# The instability of the period-two cycles of Newton's method

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*Abstract*. B. BARNA has proven that there exist unstable period-two cycles when applying Newton's method to a real polynomial having all real roots and at least four distinct ones. In this note, a lower bound, independent of the degree of polynomial, is found for the derivative of the second iteration of Newton's method evaluated at the period-two cycles.

## Introduction

In [1] and [2], B. BARNA proves that if $f$ is a real polynomial having all real roots and at least four distinct ones, then the set of initial values for which Newton's method does not yield a root of $f$ is homeomorphic to a Cantor set. (This result is proven also in [5] using symbolic dynamics.) Furthermore in [3], he proves that the set of exceptional initial values is of Lebesgue measure zero. In his discussion, Barna proves that for each root, other than the smallest and the largest ones, there is an open interval about it in which each initial value converges to the root using Newton's method. The interval has as its boundary an unstable period-two cycle. The purpose of this note is to establish a lower bound on the derivative of the second iteration of Newton's method evaluated at the period-two cycle, independent of the degree of the polynomial. The determination of the bound is based on the proof that Barna uses to prove the instability of the period-two cycle.

## Preliminaries

Let $f(x)=(x-A_1)^{M_1}(x-A_2)^{M_2}\cdot\ldots\cdot(x-A_k)^{M_k}$, where $A_1<A_2<\ldots<A_k$, all real numbers, $k\geqq 4$, and $M_1+M_2+\ldots+M_k=m$, the degree of $f$. Define the Newton transform of $f$ by $N(x)=x-\dfrac{f(x)}{f'(x)}$, where $f'$ is the derivative of $f$ with respect to $x$.

By [1], about each $A_i$, $i=2, 3, \ldots, (k-1)$, there exists a unique unstable period-two cycle $\{E_i, F_i\}$, where for each $x$ with $E_i<x<F_i$, $N^l(x)\to A_i$ as $l\to\infty$. ($N^l$ denotes the $l$th iterate of $N$.) By an unstable period-two cycle $\{E_i, F_i\}$, one means that $N(E_i)=F_i$ and $N(F_i)=E_i$ with $|N'(E_i)|>1$ and $|N'(F_i)|>1$.

By defining

$$\Phi(x)=\frac{f'(x)}{f(x)},\quad \frac{1}{x-N(x)}=\Phi(x)=\sum_{j=1}^{k}\frac{M_j}{x-A_j}$$

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and for $\{E_i, F_i\}$,

$$\Phi(E_i) = \frac{1}{E_i - F_i} = -\Phi(F_i).$$

Moreover

$$\frac{1}{E_i - F_j} = \sum_{j=1}^{k} \frac{M_j}{E_i - A_j} \quad \text{and} \quad \frac{1}{F_i - E_i} = \sum_{j=1}^{k} \frac{M_j}{F_i - A_j}.$$

By denoting $\frac{E_i - A_j}{F_i - A_j}$ by $D_j$ for a fixed $i$, one obtains $A_j = \frac{E_i - F_i D_j}{1 - D_j}$, and hence

$$\text{(1)} \qquad \frac{M_1}{D_1} + \dots + \frac{M_{i-1}}{D_{i-1}} + \frac{M_{i+1}}{D_{i+1}} + \dots + \frac{M_k}{D_k} = m - 1 + \frac{M_i}{|D_i|},$$

$$\text{(2)} \qquad M_1 D_1 + \dots + M_{i-1} D_{i-1} + M_{i+1} D_{i+1} + \dots + M_k D_k = m - 1 + M_i |D_i|,$$

since $D_j > 0$ for $j \neq i$ and $D_i < 0$.

Barna proves the instability of $\{E_i, F_i\}$ by using (1) and (2), and the following key lemma:

**Lemma [2].** *If $m$ and $p$ are positive integers, $m - p \geqq 3$, $S_q$ $(q = 1, 2, \dots, m-p)$ and $T$ are positive numbers for which*

$$\text{(a)} \qquad S_1 + S_2 + \dots + S_{m-p} = m - 1 + pT,$$

$$\text{(b)} \qquad \frac{1}{S_1} + \frac{1}{S_2} + \dots + \frac{1}{S_{m-p}} = m - 1 + \frac{p}{T},$$

then $S_1^2 + S_2^2 + \dots + S_{m-p}^2 + pT^2 > m$.

By an examination of the proof to this lemma, one can determine a lower bound for the derivative of the second iteration of the Newton transform for $\{E_i, F_i\}$ independent of the degree of f.

## Statement and proof of theorem

**Theorem.** *Let $f(x) = (x - A_1)^{M_1}(x - A_2)^{M_2} \cdot \ldots \cdot (x - A_k)^{M_k}$, where $A_1 < A_2 < \dots < A_k$, all real numbers, $k \geqq 4$, and $M_1 + M_2 + \dots + M_k = m$, the degree of f. If $N(x)$ and $\{E_i, F_i\}$ for a fixed $A_i$ are as described above, then $(N^2)' E_i = (N^2)' F_i > 4$, independent of $m$.*

Proof. Assume the notation used in the lemma stated above. In [2], Barna considers two cases: $T \leqq \min S_q$ and $T > \min S_q$. The necessary estimates to arrive at the conclusion are found in the proof of each case.

*Case I:* Suppose that $T \leqq \min_q S_q = S_1$ and that $S_{m-p} = \max_q S_q$. First consider $\sum_{q=1}^{m-q} (S_q-1)^2$:

$$\sum_{q=1}^{m-p} (S_q-1)^2 = \sum_{q=1}^{m-p} S_q^2 - 2\sum_{q=1}^{m-p} S_q + (m-p) = \sum_{q=1}^{m-p} S_q^2 - 2(m-1+pT)+(m-p)$$

or

$$\sum_{q=1}^{m-p} S_q^2 = \sum_{q=1}^{m-p} (S_q-1)^2 + (m-2+p+2pT) =$$

$$= \sum_{q=1}^{m-p} S_q\left(S_q-2+\frac{1}{S_q}\right) + (m-2+p+2pT) \geqq T\sum_{q=1}^{m-p}\left(S_q-2+\frac{1}{S_q}\right)+(m-2+p+pT) =$$

$$= T\left[(m-1+pT)-2(m-p)+\left(m-1+\frac{p}{T}\right)\right]+(m-2+p+pT) =$$

$$= 4pT+pT^2-2T+m-2+2p.$$

Consequently

$$\sum_{q=1}^{m-p} S_q^2 + pT^2 \geqq 4pT+2pT^2-2T+m-2+2p. \tag{3}$$

Next consider $\sum_{q=1}^{m-q} \left(\frac{1}{S_q}-1\right)^2$:

$$\sum_{q=1}^{m-p} \left(\frac{1}{S_q}-1\right)^2 = \sum_{q=1}^{m-p} \frac{1}{S_q^2} - 2\left(m-1+\frac{p}{T}\right)+(m-p)$$

or

$$\sum_{q=1}^{m-p} \frac{1}{S_q^2} = \sum_{q=1}^{m-p} \frac{1}{S_q}\left(\frac{1}{S_q}-2+S_q\right)+\left(m-2+p+\frac{2p}{T}\right) \geqq$$

$$\geqq \frac{1}{S_{m-p}}\left[\left(m-1+\frac{p}{T}\right)-2(m-p)+(m-1+pT)\right]+\left(m-2+p+\frac{2p}{T}\right)$$

$$> m-2+p+\frac{2p}{T}.$$

Thus

$$\sum_{q=1}^{m-p} \frac{1}{S_q^2} + \frac{p}{T^2} > m-2+p+\frac{2p}{T}+\frac{p}{T^2}. \tag{4}$$

*Case II:* Suppose that $T > \min_q S_q = S_1$ and again that $S_{m-p} = \max_q S_q$. If $T \geqq S_{m-p}$, then $\frac{1}{T} \leqq \min_q \frac{1}{s_q}$, which is Case I. Hence suppose that $S_{m-p} > T > S_1$:

From

$$\left(\frac{1}{m-p-1}\sum_{q=2}^{m-p} S_q^2\right)^{1/2} \geqq \frac{1}{m-p-1}\sum_{q=2}^{m-p} S_q = \frac{m-1+pT-S_1}{m-p-1},$$

$$\sum_{q=2}^{m-p} S_q^2 \geqq \frac{1}{m-p-1}(m-1+pT-S_1)^2 = \frac{1}{m-p-1}[(m-p-1)+p+pT-S_1]^2 =$$

$$= (m-p-1)+2(p+pT-S_1)+\frac{1}{m-p-1}(p+pT-S_1)^2 >$$

$$> (m-p-1)+2p+2(p-1)T+\frac{1}{m-p-1}[p+(p-1)T+(T-S_1)]^2 >$$

$$> (m+p-1)+2(p-1)T+\frac{1}{m-p-1}[p+(p-1)T]^2,$$

or

$$\sum_{q=1}^{m-p} S_q^2+pT^2 > (m+p-1)+2(p-1)T+pT^2+\frac{1}{m-p-1}[p+(p-1)T]^2. \tag{5}$$

Because $S_{m-p}>T>S_1$ implies that $\frac{1}{S_1}>\frac{1}{T}>\frac{1}{S_{m-p}}$, one concludes that

$$\sum_{q=1}^{m-p}\frac{1}{S_q^2}+\frac{p}{T^2} > (m+p-1)+\frac{2(p-1)}{T}+\frac{p}{T^2}+\frac{1}{m-p-1}\left[p+\frac{(p-1)}{T}\right]^2. \tag{6}$$

Inequalities (3)—(6) are used to obtain the lower bound sought.

Since $N(x)=x-\frac{1}{\Phi(x)}$, one has that $N'(x)=1-+\frac{\Phi'(x)}{\Phi(x)^2}$. Also since $\Phi'(E_i)<0$ and $\Phi'(F_i)<0$, $N'(E_i)<0$ and $N'(F_i)<0$. Consequently, $|N'(E_i)|= =\sum_{j=1}^{k}\frac{M_j}{D_j}-(m-1)$ and $|N'(F_i)|=\sum_{j=1}^{k} M_jD_j-(m-1)$. To arrive at the lower bound, it is sufficient to estimate $|N'(F_i)|\cdot|N'(E_i)|$ because $(N^2)'(E_i)=|N'(F_i)|\cdot \cdot|N'(F_i)|=(N^2)'(F_i)$.

To apply the inequalities, let $T=|D_i|$ for a fixed $i$ and $p=1$. If $|D_i|<\min_{j\neq i} D_j$, then, from (3) and (4),

$$|N'(F_i)|\cdot|N'(E_i)| > 2D_i+2D_i^2+1)\left(\frac{2}{|D_i|}+\frac{1}{D_i^2}\right) \tag{7}$$

$$= 6+\frac{2}{D_i^2}+\frac{3}{|D_i|}+4|D_i|.$$

Because $\min_{j\neq i} D_j<1$ and the right-hand portion of (7) is a decreasing function on (0, 1], the minimum value occurs at 1. Thus *)

$$(N^2)'(E_i) = (N^2)'(F_i) > 15. \tag{8}$$

*) Here is $(N^2)(E_i) \equiv \left[\frac{dN[N(x)]}{dx}\right]_{x=E_i}$.

If $\min_{j\neq i} D_j < |D_i| < \max_{j\neq i} D_j$, then, from (5) and (6),

$$(9)\qquad |N'(F_i)|\cdot|N'(E_i)| > \left(1+\frac{1}{m-2}+D_i^2\right)\left(1+\frac{1}{m-2}+\frac{1}{D_i^2}\right) =$$

$$= \left(1+\frac{1}{m-2}\right)^2 + \left(1+\frac{1}{m-2}\right)\left(D_i^2+\frac{1}{D_i^2}\right)+1.$$

Because $0 < \min_{j\neq i} D_j < \max_{j\neq i} D_j < \infty$, one seeks the minimum value of $D_i^2+\frac{1}{D_i^2}$ for $|D_i|>0$ which occurs at 1. From (9),

$$(10)\qquad (N^2)'(E_i) = (N^2)(F_i) > \left(1+\frac{1}{m-2}\right)^2 + 2\left(1+\frac{1}{m-2}\right)+1 =$$

$$= \left(2+\frac{1}{m-2}\right)^2 > 4.$$

Therefore from (8) and (10), one sees that a lower bound for $(N^2)'(E_i)$ and $(N^2)'(F_i)$ is 4, independent of $m$. □

## Closing remarks

One immediately notices that the lower bound increases with the multiplicity of the root by examining (3) through (6). However as pointed out in [5] if the multiplicity is greater than 1, the root is no longer superstable, i.e., the derivative of the Newton transform at the root is no longer zero, and Newton's method does not converge quadratically to this root (see e.g. [4]). There is a compromise between a larger lower bound for the unstable period-two cycle's derivative and the quadratic convergence.

Clearly the estimate of (8) is more desirable, but, in order to use it, one must know quite a lot about the polynomial already, in particular the relative positions of the roots and more importantly of the unstable period-two cycle which is difficult to find even when the degree of $f$ is 4 with all distinct roots.

Lastly in the establishment of (5) and (6), certain quantities are totally eliminated, and therefore the estimate is far from being the best.

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