Conditions for compactness of integral operator on Musielak—Orlicz space of vector-valued functions

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Abstract. There is given a criterion for compactness of the linear integral operator

$$Ax(t) = \int_{T_1} \langle K(t, s), x(s)\rangle d\mu_1$$

acting from a Musielak—Orlicz space of vector-valued functions into a Musielak—Orlicz space of real-valued functions. Combining this result with results of papers [6] and [7], there is obtained a condition for compactness of the Hammerstein operator in the form

$$Hx(t) = \int_{T_1} \langle K(t, s), f(s, x(s))\rangle d\mu_2.$$

The application of the above-mentioned results leads to a theorem on the existence of solutions of linear integral equations and integral equations of Hammerstein type in Musielak—Orlicz space.

0. Introduction. (T, Σ, μ) is a space of non-negative, non-atomic, complete and σ-finite measure; $(X, \|\cdot\|_X)$ denotes a reflexive and separable real Banach space and $(Y, \|\cdot\|_Y)$ denotes the dual space to the space X. We denote by $S(X)=S(T, \Sigma, X)$ the linear space obtained from the set of all strongly μ-measurable functions $x\colon T\to X$ by identifying the functions which are equal μ-almost everywhere. Similarly, we denote by $S(Y)=S(T, \Sigma, Y)$ the space of strongly measurable functions from T into Y. Moreover, let $\langle y, x\rangle$ stand for the value of the functional $y\in Y$ at the point $x\in X$. Obviously, for $x\in S(X)$ and $y\in S(Y)$ the function $\langle y(\cdot), x(\cdot)\rangle$ is μ-measurable.

A function $\Phi\colon X\times T\to[0, \infty]$ is called an $\mathcal{N}$-function if:

a) Φ is $\mathcal{B}\times\Sigma$-measurable ($\mathcal{B}$ is the Borel σ-algebra in X),

b) for a.a. $t\in T$ the function $\Phi(\cdot, t)\colon X\to[0, \infty]$ is lower semicontinuous, convex, even for $x\in X$ and $\Phi(0, t)=0$,

c) there exists a measurable function $\alpha\colon T\to(0, \infty)$ such that for a.a. $t\in T$, if $x\in X$ and $\|x\|_X\geqq\alpha(t)$, then $\Phi(x, t)\geqq 1$.

The functional $I_\Phi\colon S(X)\to[0, \infty]$ defined by

$$I_\Phi(x) = \int_T \Phi(x(t), t)\, d\mu$$

is a convex pseudomodular (see [4]) on $S(X)$. The pseudomodular I_Φ determines the Musielak—Orlicz space

$$L_\Phi = L_\Phi(T, \Sigma, X) = \{x\in S(X)\colon\ I_\Phi(rx) < \infty \quad \text{for some} \quad r > 0\}.$$

The set

$$\operatorname{dom} I_\Phi = \{x \in S(X):\ I_\Phi(x) < \infty\}$$

is called the Orlicz class. Using the pseudomodular I_Φ, we denote the Luxemburg norm $\|\cdot\|_\Phi: L_\Phi \to [0, \infty)$ by

$$\|x\|_\Phi = \inf\{r > 0:\ I_\Phi(r^{-1}x) \leqq 1\}.$$

For every $\mathcal{N}$-function Φ we define the complementary function $\Psi: T \times Y \to [0, \infty]$ by the formula

$$\Psi(y, t) = \sup\{\langle y, x\rangle - \Phi(x, t):\ x \in X\}$$

for every $t \in T$, $y \in Y$. The function Ψ is an $\mathcal{N}$-function, too. The Musielak—Orlicz space generated by the function Ψ is denoted by L_Ψ and it is called conjugate to the space L_Φ. It is worth to accentuate that the space L_Ψ is not the dual space to the space L_Φ in general (see [1]). The Luxemburg norm for L_Ψ is denoted $\|\cdot\|_\Psi$.

One can consider another norm in the space L_Φ which is defined by the formula

$$\|x\|_\Phi^0 = \sup_{I_\Psi(y) \leqq 1} \left|\int_T \langle y(t), x(t)\rangle\, d\mu\right|$$

where

$$I_\Psi(y) = \int_T \Psi(y(t), t)\, d\mu.$$

The norm $\|\cdot\|_\Phi^0$ is called the Orlicz norm. The Orlicz norm for $\mathcal{N}$-function Ψ, we define

$$\|y\|_\Psi^0 = \sup_{I_\Phi(x) \leqq 1} \left|\int_T \langle y(t), x(t)\rangle\, d\mu\right|.$$

The Orlicz and Luxemburg norms are equivalent; in fact,

$$\|x\|_\Phi \leqq \|x\|_\Phi^0 \leqq 2\|x\|_\Phi \quad \text{for every} \quad x \in L_\Phi$$

and

$$\|y\|_\Psi \leqq \|y\|_\Psi^0 \leqq 2\|y\|_\Psi \quad \text{for every} \quad y \in L_\Psi.$$

Moreover, for any function $x \in L_\Phi$ and $y \in L_\Psi$ the function $\langle y(\cdot), x(\cdot)\rangle: T \to (-\infty, \infty)$ is integrable and the following inequalities

$$\text{a)} \quad \left|\int_T \langle y(t), x(t)\rangle\, d\mu\right| \leqq \|x\|_\Phi^0 \|y\|_\Psi$$

$$\text{b)} \quad \left|\int_T \langle y(t), x(t)\rangle\, d\mu\right| \leqq \|x\|_\Phi \|y\|_\Psi^0$$

hold. Hereunder, let the following condition for $\mathcal{N}$-function Φ be satisfied:

(B): There exists an increasing sequence $\{T_i\}$ $(i=1, 2, \dots)$ of sets from Σ with $\mu(T_i) < \infty$ for $i=1, 2, \dots$ such that

$$\mu\Big(T \setminus \bigcup_{i=1}^{\infty} T_i\Big) = 0$$

and there is a sequence $\{f_n\}$ $(n=1, 2, \dots)$ of μ-measurable non-negative functions such that

$$\int_{T_i} f_n(t)\, d\mu < \infty$$

for every $i, n=1, 2, \ldots$ and $\Phi(x, t) \leqq f_n(t)$ for $x \in X$, $\|x\|_X \leqq n$, $n=1, 2, \ldots$ and for a.a. $t \in T$.

We shall denote by E_Φ the closure in L_Φ (with respect to the norm topology in L_Φ) of the set of all simple functions from T into X vanishing outside a subset, which is included in T_i for some natural number i.

Note that E_Φ is well defined, that is, the dependence on $\{T_i\}$ and $\{f_n\}$ from Condition (B) is not essential (see [1]). Moreover, we put

$$d(x, E_\Phi) = \inf \{\|x-y\|_\Phi \colon y \in E_\Phi\} \quad \text{for} \quad x \in L_\Phi$$

and

$$\Pi(E_\Phi, r) = \{x \in L_\Phi \colon d(x, E_\Phi) < r\}.$$

We shall say that the function $x \in L_\Phi$ has an absolutely continuous norm, if $\|x\chi_{C_n}\|_\Phi \to 0$ for every decreasing sequence of measurable sets $C_n \downarrow \phi$ as $n \to \infty$ and $C_n \subset T$ $(n=1, 2, \ldots)$.

0.1. Theorem. *A function $x \in L_\Phi$ has an absolutely continuous norm if and only if $x \in E_\Phi$ (see* [7], Th. 1.2.*).*

We shall say that the family $\mathscr{R}$ of functons $x \in L_\Phi$ has equi-absolutely continuous norms if for every $\varepsilon > 0$ and for every decreasing sequence of sets $C_n \downarrow \phi$ as $n \to \infty$, an n_0 can be found such that

$$\|x\chi_{C_n}\|_\Phi < \varepsilon$$

for all functions of the family $\mathscr{R}$, provided $n > n_0$.

If $X=Y=\mathbf{R}$, then the definition of $\mathscr{N}$-function becomes simplified. In this case we will denote M and N instead of Φ and Ψ, respectively. Then the definition of $\mathscr{N}$-function is of the form:

A function $M\colon \mathbf{R} \times T \to [0, \infty]$ is called an $\mathscr{N}$-function, if there exists a null set $A \subset T$ such that the following conditions are satisfied:

a) for all $t \in T \setminus A$ the function $M(\cdot, t)\colon \mathbf{R} \to [0, \infty]$ is convex, nondecreasing, even and $M(0, t)=0$,

b) $M(u, \cdot)\colon T \to [0, \infty]$ is measurable for every $u \in \mathbf{R}$.

In this case Condition (B) is equivalent to the condition

$$\int_{T_i} M(u, t) d\mu < \infty$$

for every $u \in \mathbf{R}$ and for each T_i defined as in the vector case.

The complementary function we define

$$N(v, t) = \sup_{u \in \mathbf{R}} \{uv - M(u, t)\} \quad v \in \mathbf{R}, t \in T.$$

The pseudomodulars I_M and I_N, Musielak—Orlicz spaces L_M and L_N, norms $\|\cdot\|_M$, $\|\cdot\|_M^0$, $\|\cdot\|_N$, $\|\cdot\|_N^0$ we define analogously as in case of vector-valued functions. The Hölder inequality and equivalence of the norms for $\mathscr{N}$-functions M and N remain also true.

1. Continuity of linear integral operators

Let μ_1 and μ_2 be non-negative, non-atomic, complete and σ-finite measures on T_1 and T_2 respectively, and assume that $\Phi\colon X\times T_1\to[0,\infty]$ and $M\colon \mathbf{R}\times T_2\to[0,\infty]$ are $\mathcal{N}$-functions. Let L_Φ and L_M be Musielak—Orlicz spaces generated by Φ and M, respectively. Let the function $K\colon T_2\times T_1\to Y$ be strongly $\mu_2\times\mu_1$-measurable. Then for every strongly μ_1-measurable vector-valued function $x\colon T_1\to X$, it is evident that $\langle K(\cdot\cdot,\cdot), x(\cdot)\rangle$ is strongly $\mu_2\times\mu_1$-measurable on $T_2\times T_1$. We shall consider an integral operator of the form

$$Ax(t)=\int_{T_1}\langle K(t,s),x(s)\rangle\,d\mu_1. \tag{1.1}$$

We shall naturally search for conditions for the continuity of A in terms of the properties of the kernel $K(t,s)$.

1.1. Theorem. *If*

$$\|K(t,\cdot)\|_\Psi<\infty \quad \text{a.e. in } T_2 \text{ and } \quad \|K(\cdot\cdot,\cdot)\|_\Psi\in L_M,$$

then A is a continuous operator from the space L_Φ into the space L_M.

Remark. The symbol $\|K(\cdot\cdot,\cdot)\|_\Psi$ denotes that the norm is calculated for K as a function of this variable which is denoted by one full-stop. $\|K(\cdot\cdot,\cdot)\|_\Psi$ is an element of L_M as a function of this variable which is symbolized by two full-stops.

PROOF. In view of the Hölder inequality, we have

$$\Big|\int_{T_2}Ax(t)y(t)\,d\mu_2\Big|\le\int_{T_2}\Big|\int_{T_1}\langle K(t,s),x(s)\rangle\,d\mu_1\Big|\,|y(t)|\,d\mu_2\le$$

$$\le\int_{T_2}\|K(t,\cdot)\|_\Psi\|x\|_\Phi^0|y(t)|\,d\mu_2=\|x\|_\Psi^0\int_{T_2}\|K(t,\cdot)\|_\Psi|y(t)|\,d\mu_2$$

for $x\in L_\Phi$ and $y\in L_N$. Hence, applying the definition of the Orlicz norm, we obtain

$$\|Ax\|_M^0=\sup_{I_N(y)\le1}\Big|\int_{T_2}Ax(t)y(t)\,d\mu_2\Big|\le$$

$$\le\|x\|_\Phi^0\sup_{I_N(y)\le1}\int_{T_2}\|K(t,\cdot)\|_\Psi|y(t)|\,d\mu_2\le\|x\|_\Phi^0\Big|\big\|K(\cdot\cdot,\cdot)\|_\Psi\Big|_M^0$$

for $x\in L_\Phi$. From the assumption of the theorem follows that

$$\Big|\big\|K(\cdot\cdot,\cdot)\|_\Psi\Big|_M^0<\infty,$$

and consequently, that A is a continuous operator from L_Φ into L_M. ∎

2. Compactness of the linear integral operator

In the present section, we shall study the problem of conditions for the compactness of the operator (1.1). We put

$$S_\Phi(r) = \{x \in L_\Phi : \|x\|^0_\Phi < r\}.$$

2.1. Theorem. *(A criterion for compactness of the linear integral operator.) Let $K(t, \cdot) \in E_\Psi$ for a.a. $t \in T_2$ and*

$$\int_{T_2} \langle K(\cdot, s), x(s)\rangle \, d\mu_1 \in E_M \quad \text{for} \quad x \in L_\Phi.$$

The operator A is compact if and only if the set

$$A[S_\Phi(1)] = \{Ax : x \in S_\Phi(1)\}$$

has equi-absolutely continuous norms.

PROOF OF NECESSITY. Assume that A is compact, but the set $A[S_\Phi(1)]$ has not equi-absolutely continuous norms. This means that there exist a sequence of μ_2-measurable sets $C_n \subset T$ with $C_n \downarrow \phi$ as $n \to \infty$, a number $\varepsilon_0 > 0$ and a sequence of functions $x_n \in S_\Phi(1)$ such that

$$\|Ax_n \chi_{C_n}\|_M > \varepsilon_0 \quad \text{for} \quad n = 1, 2, \ldots.$$

Since $A[S_\Phi(1)]$ is conditionally compact, we can extract a subsequence $\{Ax_{n_k}\}$ from the sequence $\{Ax_n\}$, which is convergent to an element $y \in L_M$. Obviously, y belongs to E_M as the limit of a sequence of the functions from E_M. Hence y has absolutely continuous norm. This implies that there exists a number k_1 such that

$$\|y\chi_{C_{n_k}}\|_M < \frac{\varepsilon_0}{2} \quad \text{for} \quad k > k_1.$$

Moreover, by convergence of the subsequence $\{Ax_{n_k}\}$ in E_M, a number k_2 can be found such that

$$\|Ax_{n_k} - y\|_M < \frac{\varepsilon_0}{2} \quad \text{for} \quad k > k_2.$$

Therefore,

$$\|Ax_{n_k}\chi_{C_{n_k}}\|_M \leqq \|Ax_{n_k} - y\|_M + \|y\chi_{C_{n_k}}\|_M < \varepsilon_0$$

for $k > \max\{k_1, k_2\}$, and we obtain a contradiction.

PROOF OF SUFFICIENCY. Suppose $K(t, \cdot) \in E_\Psi$ and the set $A[S_\Phi(1)]$ has equi-absolutely continuous norms. Let $\{T_{2_n}\}$ be an increasing subsequence of μ_2-measurable sets such that

$$\bigcup_{n=1}^{\infty} T_{2_n} = T_2 \quad \text{and} \quad \mu_2(T_{2_n}) < \infty \quad \text{for} \quad n = 1, 2, \ldots.$$

We choose $\varepsilon > 0$. Since $T_2 \setminus T_{2_n} \downarrow \phi$ as $n \to \infty$, a natural number c can be found such that

$$\|Ax\chi_{T/T_{2_c}}\|_M < \frac{\varepsilon}{4}$$

for $x \in S_\Phi(1)$. In virtue of Theorem 2.2 from [9], a ball in the space L_Φ is E_Ψ-weakly compact. Therefore, every sequence of elements of a ball contains an E_Ψ-weakly convergent subsequence. It suffices to prove that A transforms an E_Ψ-weakly convergent sequence into a sequence which converges in norm. Suppose the sequence $x_n \in S_\Phi(1)$ $(n=1, 2, \ldots)$ is E_Ψ-weakly convergent to the function $x_0 \in S_\Phi(1)$. By definition of E_Ψ-weak convergence and by the fact that $K(t, \cdot) \in E_\Psi$ a.e. in T_1, we have

$$Ax_n(t) = \int_{T_1} \langle K(t,s), x_n(s)\rangle \, d\mu_1 \xrightarrow[n\to\infty]{} \int_{T_1} \langle K(t,s), x_0(s)\rangle \, d\mu_1$$

a.e. in T_2. Hence $\chi_{T_{2_c}} Ax_n$ is convergent to $\chi_{T_{2_c}} Ax_0$ everywhere, and consequently, it converges to this function in measure. We note that $\chi_{T_{2_c}}(t) Ax(t) = A[\chi_{T_{2_c}} x](t)$. Obviously, $\chi_{T_{2_c}} x \in S_\Phi(1)$ for $x \in S_\Phi(1)$. From this it follows that the set

$$\{\chi_{T_{2_c}} Ax \colon \ x \in S_\Phi(1)\}$$

has equi-absolutely continuous norms, too. It is know that if a sequence convergent in measure has equi-absolutely continuous norms, then it is convergent in norm (see Lemma 2.3 in [8]). Thus

$$\|\chi_{T_{2_c}} Ax_n - \chi_{T_{2_c}} Ax_0\|_M < \frac{\varepsilon}{2}$$

for $n > n_0$. Therefore, for an arbitrary sequence $\{x_n\}$ of elements of the ball $S_\Phi(1)$, which is E_Ψ-weakly convergent to the function $x_0 \in S_\Phi(1)$, we have

$$\|Ax_n - Ax_0\|_M \leqq \|\chi_{T_{2_c}}(Ax_n - Ax_0)\|_M + \|\chi_{T_2 \setminus T_{2_c}}(Ax_n - Ax_0)\|_M <$$

$$< \frac{\varepsilon}{2} + \|\chi_{T_2 \setminus T_{2_c}} Ax_n\|_M + \|\chi_{T_2 \setminus T_{2_c}} Ax_0\|_M < \varepsilon,$$

provided $n > n_0$, and this completes the proof. ∎

2.2. Corollary. *If the kernel* $K(t, s)$ *as a function of the variable* s *belongs to* E_Ψ *for a.e.* $t \in T_2$ *and* $\|K(\cdot\cdot, \cdot)\|_\Psi \in E_M$, *then the linear operator* (1.1) *is a compact operator from the space* L_Φ *into the space* E_M.

PROOF. It sufficies to verify that the assumptions of Theorem 2.1 are satisfied. We note that, by Hölder inequality, the operator A acts from L_Φ into E_M. Indeed, let $x \in L_\Phi$. Then

$$|Ax(t)| = \Big|\int_{T_1} \langle K(t,s), x(s)\rangle \, d\mu_1\Big| \leqq \|K(t, \cdot)\|_\Psi \|x\|_\Phi^0 \quad \text{a.e. in} \quad T_2.$$

From this and by the assumption $\|K(t, \cdot)\|_\Psi \in E_M$ follows that $Ax \in E_M$.

Now, we shall show that the set $A[S_\Phi(1)]$ has equi-absolutely continuous norms. Let $\{C_n\}$ be a decreasing sequence which is convergent to the empty set. For $x \in S_\Phi(1)$ and $y \in L_N$, applying the Hölder inequality, we obtain

$$\Big|\int_{C_n} Ax(t) y(t) \, d\mu_2\Big| \leqq \int_{C_n} \Big|\int_{T_2} \langle K(t,s), x(s)\rangle d\mu_1\Big| \, |y(t)| \, d\mu_2 \leqq$$

$$\leqq \int_{C_n} \|K(t, \cdot)\|_\Psi \|x\|_\Phi^0 |y(t)| \, d\mu_2 \leqq \int_{C_n} \|K(t, \cdot)\|_\Psi |y(t)| \, d\mu_1.$$

In virtue of Theorem 0.1, for every $\varepsilon>0$ there exists a positive integer n_0 such that for every $x\in S_\Phi(1)$ and $n>n_0$, we have

$$\|Ax\chi_{C_n}\|_M^0 = \sup_{I_N(y)\leq 1} \Big|\int_{C_n} Ax(t)y(t)\,d\mu_2\Big| \leqq A\|K(\cdot\,\cdot,\cdot)\|_\Psi \chi_{C_n}\|_M^0 < \varepsilon.$$

Hence, the set $A[S_\Phi(1)]$ has equi-absolutely continuous norms. Thus, Theorem 2.1 implies the conclusion. ∎

3. Some remarks on compactness of Hammerstein operators

3.1. Definition. Suppose the function $f\colon T_1\times X\to X$ satisfies the Carathéodory conditions, i.e. it is continuous in X for almost all $s\in T_1$ and measurable for every fixed $u\in X$. The operator F, defined by the formula

$$[Fx](s) = f(s, x(s)),$$

where $x\in S(X)$, $s\in T_1$, is called a superposition operator.

The fundamental properties of the superposition operator are presented in papers [6] and [7]. Among other things, the following results are presented there:

3.2. Theorem. *If the operator F acts from $\Pi(E_{\Phi_1}, r)$ into E_{Φ_2} then F is continuous at every point of $\Pi(E_{\Phi_1}, r)$.*

3.3. Theorem. *Suppose the operator F acts from the ball $S_{\Phi_1}(r)$ into* $\operatorname{dom} I_{\Phi_2}$. *Then F is bounded on any ball $S_{\Phi_1}(r_0)$ for $r_0<r$, i.e.*

$$\sup_{\|x\|_{\Phi_1}^0<r_0} \|Fx\|_{\Phi_2} < \infty.$$

3.4. *Definition.* The nonlinear integral operator

$$Hx(t) = \int_{T_2} \langle K(t,s), f(s, x(s))\rangle\, d\mu_1 \tag{3.1}$$

is called the Hammerstein operator.

This operator can be represented as the composition of the nonlinear operator F and a linear operator (1.1). Combining the conditions under which the operator F acts from the space L_{Φ_1} into L_{Φ_2} and is continuous and bounded on L_{Φ_1} with the conditions under which the operator (1.1) acts from L_{Φ_2} into E_M and is compact, we arrive at the conditions for compactness of the operator (3.1).

Let Ψ_1 and Ψ_2 be complementary functions to Φ_1 and Φ_2, respectively.

3.5. Theorem. *Let Φ_1 satisfy the Δ_2 condition. If the kernel $K(t,s)$ of the Hammerstein operator as a function of s belongs to E_{Ψ_2} for almost all $t\in T_2$, and $\|K(\cdot\cdot,\cdot)\|_{\Psi_2}\in E_M$, and F acts from L_{Φ_1} into E_{Φ_2}, then the Hammerstein operator* (3.1) *is a continuous and compact operator from L_{Φ_1} into E_M.* ∎

4. The existence of solutions of integral equations in Musielak—Orlicz space

Now, we shall apply the results from preceding sections to the theorem on existence of solutions of integral equations in Musielak—Orlicz space. To this end, we suppose: $T_1=T_2=T$, $\mu_1=\mu_2=\mu$, $\Phi_1=\Phi_2=M$, $\Psi_1=\Psi_2=N$, and $X=Y=\mathbf{R}$. We shall consider the integral equation

$$x(t) = \varkappa \int_T K(t,s)x(s)\,d\mu + z(t) \quad \text{for} \quad x\in E_M, \tag{4.1}$$

where $\varkappa$ is a real number, z is an element of the space E_M and the kernel $K\colon T\times T\to\mathbf{R}$ is a $\mu\times\mu$-measurable function.

4.1. Theorem. *Let the kernel* $K(t,s)$ *as a function of the variable* s *belong to the space* E_N *for almost all* $t\in T$ *and let* $\|K(\cdot\cdot,\,\cdot)\|_N$ *be an element of the space* E_M. *Then the integral equation* (4.1) *has at least one solution in the ball with radius*

$$r > \frac{\|z\|_M^0}{1-2|\varkappa|\,\|\,\|K(\cdot\,\cdot,\,\cdot)\|_N\|_M}, \tag{4.2}$$

for

$$|\varkappa| \leqq \frac{1}{2\|\,\|K(\cdot\,\cdot,\,\cdot)\|_N\|_M}. \tag{4.3}$$

Proof. Let A be an integral operator defined by the formula (1.1). By Corollary 2.2, the operator A is a compact operator from the space E_M into itself, so the operator B defined by the formula

$$Bx = \varkappa Ax + z,$$

where $\varkappa$ and z are as in the equation (4.1), is also a compact operator from the space E_M into itself. First, we shall estimate the norm of elements of the set of values of the operator A on some ball $S_M^E(r)=\{x\in E_M\colon \|x\|_M^O<r\}$. Using in turn the Fubini theorem and twice the Hölder inequality, we obtain

$$\Big|\int_T Ax(t)y(t)\,d\mu\Big| < r\|y\|_N\,\|\,\|K(\cdot\,\cdot,\,\cdot)\|_N\|_M^0,$$

for $x\in S_M^E(r)\subset E_M$ and $y\in L_N$. Hence, by the definition of the Orlicz norm and by the equivalence of Orlicz and Luxemburg norms, we have

$$\|Ax\|_M^0 = \sup_{I_N(y)\leqq 1}\Big|\int_T Ax(t)y(t)\,d\mu\Big| < 2r\|\,\|K(\cdot\,\cdot,\,\cdot)\|_N\|_M,$$

for $x\in S_M^E(r)$. Resuming to the operator B, let us note that the inclusion

$$B[S_M^E(r)] \subset S_M^E(r)$$

follows from the inequality

$$\|Bx\|_M^0 \leqq |\varkappa|\,\|Ax\|_M^0 + \|z\|_M^0 < 2r|\varkappa|\,\|\,\|K(\cdot\,\cdot,\,\cdot)\|_N\|_M + \|z\|_M^0 < r.$$

It is easy to verify that the above inequality ist rue if r and $\varkappa$ satisfy the inequalities (4.2) and (4.3), respectively. Thus, in virtue of Schauder's fixed-point principle, there exists at least one element $x_0 \in S_M^E(r)$ such that $Bx_0 = x_0$. Therefore

$$x_0(t) = Bx_0(t) = \varkappa \int_T K(t, s) x_0(s)\, d\mu + z(t),$$

so x_0 is the solution of the integral equation (4.1). ▮

Finishing this note, we shall consider the integral equation of Hammerstein type in form

$$x(t) = \varkappa \int_T K(t, s) f(s, x(s))\, d\mu + z(t), \tag{4.4}$$

where $z \in E_M$ and the functions K and f are as in Definition 3.4.

Puting $B = \varkappa Hx + z$ and reasoning similarly as in the proof of Theorem 4.1, we obtain a theorem on existence of solutions of integral equations of Hammerstein type.

4.2. Theorem. *Let an $\mathcal{N}$-function M satisfy the Δ_2 condition and let the superposition operator F act from L_M into itself. If $K(t, \cdot) \in E_N$ for almost all $t \in T$ and $\|K(\cdot\cdot, \cdot)\|_N \in L_M$, then for every $r > \|z\|_M^0$ the integral equation* (4.4) *has at least one solution in the ball $S_M(r)$ for*

$$|\varkappa| < \frac{r - \|z\|_M^0}{a \|\|K(\cdot\cdot, \cdot)\|_N\|_M},$$

where

$$a = \sup_{x_M \in S(r)} \|Fx\|_M^0. \quad ▮$$

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