Some applications of the generalized hypergeometric function involving certain subclasses of analytic functions *

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Abstract

Two familiar subclasses of analytic functions are considered here: the class $\mathcal{S}(\alpha)$ of analytic functions $f(z)$ satisfying the inequality

$$\operatorname{Re}\{f(z)/z\} > \alpha \qquad (0 \leqq \alpha < 1),$$

and the class $\mathcal{R}$ of analytic functions whose derivative has a positive real part. The object of this paper is to present several interesting applications of the generalized hypergeometric function, which involve the classes $\mathcal{S}(\alpha)$ and $\mathcal{R}$, and the concept of subordination between analytic functions. A theorem on the radius of univalence for a certain class of generalized hypergeometric functions is also established.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha)$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for $0 \leqq \alpha < 1$.

The class $\mathcal{S}(\alpha)$ was introduced by GOEL [5], and was studied subsequently by CHEN ([2], [3]). In particular, the class $\mathcal{S}(0)$ was studied by GOEL [6] and YAMAGUCHI [18].

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be in the class $\mathcal{R}$ if it satisfies the following inequality:

$$\operatorname{Re}\{f'(z)\} > 0 \qquad (z \in \mathcal{U}). \tag{1.3}$$

The class $\mathcal{R}$ was introduced by MACGREGOR [11].

* This research was carried out at the University of Victoria while the second author was on study leave from Kinki University, Osaka, Japan.

** Supported, in part, by NSERC (Canada) under Grant A—7353.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A30; Secondary 30C45.

In order to recall the concept of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$. The function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $h(z)$ analytic in the unit disk $\mathcal{U}$, with $h(0)=0$ and $|h(z)|<1$, such that

$$f(z) = g(h(z)) \tag{1.4}$$

for $z \in \mathcal{U}$. We denote this subordination by

$$f(z) \prec g(z). \tag{1.5}$$

In particular, if $g(z)$ is univalent in the unit disk $\mathcal{U}$, the subordination (1.5) is equivalent to (*cf.* [4], p. 190).

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \tag{1.6}$$

The concept of subordination can be traced back to LINDELÖF [8], although LITTLEWOOD ([9], [10]) and ROGOSINSKI ([14], [15]) introduced the term and established the basic results involving subordination. More recently, SUFFRIDGE [17], and HALLENBECK and RUSCHEWEYH [7], studied various families of subordinate functions.

Finally, let α_j $(j=1, \ldots, p)$ and β_j $(j=1, \ldots, q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, \ldots, q.$$

Also let $(\lambda)_n$ denote the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \text{if} \quad n \in \mathcal{N} = \{1, 2, 3, \ldots\}. \end{cases} \tag{1.7}$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by *(cf., e.g.,* [16], p. 33 *et seq.)*

$$ {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \tag{1.8}$$

$$(p \leqq q+1).$$

It may be recalled that the ${}_pF_q(z)$ series in (1.8) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathcal{U}$ if $p = q+1$. Furthermore, if we set

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.9}$$

then the ${}_pF_q(z)$ series (1.8), with $p=q+1$, is absolutely convergent for

$$|z| = 1 \quad \text{if} \quad \operatorname{Re}(\omega) > 0, \tag{1.10}$$

and conditionally convergent for

$$|z| = 1, \quad z \neq 1, \quad \text{if} \quad -1 < \operatorname{Re}(\omega) \leqq 0. \tag{1.11}$$

We now introduce the following class of generalized hypergeometric functions:

Definition. The generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) is said to be in the class $\mathscr{A}(p; q; \alpha)$ if it satisfies the following inequality:

$$\text{Re}\,\{{}_pF_q(\alpha_1, \ldots, \alpha_p;\ \ \beta_1, \ldots, \beta_q;\ z)\} > \alpha \qquad (z\in\mathscr{U}) \tag{1.12}$$

for $0\leqq\alpha<1$.

Making use of the above definitions, we present several interesting applications of the generalized hypergeometric function ${}_pF_q(z)$, which involve the classes $\mathscr{S}(\alpha)$ and $\mathscr{R}$, and the concept of subordination between analytic functions. We also establish a theorem on the radius of univalence for the above class of generalized hypergeometric functions.

2. Application involving subordination between analytic functions

Our first application of the generalized hypergeometric function ${}_pF_q(z)$ depends upon a result due to Nehari [12, p. 168], which we recall here as

Lemma 1. *Let $\varphi(z)$ be analytic in the unit disk $\mathscr{U}$ and satisfy $|\varphi(z)|\leqq 1$ for $z\in\mathscr{U}$. Then*

$$|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z\in\mathscr{U}). \tag{2.1}$$

By using Lemma 1 and the concept of subordination, we shall prove

Theorem 1. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by* (1.8) *belong to the class $\mathscr{A}(p; q; \alpha)$. Then*

$$|{}_pF_q(\alpha_1+1, \ldots, \alpha_p+1;\ \beta_1+1, \ldots, \beta_q+1;\ z)| \leqq \Big(\Big|\prod_{j=1}^{q}\beta_j\Big|\Big/\Big|\prod_{j=1}^{p}\alpha_j\Big|\Big)\left(\frac{2(1-\alpha)}{(1-|z|)^2}\right), \tag{2.2}$$

where

$$\prod_{j=1}^{p}\alpha_j \prod_{j=1}^{q}\beta_j \neq 0. \tag{2.3}$$

The result (2.2) *is sharp.*

PROOF. We note that

$$\text{Re}\left(\frac{1+(1-2\alpha)z}{1-z}\right) > \alpha \qquad (z\in\mathscr{U}). \tag{2.4}$$

Hence, by virtue of the definition of subordination, we obtain

$${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) \prec \frac{1+(1-2\alpha)\,z}{1-z} \tag{2.5}$$

for $z\in\mathscr{U}$. Thus we may write

$${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) = \frac{1+(1-2\alpha)\,w(z)}{1-w(z)}, \tag{2.6}$$

where $w(z)$ is analytic in the unit disk $\mathscr{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1. \tag{2.7}$$

Differentiating both sides of (2.6), we get

$$(2.8)\qquad {}_pF_q'(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) = \frac{2(1-\alpha)\, w'(z)}{[1-w(z)]^2}.$$

Note that

$$(2.9)\qquad {}_pF_q'(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) =$$

$$= \Big(\prod_{j=1}^{p} \alpha_j \Big/ \prod_{j=1}^{q} \beta_j\Big)\, {}_pF_q(\alpha_1+1, \ldots, \alpha_p+1;\ \beta_1+1, \ldots, \beta_q+1;\ z),$$

and that

$$(2.10)\qquad |w(z)| \leqq |z| \quad \text{for} \quad z \in \mathcal{U},$$

by the Schwarz lemma.

Applying Lemma 1 to $w(z)$, we find that

$$(2.11)\qquad |{}_pF_q(\alpha_1+1, \ldots, \alpha_p+1;\ \beta_1+1, \ldots, \beta_q+1;\ z)| \leqq$$

$$\leqq \Big(\Big|\prod_{j=1}^{q} \beta_j\Big| \Big/ \Big|\prod_{j=1}^{p} \alpha_j\Big|\Big) \cdot \frac{2(1-\alpha)}{[1-|w(z)|]^2} \cdot \frac{1-|w(z)|^2}{1-|z|^2} \leqq \Big(\Big|\prod_{j=1}^{q} \beta_j\Big| \Big/ \Big|\prod_{j=1}^{p} \alpha_j\Big|\Big) \left(\frac{2(1-\alpha)}{(1-|z|)^2}\right)$$

provided that the condition (2.3) holds true.

Finally, by taking the generalized hypergeometric function defined by

$$(2.12)\qquad {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) = \frac{1+(1-2\alpha)\, z}{1-z},$$

we readily verify that the result (2.2) is sharp.

3. Application involving the class $\mathcal{S}(\alpha)$

We need the following result given by Chen [3]:

Lemma 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}(\alpha)$. Then, for* $0 \leqq |z| < \frac{1}{2}$,

$$(3.1)\qquad \operatorname{Re}\{f'(z)\} \geqq \frac{1+2(2\alpha-1)|z|+(2\alpha-1)|z|^2}{(1+|z|)^2},$$

and, for $\frac{1}{2} \leqq |z| < 1$,

$$(3.2)\qquad \operatorname{Re}\{f'(z)\} \geqq \frac{\alpha-2\alpha|z|^2+(2\alpha-1)|z|^4}{(1-|z|^2)^2}.$$

The results (3.1) *and* (3.2) *are sharp.*

By using Lemma 2, we now prove

Theorem 2. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.8) *belong to the class* $\mathscr{A}(p;q;\alpha)$. *Then, for* $0 \leqq |z| < \frac{1}{2}$,

$$(3.3) \qquad \operatorname{Re}\{{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 2;\, \beta_1, \ldots, \beta_q, 1;\, z)\} \geqq \frac{1+2(2\alpha-1)|z|+(2\alpha-1)|z|^2}{(1+|z|)^2},$$

and, for $\frac{1}{2} \leqq |z| < 1$,

$$(3.4) \qquad \operatorname{Re}\{{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 2;\, \beta_1, \ldots, \beta_q, 1;\, z)\} \geqq \frac{\alpha-2\alpha|z|^2+(2\alpha-1)|z|^4}{(1-|z|^2)^2}.$$

The results (3.3) *and* (3.4) *are sharp.*

Proof. Define a function $H(z)$ by

$$(3.5) \qquad H(z) = z\,{}_pF_q(\alpha_1, \ldots, \alpha_p;\, \beta_1, \ldots, \beta_q;\, z)$$

for $z \in \mathscr{U}$. Since ${}_pF_q(z)$ is in the class $\mathscr{A}(p;q;\alpha)$, we have

$$(3.6) \qquad \operatorname{Re}\left\{\frac{H(z)}{z}\right\} > \alpha \qquad (z \in \mathscr{U}),$$

which implies that $H(z) \in \mathscr{S}(\alpha)$.

Applying Lemma 2 to $H(z)$, we find that

$$(3.7) \qquad \operatorname{Re}\{H'(z)\} \geqq \frac{1+2(2\alpha-1)|z|+(2\alpha-1)|z|^2}{(1+|z|)^2}$$

for $0 \leqq |z| < \frac{1}{2}$, and that

$$(3.8) \qquad \operatorname{Re}\{H'(z)\} \geqq \frac{\alpha-2\alpha|z|^2+(2\alpha-1)|z|^2}{(1-|z|^2)^2}$$

for $\frac{1}{2} \leqq |z| < 1$.

Now it is not difficult to verify that

$$(3.9) \qquad \begin{aligned} H'(z) &= z\,{}_pF_q'(\alpha_1, \ldots, \alpha_p;\, \beta_1, \ldots, \beta_q;\, z) + {}_pF_q(\alpha_1, \ldots, \alpha_p;\, \beta_1, \ldots, \beta_q;\, z) = \\ &= {}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 2;\, \beta_1, \ldots, \beta_q, 1;\, z), \end{aligned}$$

which, in conjunction with (3.7) and (3.8), yields the assertions (3.3) and (3.4) of Theorem 2.

Finally, by taking the generalized hypergeometric function defined by

$$(3.10) \qquad \begin{aligned} &{}_pF_q(\alpha_1, \ldots, \alpha_p;\, \beta_1, \ldots, \beta_q;\, z) * {}_1F_0(2;\, -;\, z) = \\ &= \begin{cases} \dfrac{1+2(\alpha-1)z+(2\alpha-1)\,z^2}{(1+z)^2} & \left(0 \leqq |z| < \dfrac{1}{2}\right), \\ \dfrac{\alpha-2\alpha z^2+(2\alpha-1)\,z^4}{(1-z^2)^2} & \left(\dfrac{1}{2} \leqq |z| < 1\right), \end{cases} \end{aligned}$$

we can show that the results (3.3) and (3.4) are sharp; here $f(z) * g(z)$ denotes the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$.

4. Application involving the class $\mathcal{R}$

In order to apply the generalized hypergeometric function ${}_pF_q(z)$ to the class $\mathcal{R}$, we require the following lemma due to MacGregor [11]:

Lemma 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{R}$. Then*

$$|a_n| \leqq \frac{2}{n} \qquad (n \geqq 2), \tag{4.1}$$

$$|f'(z)| \leqq \frac{1+|z|}{1-|z|} \qquad (z \in \mathcal{U}), \tag{4.2}$$

$$\mathrm{Re}\,\{f'(z)\} \geqq \frac{1-|z|}{1+|z|} \qquad (z \in \mathcal{U}), \tag{4.3}$$

$$|f(z)| \geqq -|z| + 2\log(1+|z|) \qquad (z \in \mathcal{U}), \tag{4.4}$$

and

$$|f(z)| \leqq -|z| - 2\log(1-|z|) \qquad (z \in \mathcal{U}). \tag{4.5}$$

We now establish

Theorem 3. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by* (1.8) *belong to the class $\mathcal{A}(p; q; \alpha)$. Then*

$$\left|\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\right| \leqq 2(1-\alpha) \cdot n! \qquad (n \geqq 1), \tag{4.6}$$

$$|{}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z)| \leqq \frac{1+(1-2\alpha)|z|}{1-|z|} \qquad (z \in \mathcal{U}), \tag{4.7}$$

$$\mathrm{Re}\,\{{}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z)\} \geqq \frac{1-(1-2\alpha)|z|}{1+|z|} \qquad (z \in \mathcal{U}), \tag{4.8}$$

$$|{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 1;\ \beta_1, \ldots, \beta_q, 2;\ z) - \alpha| \geqq (\alpha-1) + \frac{2(1-\alpha)\log(1+|z|)}{|z|} \qquad (z \in \mathcal{U} - \{0\}), \tag{4.9}$$

and

$$|{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 1;\ \beta_1, \ldots, \beta_q, 2;\ z) - \alpha| \leqq (\alpha-1) + \frac{2(\alpha-1)\log(1-|z|)}{|z|} \qquad (z \in \mathcal{U} - \{0\}). \tag{4.10}$$

The results (4.6) *to* (4.10) *are sharp.*

Proof. We introduce a function $G(z)$ defined by

(4.11)

$$G(z) = \int_0^z {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ t)\,dt = z\,{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 1;\ \beta_1, \ldots, \beta_q, 2;\ z)$$

for $z\in\mathcal{U}$. It follows from (4.11) that

$$\mathrm{Re}\,\{G'(z)\} = \mathrm{Re}\,\{{}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z)\} > \alpha. \tag{4.12}$$

Next we define a function $G_1(z)$ by

$$G_1(z) = \frac{G(z)-\alpha z}{1-\alpha} \tag{4.13}$$

for $z\in\mathcal{U}$. Then it is easily verified that

$$\mathrm{Re}\,\{G_1'(z)\} = \mathrm{Re}\left\{\frac{G'(z)-\alpha}{1-\alpha}\right\} > 0, \qquad z\in\mathcal{U}; \tag{4.14}$$

which, by the definition (1.3), implies that

$$G_1(z)\in\mathcal{R}.$$

Noting that

$$G_1(z) = z+\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n(1)_n}{(\beta_1)_n\cdots(\beta_q)_n(2)_n(1-\alpha)}\,z^{n+1}, \tag{4.15}$$

and applying Lemma 3 to $G_1(z)$, we immediately get the assertions (4.6) to (4.10) of Theorem 3.

Finally, taking the generalized hypergeometric function defined by

$${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) = 1+2(1-\alpha)\sum_{n=1}^{\infty} z^n, \tag{4.16}$$

we see that each of the results (4.6) to (4.10) is sharp.

5. Univalence of the generalized hypergeometric function

Carlson and Shaffer [1] presented a study of various interesting classes of starlike, convex, and prestarlike hypergeometric functions by applying a linear operator defined by a certain convolution. Recently, Owa and Srivastava [13] derived several interesting results concerning univalent generalized hypergeometric functions, starlike generalized hypergeometric functions of order α, and convex generalized hypergeometric functions of order α. In this section we determine the radius of univalence for the generalized hypergeometric functions belonging to the class $\mathcal{A}(p; q; \alpha)$ with the aid of the following lemma due to Chen [3]:

Lemma 4. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}(\alpha)$ with $0\leqq\alpha<\frac{1}{10}$. Then $f(z)$ is univalent in $|z|<r_1$, where r_1 is given by*

$$r_1 = \sqrt{\frac{2(1-\alpha)}{1-2\alpha}}-1. \tag{5.1}$$

The result is sharp.

By using Lemma 4, we shall derive

Theorem 4. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.8) *belong to the class* $\mathscr{A}(p;\, q;\, \alpha)$ *with* $0 \leqq \alpha < \frac{1}{10}$. *Then the function*

$$z\, {}_pF_q(\alpha_1, \ldots, \alpha_p;\; \beta_1, \ldots, \beta_q;\; z)$$

is univalent in $|z| < r_1$, *where* r_1 *is given by* (5.1). *The result is sharp.*

PROOF. The hypothesis that ${}_pF_q(z)$ is in the class $\mathscr{A}(p;\, q;\, \alpha)$ implies that

$$z\, {}_pF_q(\alpha_1, \ldots, \alpha_p;\; \beta_1, \ldots, \beta_q;\; z) \in \mathscr{S}(\alpha), \tag{5.2}$$

and the proof of Theorem 4 follows easily from Lemma 4.

The assertion of Theorem 4 is sharp for the generalized hypergeometric function defined by (3.10).

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(Received August 30, 1985.)