat{\beta}=\Sigma b_i\tilde{\beta}_i$. By (10.11) and (10.14) each b_i is a rational integer. By (10.5) $(\hat{\beta}, \hat{\beta})_{D\cap Z}=1$. Thus $\hat{\beta}=\tilde{\beta}_k$ for some k. Hence $\beta=\beta_k$ and so $\theta=\beta_k\alpha$. Therefore by a suitable choice of extension $\tilde{\theta}=\tilde{\beta}_k\alpha$. Hence (10.13) implies that

$$\eta(\sigma) = m\tilde{\beta}_k(\sigma) = m\tilde{\theta}(\sigma).$$

Thus (10.2) follows from (10.9). □

We can now summarize the results proved in this section so far as follows.

Theorem 10J. *Let χ_θ be a cuspidal character of G with $\chi_\theta^\tau=\chi_\theta$. Let $\tilde{\chi}_\theta$ denote an extension of χ_θ to H. Let $w_0\in G$ with $t^{w_0}=t^{q^n}$. Let $D=\langle\tau w_0, y\rangle$ where $\langle y\rangle$ is a S_2-group of T. Let $\sigma x\in H-G$ where σ is the 2-part and x the 2'-part of σx.*

If σ is not conjugate to an element of $D-Y$ then $\tilde{\chi}_\theta(\sigma)=0$.

Suppose that $\sigma\in D-Y$. Then $Z_H(\sigma)$ is discribed in Lemma 10C. *Let ζ_θ be the cuspidal character of $Z_G(\sigma)$ corresponding to θ. Then*

$$\tilde{\chi}_\theta(\sigma x) = \tilde{\theta}(\sigma)\zeta_\theta(x),$$

where the choice of extension $\tilde{\theta}$ of θ to $(T\cap Z_G(\sigma))\langle\sigma\rangle$ depends only on the choice of the extension $\tilde{\chi}_\theta$ of χ_θ to H.

Corollary 10K. *Let the notation be as in Theorem* 10J. *Let $\tilde{\omega}_\theta$ be the central character corresponding to $\tilde{\chi}_\theta$.*

(i) *If σ is an involution then*

$$\tilde{\chi}_\theta(\sigma) = \pm \prod_{i=1}^{n-1} (q^{2i}-1).$$
$$\tilde{\omega}_\theta(\sigma) = \pm q^{n^2}(q^n-1).$$

(ii) *If σ has order* 4 *then*

$$\tilde{\chi}_\theta(\sigma) = \pm(q^n-1) \prod_{i=1}^{n-1} (q^{2i}-1).$$
$$\tilde{\omega}_\theta(\sigma) = \pm q^{n^2-n}(q^n-1).$$

(iii) *If σ has order $2^k>4$ then n is odd and*

$$\tilde{\chi}_\theta(\sigma) = \varepsilon \prod_{i=1}^{n-1} ((-q^i)-1).$$
$$\tilde{\omega}_\theta(\sigma) = \varepsilon q^{(3n^2-n)/2}(q^n-1).$$

where ε is a $(2^{k-1})^{\text{th}}$ root of 1.

(iv) *Suppose that σ has order* 4 *and u is a unipotent element in $Z_G(\sigma)$ which has an $m-1$ dimensional invariant subspace. Then $Z_G(\sigma u)$ is a parabolic subgroup of $Z_G(\sigma)$ with $|Z_G(\sigma)\colon Z_G(\sigma u)|=q^{2n}-1$ and*

$$\tilde{\chi}_\theta(\sigma u) = \pm \prod_{i=1}^{n-1} (q^{2i}-1).$$
$$\tilde{\omega}_\theta(\sigma u) = \pm q^{n^2-n}(q^{2n}-1).$$

PROOF. (i) (ii) (iii). By Theorem 10J $\tilde{\chi}_\theta(\sigma)=\varepsilon\zeta_\theta(1)$. Thus $\tilde{\chi}_\theta(\sigma)$ has the required value in each case by Lemma 10C. By (6.11)

$$\tilde{\omega}_\theta(\sigma)=\varepsilon\frac{|H|}{2|Z|}\frac{|T|}{|G|_{r'}}\frac{|Z|_{r'}}{|T\cap Z|}=\varepsilon\frac{|G|_r}{|Z|_r}(q^n-1).$$

The result follows.

(iv) By Theorem 10J $\tilde{\chi}_\theta(\sigma)=\pm\zeta_\theta(u)$ where $Z\cong Sp_{2n}(q)$. By [11] (38.4) $\tilde{\chi}_\theta(\sigma)$ has the required value. Since $Z_{Sp_{2n}(q)}(u)$ is parabolic in Z, $|Z_G(\sigma)\colon Z_G(\sigma u)|$ has the required value. Thus

$$\tilde{\omega}_\theta(\sigma)=\pm\frac{|H|\,|T|}{2|Z_G(\sigma u)|\,|G|_{r'}}\prod_{i=1}^{n-1}(q^{2i}-1)=\pm\frac{|G|_r}{|Z|_r}(q^{2n}-1)=\pm q^{n^2-n}(q^{2n}-1)\quad\square$$

Corollary 10L. *Let the notation be as in Theorem* 10J. *Let A be a complex representation which affords* $\tilde{\chi}_\theta$.

(i) $\Big\{\sum_{x'=x\in GL_m(q)}A(x)\Big\}^2=q^{2n^2}(q^n-1)^2I.$

(ii) *Let S be defined as in Lemma* 3F. *Then*

$$\Big\{\sum_{x\in S}A(x)\Big\}^2=q^{2(n^2-n)}(q^{2n}-1)^2I.$$

PROOF. Immediate from Lemmas 3F, 3G and Corollary 10K. $\square$

References

[1] E. ARTIN, The orders of the linear groups, *Comm. Pure Appl. Math.* **8** (1955), 355—366.

[2] P. DELIGNE and G. LUSZTIG, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103—161.

[3] W. FEIT, The representation theory of finite groups, *North-Holland Matematical Library vol.* **25** (1982), *Amsterdam, New York, Oxford.*

[4] P. FONG and B. SRINIVASAN, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** (1982) 109—153.

[5] D. GORENSTEIN, Finite groups, *Harper and Row* (1968) *New York, Evanston and London.*

[6] J. A. GREEN, The characters of the finite general linear groups *Trans. A. M. S.* **80** (1955), 402—447.

[7] R. HOWE and A. MOY, Personal Communication.

[8] I. M. ISAACS and D. PASSMAN, A characterization of groups in terms of the degrees of their characters II, *Pacific J. Math.* **24** (1968), 467—510.

[9] N. KAWANAKA, On the irreducible characters of the finite unitary groups, *J. Math. Soc. Japan* **29** (1977), 425—450.

[10] N. KAWANAKA, Liftings of irreducible characters of finite classical groups II, *J. Fac. of Sci., University of Tokyo* **30** (1984), 499—516.

[11] G. LUSZTIG, On the Green polynomials of classical groups, *Proc. London. Math. Soc.* **33** (1976). 443—475.

WALTER FEIT
DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
BOX 2155, YALE STATION
NEW HAVEN, CONNECTICUT 06520

(Received September 5, 1985.)