Injective and projective graphs

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Introduction

The category of graphs*), G, has as objects all finite graphs and as morphisms all graph homomorphisms. In this paper we beging a study of G by establishing that G has enough injective objects, and then demonstrate the existence of an injective hull of a graph. Dually, we then describe the projective objects of G.

Injective Graphs and Injective Hulls

The injectivity of graph I is concerned with the lifting of graph homomorphism into I.

Definition. A graph I is *injective* if, and only if, given any graph B and any subgraph A of B, each graph homomorphism $f: A \to I$ has an extension $\hat{f}: B \to I$ which is also a graph homomorphism.

The characterization of injective graphs is straightforward.

Theorem 1. *A graph G is injective if, and only if, G is a complete graph.*

PROOF. "if": Let A be a subgraph of B and $f: A \to K_n$ be a graph homomorphism. Extend f to $\hat{f}: B \to K_n$ by choosing some $k \in K_n$ and defining $\hat{f}(b) = k$ for all $b \in B \setminus A$. Clearly, $\hat{f}$ preserves edges since K_n is complete.

"only if": Let G be injective, $|G| = n$ and suppose there are $a, b \in G$ not connected by an edge. Let $A = \{a, b\}$ be the discrete subgraph of $B = K_2$ and define $f(a) = a$ and $f(b) = b$. Since G is injective, f has a graph homomorphism extension $\hat{f}: K_2 \to G$ and since there is an edge connecting a and b in K_2, there must be an edge connecting $a = \hat{f}(a)$ and $b = \hat{f}(b)$ in G. This is a contradiction so $G = K_n$.

Definition. A graph G is *self-injective* if, and only if, each graph homomorphism $f: A \to G$ from a subgraph A of G can be extended to a graph endomorphism of G.

*) All of our graphs may have loops.

Corollary 2. *A graph G is self-injective if, and only if, $G=K_n$, or G is totally disconnected.*

PROOF. "if": If G is totally disconnected, G is self-injective (but *not* injective unless $G=K_1$). If $G=K_n$, G is injective and so is certainly self-injective.

"only if": If G is self-injective it is either totally disconnected or has an edge. Let the edge join a and b. If $x, y \in G$ are not connected by an edge, let $A=\{a, b\}$ have no edges and define $f\colon A \to G$ by $f(a)=x, f(b)=y$. For G to be self-injective there is $g\colon G \to G$ and so $x=g(a)$ and $y=g(b)$ are joined by an edge.

We now turn out attention to the existence of injective hulls.

Definition. A subgraph A of B is *essential* in B if, and only if, for any graph G, if $f\colon B \to G$ is a graph homomorphism whose restriction to A is one-to-one, then $f\colon B \to G$ is also one-to-one. B is also called an *essential extension* of A.

Lemma 3. *Let A be a subgraph of B which is a subgraph of C, then A is essential in C if, and only if, A is essential in B and B is essential in C.*

PROOF. "if": Let $f\colon C \to G$ be a graph homomorphism whose restriction to A is one-to-one. Then, since A is essential in B, $f_B\colon B \to C$, the restriction of f to B, is one-to-one. Since B is essential in C, f is one-to-one. Hence A is essential in C.

"only if": Let $f\colon C \to G$ be a graph homomorphism whose restriction f_B to B is one-to-one. Then f_A is one-to-one so f is one-to-one since A is essential in C. Thus B is essential in C.

Let $f\colon B \to G$ be a graph homomorphism whose restriction to A is one-to-one. If $|G|=n$, then $G \subseteq K_n$ and since K_n is injective, f has an extension $\hat{f}\colon C \to K_n$, and $\hat{f}_A = f_A$ is one-to-one. Since A is essential in C, $\hat{f}$ is one-to-one and so f is one-to-one. Hence A is essential in B.

Theorem 4. *The following statements are equivalent for a subgraph A of B:*

(1) *B is an injective, essential extension of A,*

(2) *B is a maximal essential extension of A,*

(3) *B is a minimal injective extension of A.*

PROOF. (1)→(2): Suppose B is not maximal and $A \subseteq B \subseteq C$ with C an essential extension of A. Since B is injective, there is a graph homomorphism $g\colon C \to B$ extending the identity mapping $i\colon B \to B$. Since i is one-to-one, g is one-to-one. Since g maps C onto B, $|B|=|C|=n$. But then $B=K_n$ since B is injective so $C=B$. Thus B is a maximal essential extension of A.

(2)→(3): Let B be a maximal essential extension of A and C be a minimal injective extension of A. Then if $|A|=n$, $C=K_n$, since K_n is injective. Without loss of generality, $A \subseteq K_n$. The identity mapping $i\colon A \to A \subseteq K_n = C$ has an extension $f\colon B \to C$ which is one-to-one. Thus $|B|=n$ so, without loss of generality, $A \subseteq B \subseteq C$. However, C is essential over A since $|A|=|C|=n$. Thus $B=C$ and B is a minimal injective extension of A.

$(3) \to (1)$: Let B be a minimal injective extension of A. Thus if $|A|=n$, $B=K_n$. But K_n is essential over A so B is an injective essential extension of A.

Definition. A graph G is the *injective hull* of a subgraph A if, and only if, G is an injective essential extension of A.

Theorem 5. *Let B and C be injective hulls of A. Then there is a graph isomorphism g: $B \to C$ whose restriction to A is the identity map i: $A \to A$.*

PROOF. By the injectivity of B and C there exists a graph homomorphism g: $B \to C$ as shown in Figure 1.

$$\begin{array}{ccc} A & \subseteq & B \\ \| & & \downarrow g \\ A & \subseteq & C \end{array}$$

Figure 1.

Since $|B|=|C|=|A|=n$, g maps B onto C since g is one-to-one since B is essential over A. By Theorem 3, C is complete so g^{-1} is a bijection which preserves edges and so is a graph isomorphism.

Given a subgraph A of B and a graph homomorphism f: $A \to C$, we ask when f has a unique extension $\hat{f}$: $B \to C$.

Definition. A subgraph A is *dense* in B if, and only if, whenever f: $B \to C$ and g: $B \to C$ are graph homomorphisms which agree on A, then $f=g$.

Lemma 6. *A is dense in B if, and only if, $|A|=|B|$.*

PROOF. "only if": Let $C=K_n$, $n=|B|$ and let f: $A \to C$ be the identity mapping. Extend f to g_1: $B \to C$ by $g_1(b)=a_1$ for all $b \in B \setminus A$, and g_2: $B \to C$ be the insertion mapping. Then $g_1(a)=g_2(a)$ for $a \in A$ but $g_1(b) \neq g_2(b)$. Thus $|A|=|B|$.

"if": If f: $B \to C$ and g: $B \to C$ agree on A, then they agree on the vertex set of B and so are equal.

Corollary 7. *If A is essential in B and B is self-injective, then* $\mathrm{End}\,(A)=\mathrm{End}\,(B)$ *where* $\mathrm{End}\,(G)$ *is the collection of graph endomorphisms of G.*

Projective Graphs

In Category Theory, the dual concept to "injective" is "projective".

Definition. A graph P is *projective* if, and only if, given graphs A and B and an onto graph homomorphism f: $A \to B$, for each graph homomorphism g: $P \to B$, there exists a graph homomorphism h: $P \to A$ for which $f \cdot h=g$.

We characterize projective graphs as totally disconnected graphs in the following theorem.

Theorem 8. *A graph P is projective if, and only if, P is the disjoint union of a totally disconnected graph and finitely many copies of K_2.*

PROOF. "if": Let f: $A \to B$ be a graph epimorphism and g: $P \to B$ be a graph homomorphism. Define h: $P \to A$ by choosing for each edge pq in P, an edge xy in A for which $f(x)=g(p)$ and $f(y)=g(q)$, and set $h(p)=x$ and $h(q)=y$. This guarantees that f preserves edges. For each isolated vertex $p \in P$, choose $a \in A$ with $f(a)=g(p)$, then set $h(p)=a$. Then h is a graph homomorphism and $f \circ h=g$.

"only if": Let P be projective. We will show that P has no paths of length 2 or more. Let g: $P \to P$ be the identity homomorphism. Define the graph A as follows: (1) for each edge e in P, form the complete graph K_e with vertices a_e, b_e in A and edge $a_e b_e$, (2) for $e \neq f$ require K_e and K_f to be disjoint, (3) for each isolated vertex $p \in P$ adjoin the isolated vertex a_p to A. Then P is a homomorphic image of A by the map f: $A \to P$ given by $f(a_e)=p$, $f(b_e)=q$ where $pq=e$ is an edge in P, and $f(a_p)=p$ for isolated vertices a_p in A. Clearly, if P has a path of length 2 or more, there is no graph homomorphism h: $P \to A$ with $f \circ h=q$.

Apologia

In the last several pages we have characterized injective and projective graphs. While the problem of characterizing such graphs is *a priori* interesting, the results are somewhat disappointing. The reason that injective graphs are complete and projective graphs are totally disconnected arises from the fact that we consider all subgraphs of a graph. Perhaps it might be more interesting to consider only induced subgraphs of a graph. However, in this situation we lose the possibility of each graph possessing an injective hull as our discussion of the injective hull demonstrates.

In the next two sections we consider injectives in the categories of bipartite graphs and trees.

Bipartite graphs

Recall that a (non directed) graph $G=(V, E)$ is *bipartite* if, and only if, V is partitioned into disjoint nonempty subsets V_1 and V_2 with $E \subset V_1 \times V_2$. Let $G=(V, E)$ and $H=(U, F)$ be bipartite graphs. A *graph homomorphism* f: $G \to H$ satisfies f: $V_1 \to U_1$, f: $V_2 \to U_2$ and $v_1 E v_2$ implies $f(v_1) F f(v_2)$.

Theorem 9. *A bipartite graph $G=(V, E)$ is injective in the category of bipartite graphs if, and only if. $G=K_{m,n}$ where V is partitioned into V_1 and V_2 with $|V_1|=m$ and $|V_2|=n$.*

PROOF. "if": Let G be a bipartite graph with parts V_1 and V_2, A be a subgraph with parts $A_1 \subset V_1$ and $A_2 \subset V_2$, and let f: $A \to K_{m,n}$ be any homomrphism. Extend f to $\hat{f}: G \to K_{m,n}$ by choosing some k and k' in the two parts of $K_{m,n}$; and setting $\hat{f}(a)=k$ for all $a \in V_1 \setminus A_1$ and $\hat{f}(a)=k$ for all $a \in V_2 \setminus A_2$. Clearly $\hat{f}$ preserves edges since $K_{m,n}$ is complete.

"only if": Let $G=(V, E)$ be an injective bipartite graph with parts V_1 and V_2. Suppose there is $v_1 \in V_1$ and $v_2 \in V_2$ not connected by an edge. Let $A=\{a, b\}$ be the discrete subgraph of K_2, and define f: $A \to G$ with $f(a)=v_1$ and $f(b)=v_2$.

Since G is injective f has an extension $\hat{f}$: $K_2 \to G$. Since there is an edge joining a and b in K_2, there must be an edge connecting v_1 and v_2 in G. This is a contradiction. Thus G is a complete bipartite graph.

Similar to the case for graphs, we have the following Theorem for bipartite graphs. Since the proof for the bipartite case is similar to the general case, it is omitted.

Theorem 10. *Let G be a bipartite graphs with parts* $|G_1|=m$ *and* $|G_2|=n$. *Then G has an injective hull isomorphic to* $K_{m,n}$.

Forests and Trees

Recall that a *forest* is an acyclic graph (possibly with loops). A subobject of a forest is a forest.

In the category of forests, K_2 is the only injective object as shown by Theorem 11.

Theorem 11. *A forest F is injective if, and only if,* $F=K_1$ *or* $F=K_2$.

PROOF. "if": K_1 and K_2 are injective in the category of graphs and so are injective in the category of forests.

"only if": Let F be an injective forest. It is easily seen that F is complete. The only complete forests are K_1 and K_2.

Recall that a *Tree* is a connected acyclic graph. Theorem 11 shows that if we allow non-induced subgraphs, we require our injective objects to be complete. This contrasts with Theorem 12 below.

Notation. Let $\mathcal{T}$ be the category of the trees with induced subtrees as subobjects.

Theorem 12. *In the category* $\mathcal{T}$, *all trees are injective.*

PROOF. Let $A, B, T \in \mathcal{T}$ with A a subtree of B and f: $A \to T$ be a tree homomorphism. For each $b \in B \setminus A$, there is a unique leaf $a \in A$ for which the path from a to b is of shortest length (otherwise B would have a cycle). Extend f: $A \to T$ to $\hat{f}$: $B \to T$ by requiring $\hat{f}(b)=f(a)$. Then $\hat{f}$ is a tree homomorphism and $\hat{f}(B)$ is a subtree of T. Thus T is injective.

While in the category $\mathcal{T}$, injectives need not be complete, the fact that all trees are injective is surprising. This fact does point out that the concept of an injective hull is trivial in this category.

In Theorem 12, if we require that trees have no loops (the more commonly accepted definition), the result holds true.

If we require that all subobjects of trees be induced subforests, we have the following result.

Theorem 13. *If in the category of trees we require subobjects to be induced subthe forests, injective objects are stars.*

PROOF. Let P_3 be the path with three vertices as pictured in Figure 2 and denote the injective tree as I.

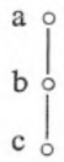

Figure 2.

Let $S=\{a, c\}$ be the discrete induced subforest of P_3. Given $x, y \in I$, f: $S \to I$ defined by $f(a)=x, f(c)=y$, has an extension $\hat{f}$: $P_3 \to I$. Thus x and y are either adjacent or connected by a path of length 2. This says that I is a tree of diameter 2, or a star. (Note that if $f(a)=f(c)$, the star has a loop on its center.)

Conversely let S be a star (possibly with a loop on its center) B be a tree, and A an induced subforest of B. Suppose f: $A \to S$ is a graph homomorphism. Define $\hat{f}$: $B \to S$, an extension of f, by $\hat{f}(b)=c$, the center of S, for $b \in B \setminus A$. Then f is a graph homomorphism for if $b_1 b_2$ is a edge of B, without loss of generality $f(b_2)=c$ and $f(b_1)$ is either a leaf of S or c. In either case, $f(b_1)f(b_2)$ is an edge of S.

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