A variant of Kátai's minimax theorem for additive functions

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Let $f(n)$ be a nonnegative, strongly additive function, which tends to 0 monotonically on the sequence of primes. Assume

$$\sum_{y<p\leqq 2y} f(p)=o(1), \quad y\to\infty. \tag{1}$$

For $C>0$, let $n_1<\ldots<n_i<\ldots$ be the sequence of integers determined by the condition $f(n_i)\leqq C$.

In this note we investigate the gaps $(n_{i+1}-n_i)$.

If $f(n)$ has a limiting distribution $F(x)$ and t is determined by

$$n_1<\ldots<n_t\leqq x,\quad f(n_i)\leqq C \tag{2}$$

then
$$\lim_{x\to\infty}\frac{t}{x}=F(C).$$

If $F(C)\neq 0$ and $t\neq 0$, it follows that for large x, $\frac{x}{t}\sim\frac{1}{F(C)}$. Since $n_t=n_1+\sum_{i=1}^{t-1}(n_{i+1}-n_i)$ and $n_t\sim x$, one can conclude that the gaps in (2) are bounded in average.

A well-known theorem by ERDŐS and WINTNER gives the following criterion to determine if an additive function has a limiting distribution.

Theorem 1. *In order that an additive function $f(n)$ should possess a limiting distribution, it is both necessary and sufficient that the three series*

$$\sum_{|f(p)|>1}\frac{1}{p},\quad \sum_{|f(p)|\leqq 1}\frac{f(p)}{p}\quad\text{and}\quad \sum_{|f(p)|\leqq 1}\frac{f^2(p)}{p} \tag{3}$$

converge.

One can use theorem 1 to prove the following lemma.

Lemma. *If $f(n)$ is a nonnegative, additive function which tends monotonically to 0 on the sequence of primes, and which satisfies* (1), *then $f(n)$ has a limiting distribution.*

PROOF. The first series in (3) is a finite sum since $f(p)$ tends to 0 monotonically.

The convergence of the third series in (3) will be a consequence of the convergence of the middle series and the fact that $f(p)\geqq 0$.

The convergence of the middle series in (3) follows from the assumption (1), which implies that there exists $N>0$ such that for all $n\geqq N$

$$\sum_{n<p\leqq 2n} f(p) \leqq \frac{1}{2};$$

Therefore

$$\sum_{|f(p)|\leqq 1} \frac{f(p)}{p} = \sum_{\substack{|f(p)|\leqq 1\\ p\leqq N}} \frac{f(p)}{p} + \sum_{i=0}^{\infty} \sum_{2^iN<p\leqq 2^{i+1}N} \frac{f(p)}{p}.$$

The first sum is a finite sum and the double sum can be majorized by

$$\sum_{i=0}^{\infty} \frac{1}{2^iN} \sum_{2^iN<p\leqq 2^{i+1}N} f(p) \leqq \sum_{i=0}^{\infty} \frac{1}{2^iN}\frac{1}{2} = \frac{1}{N}.$$

Therefore each series in (3) converges and the lemma has been proved.

Application of the lemma therefore implies that the gaps $(n_{i+1}-n_i)$ are bounded in average. But this does not exclude the possibility that $\limsup_{i\to\infty} (n_{i+1}-n_i)=\infty$, which in fact is often the case. But it is possible, and this is the purpose of the present note, to show that there is a function $k(x)$ such that $\limsup_{x\to\infty} \frac{n_{i+1}-n_i}{k(x)} \leqq 1$. The function $k(x)$ will be determined through the method of proof which Kátai developed for his minimax theorem. Let us first state the result of Kátai [2].

Theorem 2. *Let $f(n)$ be a nonnegative additive function, which tends monotonically to 0 on the sequence of primes. Let*

$$\psi(y) = \sum_{p\leqq y} f(p)$$

$$\beta(p) = \sup_{\alpha\geqq 1} f(p^\alpha)$$

For $k\geqq 1$, let $E_k(x)=\max_{n\leqq x} \min\{f(n+1), f(n+2), \dots, f(n+k)\}$. Assume that

$$\psi(2y)-\psi(y) = o(1), \quad y\to\infty$$

$$\psi(y)\to\infty$$

$$\sum_p |\beta(p)-f(p)| < \infty$$

Then

$$\lim_{x\to\infty}\left\{E_k(x)-\frac{\psi(\log x)}{k}\right\} = B_k+C_k-\frac{\psi(k)}{k}$$

where $B_k = \sup_{n\geqq 1} \frac{1}{k}\sum_{j=1}^{k} \sum_{\substack{p\leqq k\\ p^\alpha \| n+j}} f(p^\alpha)$ and $C_k = \frac{1}{k}\sum_{p\leqq k} (\beta(p)-f(p))$.

In the proof of theorem 2, k is considered fixed. But as long as $k(x)<\log x$, one can let $k=k(x)$ depend upon x without altering the proof. We now prove the following result:

Theorem 3. *Let $f(n)$ be a nonnegative, strongly additive function which tends monotonically to 0 on the sequence of primes. Let* $A=\sum_p \frac{f(p)}{p}$ *and let* $C>A$ *determine a sequence of integers* $n_1<\ldots<n_i<\ldots$ *by the condition* $f(n_i)\leqq C$. Let

$$k(x)=\frac{1}{C-A}\sum_{p\leqq \log x} f(p).$$

Then
$$\limsup_{x\to\infty}\frac{(n_{i+1}-n_i)}{k(x)}\leqq 1.$$

PROOF. During the proof of the minimax theorem KÁTAI shows that for fixed $k\geqq 1$

$$E_k(x)\leqq B_k+C_k+\frac{\psi(\log x)}{k}-\frac{\psi(k)}{k}+\big(\psi(4\log x)-\psi(\log x)\big).$$

For a strongly additive function $C_k=0$, and

$$B_k=\frac{1}{k}\sum_{j=1}^{k}\sum_{\substack{p\leqq k\\ p\mid n+j}} f(p)\leqq\frac{1}{k}\sum_{\substack{p\leqq k\\ p\mid\prod_{j=1}^{k} n+j}} f(p)\left\{\left[\frac{k}{p}\right]+1\right\}.$$

If one drops the divisibility requirement and replaces $\left[\frac{k}{p}\right]$ by $\frac{k}{p}$ one obtains the further inequality

$$B_k<\frac{1}{k}\left\{\sum_{p\leqq k} f(p)\frac{k}{p}+\sum_{p\leqq k} f(p)\right\}=\sum_{p\leqq k}\frac{f(p)}{p}+\frac{\psi(k)}{k}.$$

This implies that

$$E_k(x)<\frac{\psi(\log x)}{k}+\sum_{p\leqq k}\frac{f(p)}{p}+o(1).$$

Although the inequalities were obtained under the assumption that k is fixed, they are equally valid if k depends upon x as long as $k(x)<\log x$. This will be true for large x if we define

$$k(x)=\frac{1}{C-A}\sum_{p\leqq\log x} f(p)=\frac{1}{C-A}\psi(\log x).$$

For this $k(x)$, $E_k(x)<C+o(1)$ and $\limsup_{x\to\infty} E_k(x)\leqq C$.

But if $E_k(x)<C$, then every interval of length $k(x)$ contains at least one n_i such that $f(n_i)\leqq C$. Therefore, for all $n_i\leqq x$, $(n_{i+1}-n_i)\leqq k(x)$ which finishes the proof.

A similar result has been obtained by GALAMBOS [1] for the case where $f(p)=\frac{1}{p}$ and $C=2$. In this case A can be shown to be less than 1 and $k(x)=\log\log\log x$.

References

[1] J. GALAMBOS, On a conjecture of Kátai concerning weakly composite numbers. *Proc. Amer. Math. Soc.* **96** (1986), 215—216.

[2] I. KÁTAI, A minimax theorem for additive functions. *Publ. Math. (Debrecen)* **30** (1983), 249—252

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