On reduction of bivector connections

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In this paper some reduction problems are investigated for connections of bivectors, i.e. of contravariant antisymmetric tensors of order 2. This class of tensors has an important geometrical content, namely, there is a close relation between the simple (in other words, decomposable) bivectors and the 2-dimensional plane directions. Furthermore, in the geometry of areal spaces simple bivectors (in general, m-vestors) are fundamental objects of study.

Given a linear connection in a vector bundle ξ, one can easily construct an induced linear connection in the bivector bundle $\wedge^2\xi$. It can be also said that the bivector connection obtained in this manner reduces to a vector connection. An induced bivector connection has naturally the property that the parallel transport of bivectors maps simple bivectors into simple ones. S. STEINER [1] proved that this property is not only necessary but sufficient for reduction of a linear bivector connection to a vector connection. This is also an immediate consequence of a result of L. TAMÁSSY [2]. Our aim is to find other conditions for reduction of bivector connections.

In the first section the preliminary notions and the notations used are fixed. Then, according to various approaches of connection theory, the reduction question of linear bivector connections is investigated, in Section 2 from vector bundle viewpoint, and in Section 3 in the principal bundle framework. Finally the two lines of the study is synthetized.

1. Preliminaries

Bundles

In this paper the usual differential geometric conditions are assumed. All manifolds are paracompact, finite dimensional and of class C^∞, maps and functions are of class C^∞. $\xi=(E, \pi, B, F)$ denotes a real vector bundle of finite rank s, with the total space E, the n-dimensional base space B, the projection $\pi\colon E\to B$, the fibre type F. The fibre $\pi^{-1}(x)$ over $x\in B$ is denoted by E_x. The construction theorems for vector bundles admit to create the bivector bundle $\wedge^2\xi=(\wedge^2E, \pi_\wedge, B, \wedge^2F)$ belonging to the vector bundle ξ. The rank of the bivector bundle $\wedge^2\xi$ is $\binom{s}{2}$. A bivector in the form $a\wedge b\,(a, b\in F)$ is called simple (or decomposable). Considering all simple bivectors in the fibres we obtain a subbundle of $\wedge^2\xi$, called the Grassmann cone bundle $Z^2\xi=(Z^2E, \pi_Z, B)$ belonging to ξ. $Z^2\xi$ is not a vectorial subbundle of $\wedge^2\xi$.

$P(\xi)=(P, p, B, Gl(F))$ denotes the frame bundle belonging to ξ. Then the fibre P_x over $x\in B$ consists of linear isomorphisms $F\to E_x$. Accordingly, the frame bundle of the bivector bundle $\wedge^2\xi$ is denoted by $P(\wedge^2\xi)=(P^\wedge, p_\wedge, B, Gl(\Lambda^2\xi))$. Now we describe a subbundle of $P(\wedge^2\xi)$, which consists of the so-called induced bivector frames. A bivector frame R: $\wedge^2F\to\wedge^2E_x$ over $x\in B$ is called induced when $R=\wedge^2r$ for some $r\in P_x$, i.e. $R(a\wedge b)=r(a)\wedge r(b)$ for all $a, b\in F$. The set of all induced bivector frame is denoted by P^Λ. The action of $Gl(F)$ on $P_0^\wedge$ can be defined in a natural way: $(R, g)\mapsto\wedge^2(r\circ g)$ for $R=\wedge^2r\in P_0^\wedge$, $g\in Gl(F)$. Thus we obtain readily the following.

Proposition 1. *The principal bundle* $P_0(\wedge^2\xi)=(P_0^\wedge, p_\wedge, Gl(F))_1$ *with the structure group* $Gl(F)$ *is a reduced subbundle of the frame bundle* $P(\wedge^2\xi)$, *which is isomorphic to* $P(\xi)$.

Connections

Starting from a vector bundle ξ we can construct the canonical short exact sequence

(1) $$0\to V\xi\xrightarrow{\iota}\tau_E\xrightarrow{\widetilde{d\pi}}\pi^*(\tau_B)\to 0 \qquad (H: \pi^*(\tau_B)\to\tau_E)$$

where $V\xi$ denotes the vertical subbundle of the tangent bundle τ_E of the total space E, $\pi^*(\tau_B)$ is the pull-back bundle of the tangent bundle τ_B of the base space B by π, and $\widetilde{d\pi}\,(u)=(\pi_E(u), d\pi(u))$ for $u\in TE$. A splitting H: $\pi^*(\tau_B)\to\tau_E$ of (1), for which $\widetilde{d\pi}\circ H=id_{\pi^*(\tau_B)}$ holds, is called a *VB-horizontal map* for ξ. It is well known (see [3], [4]) that all fundamental data of connection theory can be derived from a horizontal map. The images H_z at $z\in E$, called horizontal subspaces, are supplementary subspaces to the vertical subspaces V_zE, $z\in E$. The horizontal and vertical projections are $h:=H\circ\widetilde{d\pi}$, $v:=id-h$ resp. We obtain a *linear connection* when the horizontal projection satisfies the homogeneity condition ([4]): (HC) for all real $t\in\mathbf{R}$ $d\mu_t\circ h=h\circ d\mu_t$, where μ_t: $E\to E$ is the scalar multiplication by t in the fibres. Then the covariant derivation ∇: $\mathfrak{X}(B)\times\mathrm{Sec}\,\xi\to\mathrm{Sec}\,\xi$ is given as follows: $(X,\sigma)\mapsto\nabla_X\sigma:=\alpha\cdot V\cdot d\sigma(x)$, where α: $V\xi\to\xi$ is the canonical epimorphism. A section $\sigma\in\mathrm{Sec}\,\xi$ is called parallel along a curve φ in B, if $\nabla_{\dot\varphi}\sigma=0$, where $\dot\varphi$ denotes the tangent curve in TB belonging to φ. It gives the possibility to define the parallel transport of a fiber into another along a curve of the base space. Thus we obtain a parallelism structure T_φ in ξ, which could be also a starting point of definition for linear connections (see [5]). We will need the explicite formula between the covariant derivation and the parallel transport:

(2) $$\nabla_X\sigma(x)=\lim_{t\to 0}\frac{1}{t}\left(T_\varphi^{-1}(\sigma(\varphi(t)))-\sigma(x)\right)$$

where φ: $I\to B$ is such a surve that $\dot\varphi(0)=X(x)$, $x\in B$.

The theory of linear connections can be built also upon the principal bundles. Let us consider the frame bundle $P(\xi)=(P, p, B, Gl(F))$ belonging to ξ. A splitting H^p of the short exact sequence

$$0\to VP(\xi)\xrightarrow{\iota}\tau_p\xrightarrow{\widetilde{dp}}p^*(\tau_B)\to 0 \qquad (H^p: p^*(\tau_B)\to\tau_p)$$

constructed from $P(\xi)$ is called a *PB-horizontal map* when its horizontal projection h^p is invariant with respect to the right action of the structure group $Gl(F)$:

$$dR_a \cdot h^p = h^p \cdot dR_a \quad \text{for all} \quad a \in Gl(F).$$

As it is well known [6] there is a one-to-one correspondence between the set of VB-horizontal maps satisfying the homogeneity condition (HC) and the set of PB-horizontal maps, i.e. the linear connections in ξ are just the principal connections of the frame bundle $P(\xi)$. The exact relation between H and H^p is expressed by the following commutative diagram:

$$\begin{array}{ccc} p^*(\tau_B) & \xrightarrow{\tilde{i}_a} & \pi^*(\tau_B) \\ {\scriptstyle H^p}\downarrow & & \downarrow{\scriptstyle H} \\ \tau_p & \xrightarrow{di_a} & \tau_E \end{array}$$

where for all $a \in F$

$i_a\colon P \to E \quad r \mapsto r(a)$

and $\tilde{i}_a\colon p^*(\tau_B) \to \pi^*(\tau_B) \quad (r, v) \mapsto (i_a(r), v)$.

Considering the parallel transport T^p_φ belonging to H^p in the frame bundle $P(\xi)$, an analogous commutative diagram holds for the parallelism structures

$$\begin{array}{ccc} P_{\varphi(0)} & \xrightarrow{i_a} & E_{\varphi(0)} \\ {\scriptstyle T^p_\varphi(t)}\downarrow & & \downarrow{\scriptstyle T_\varphi(t)} \\ P_{\varphi(t)} & \xrightarrow{i_a} & E_{\varphi(t)} \end{array} \tag{3}$$

for all curve φ in B and $a \in F$. The parallel transports T_φ and T^p_φ determine uniquely each other through this diagram.

2. Induced linear connections in the bivector bundle

Let us consider a linear connection in a vector bundle ξ. When denoting its covariant derivation by ∇, we can readily see that there exists a unique covariant derivation $\tilde{\nabla}\colon \mathfrak{X}(B) \times \operatorname{Sec} \wedge^2\xi \to \operatorname{Sec} \wedge^2\xi$ in the bivector bundle $\wedge^2\xi$ such that

$$\tilde{\nabla}_X(\sigma \wedge \eta) = \nabla_X \sigma \wedge \eta + \sigma \wedge \nabla_x \eta \tag{4}$$

holds for all $\sigma, \eta \in \operatorname{Sec} \xi$ and $X \in \mathfrak{X}(B)$. In fact, we may define $\tilde{\nabla}$ for simple bivector sections by (4), and then requiring the usual properties of a covariant derivation, $\tilde{\nabla}$ is extended for arbitrary bivector sections.

On the other hand, there is a second way of induction by means of parallelism structure. Considering the parallel transport T_φ belonging to the given linear connection in ξ, there is a unique parallel transport $\overline{T}_\varphi$ in the bivector bundle such that $\overline{T}_\varphi(t) = \Lambda T_\varphi(t)$ holds for all curve φ in B, i.e. for all $z_1, z_2 \in E_{\varphi(0)}$ $\overline{T}_\varphi(t)(z_1 \wedge z_2) = T_\varphi(t)(z_1) \wedge T_\varphi(t)(z_2)$.

This latter way of induction is equivalent to that one given by covariant derivation. To check it we calculate the covariant derivation $\overline{\nabla}$ belonging to the parallelism structure $\bar{T}_\varphi$ by making use of (2). For all $X\in\mathfrak{X}(B)$ and $\sigma_1, \sigma_2\in\operatorname{Sec}\xi$

$$\bar{\nabla}_x(\sigma_1\wedge\sigma_2)(x) = \lim_{t\to 0}\frac{1}{t}[\bar{T}_\varphi^{-1}(t)(\sigma_1\wedge\sigma_2)(\varphi(t))-(\sigma_1\wedge\sigma_2)(x)]$$

Since $\bar{T}_\varphi^{-1}(t)=\wedge^2 T_\varphi^{-1}(t)$ holds, we can continue as

$$= \lim_{t\to 0}\frac{1}{t}\left[\wedge^2 T_\varphi^{-1}(t)\big(\sigma_1(\varphi(t))\wedge\sigma_2(\varphi(t))\big)-\sigma_1(x)\wedge\sigma_2(x)\right] =$$

$$= \lim_{t\to 0}\frac{1}{t}\left[T_\varphi^{-1}(t)\big(\sigma_1(\varphi(t))\big)\wedge T_\varphi^{-1}(t)\big(\sigma_2(\varphi(t))\big)-T_\varphi^{-1}(t)\big(\sigma_1(\varphi(t))\big)\wedge\sigma_2(x)+\right.$$

$$\left.+T_\varphi^{-1}(t)\big(\sigma_1(\varphi(t))\big)\wedge\sigma_2(x)-\sigma_1(x)\wedge\sigma_2(x)\right] =$$

$$= \sigma_1(x)\wedge\nabla_X\sigma_2(x)+\nabla_X\sigma_1(x)\wedge\sigma_2(x).$$

This means that $\tilde{\nabla}=\bar{\nabla}$, for a covariant derivation of bivectors is uniquely determined by its action on the simple bivectors. The linear bivector connection $H^\wedge$, constructed by covariant derivation, or by parallelism structure of a linear connection H in ξ, is called an *induced linear connection* in the bivector bundle. In this situation we also say that the bivector connection reduces to a vector connection.

Now our question is inverted: under what assumption a linear bivector connection reduces to some vector connection. From the construction of an induced bivector connection by means of parallelism structure it is clear that the induced parallel transport in the bivector bundle maps simple bivectors into simple ones. The result that this condition is not only necessary but also sufficient for reduction is due to S. Steiner [1] and L. Tamássy [2].

First let us consider the horizontal subspaces of an induced linear bivector connection. Since these are spanned by the tangent vectors of parallel sections along curves, and for an induced linear bivector connection simple bivectors are mapped into simple bivectors, the horizontal subspaces at simple bivectors are in the tangent space of the total space containing all simple bivectors. In order to prove that this property not only necessary but also sufficient for reduction of a linear bivector connection we are in need of the following.

Lemma. *Let $\eta'=(E',B,\pi',F')$ be a subbundle of a vector bundle η, not necessarily a vector subbundle of η. Supposing that all the horizontal subspaces H_zE at the point $z\in E'$ of a connection are contained in the tangent space T_zE' of the total space E' belonging to η', then also the following is true: if for a parallel section $\sigma\in\operatorname{Sec}\eta$ along a curve φ in B $\sigma(\varphi(0))\in E'$ holds, then $\sigma(\varphi(t))\in E'$ for all $t\in I$.*

Proof. Let $E'|\varphi$ denote the subbundle of η' over φ: $E'|\varphi=\bigcup_{t\in I}E'_{\varphi(t)}$. From dimensionality reasons $H'_z:=H_zE\cap T_z(E'|\varphi)$ is a one-dimensional subspace for all $z\in E'|\varphi$. Therefore the distribution H'_z, $z\in E'|\varphi$ is integrable, i.e. for $z\in E'|\varphi$ there exists a curve σ' in $E'|\varphi$ starting from $z\in E'|\varphi$. By the local uniqueness of parallel sections along curves we have $\sigma'=\sigma\circ\varphi$.

Applying our lemma to the case $\eta'=Z^2\xi$ and $\eta=\wedge^2\xi$, we obtain immediately the following result.

Theorem. *The parallel transport of a (linear) bivector connection maps simple ones if and only if the horizontal subspaces belonging to the connection are contained in the tangent space of the simple bivectors.*

Combining our Theorem with the mentioned result of S. STEINER, we have a characterizing property for the reduction of linear bivector connections.

Corollary. *A linear bivector connection in $\wedge^2\xi$ reduces to some vector connection in ξ if and only if its horizontal subspaces at simple bivectors are contained in the tangent space of the total space Z^2E of the Grassmann cone bundle $Z^2\xi$ consisting of all simple bivectors.*

3. Induced principal connection in the frame bundle of bivectors

By the isomorphism j: $P(\xi)\to P_0(\wedge^2\xi)$ mentioned in Proposition 1 a principal connection H^p in $P(\xi)$ can be lifted to a principal connection $H^p_\wedge$ in $P_0(\wedge^2\xi)$. Since $P_0(\wedge^2\xi)$ is a reduced subbundle, $H^p_\wedge$ can be extended to the whole bivector frame bundle $P(\wedge^2\xi)$. The extended connection in $P(\wedge^2\xi)$ is denoted also by $H^p_\wedge$. In this manner, through principal connections, starting from a linear connection in ξ we obtain a linear connection in the bivector bundle. This method is called *PB-induction*. Now we are going to show that the two methods of induction introduced are equivalent.

Let H be a linear connection in ξ. Its corresponding principal connection in the frame bundle $P(\xi)$ is denoted by H^p. The VB-induced connection in $\wedge^2\xi$, and the PB-induced connection in $P(\wedge^2\xi)$ are denoted by $H_\wedge$, and $H^p_\wedge$, resp. Then

Theorem. *The principal connection in $P(\wedge^2\xi)$ corresponding to $H_\wedge$ is just equal to $H^p_\wedge$.*

PROOF. Since the parallel transport structure determines uniquely the connection structure, it is enough to prove that the diagram (3) commutes applied for the parallel transports $T_{\wedge,\varphi}$ belonging to $H_\wedge$ and $T^p_{\wedge,\varphi}$ belonging to $H^p_\wedge$. Let $\mathscr{A}$ be a simple bivector in the form $\mathscr{A}=a\wedge b\in Z^2F$, and let R be an induced bivector frame: $R=\wedge^2 r$. Calculate $i_{\mathscr{A}}\circ T^p_{\wedge,\varphi}(t)(R)$. First notice that the following relation is valid between the parallel transports $T^p_{\wedge,\varphi}$ and T^p_φ belonging to H^p:

$$T^p_{\wedge,\varphi}(t)(R)=\wedge^2\big(T^p_\varphi(t)(r)\big)$$

Therefore

$$i_{\mathscr{A}}\circ T^p_{\wedge,\varphi}(R)=i_{\mathscr{A}}\circ\wedge^2\big(T^p_\varphi(t)(r)\big)=\wedge^2 T^p_\varphi(t)(r)(a\wedge b)=$$

$$=\big(T^p_\varphi(t)(r)(a)\wedge T^p_\varphi(t)(r)(b)\big)=[i_a\circ T^p_\varphi(t)(r)]\wedge[i_b\cdot T^p_\varphi(t)(r)]=$$

Now using the fact that the diagram commutes for T^p_φ and T_φ, we can continue as

$$=\big[(T_\varphi(t)\circ i_a)(r)\big]\wedge\big[(T_\varphi(t)\circ i_b)(r)\big]=\wedge^2 T_\varphi(t)\big(r(a)\wedge r(b)\big)=$$

By the second definition of VB-induction $\wedge^2 T_\varphi=T_{\wedge,\varphi}$ holds, thus we obtain finally

$$=(T_{\wedge,}(t)\circ i_{\mathscr{A}})(R),$$

which means that the diagram in question is commutative. It should be remarked that it is enough to check this relation only for simple bivectors, for they generate the entire bivector space.

Our Theorem can be expressed also by the commutativity of the diagram:

$$\begin{array}{ccc} \text{linear connections in } \xi & \leftrightarrow & \text{principal connections in } P(\xi) \\ \text{VB-induction} \downarrow & & \downarrow \text{PB-induction} \\ \text{linear connections in } \wedge^2\xi & \leftrightarrow & \text{principal connections in } P(\wedge^2\xi). \end{array}$$

Finally a consequence of the Theorem is mentioned.

Corollary. *A linear connection in the bivector bundle $\wedge^2\xi$ reduces to some vector connection in ξ if and only if its corresponding principal connection in $P(\wedge^2\xi)$ is reducible to the reduced bundle $P_0(\wedge^2\xi)$.*

References

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