# Remarks on generalized homogeneous deviation means

By ZSOLT PÁLES (Debrecen)

**1. Introduction.** In a previous paper DARÓCZY and PÁLES [1] introduced the following concept:

A mean $M$ defined on the set of positive numbers is called *k-homogeneous* if

$$M_{nk}(t\circ x) = M_k(t)\,M_n(x) \tag{1}$$

holds for all

$$n\in\mathbf{N},\ x=(x_1,\ldots,x_n)\in\mathbf{R}_+^n,\ t=(t_1,\ldots,t_k)\in\mathbf{R}_+^k$$

where

$$t\circ x = (t_1x_1,\ldots,t_1x_n,\ldots,t_kx_1,\ldots,t_kx_n)\in\mathbf{R}_+^{nk}.$$

It is easy to see that 1-homogeneous means are homogeneous in the usual sense.

The problem investigated in [1] was to find all the $k$-homogeneous deviation means if $k\geqq 2$. A mean $M$ on $\mathbf{R}_+$ is called a *deviation mean* if there exists a *deviation* (i.e. a function $E\colon \mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}$ with the properties

E1) $y\to E(x,y)$ is strictly decreasing for each fixed $x\in\mathbf{R}$;

E2) $E(x,x)=0$ for all $x\in\mathbf{R}_+$)

such that $y=M_n(x_1,\ldots,x_n)$ is the unique solution of the equation

$$E(x_1,y)+\ldots+E(x_n,y)=0$$

for all $n\in\mathbf{N}$, $x_1,\ldots,x_n\in\mathbf{R}_+$. In this case we say that $M$ is generated by $E$ and we write $M=\mathfrak{M}_E$.

In [1] the following result was proved:

**Theorem 1.** *If $E$ is a differentiable deviation on $\mathbf{R}_+$, that is*

$$E_2(x,y)=\frac{\partial}{\partial y}E(x,y)$$

*exists for each $x,y\in\mathbf{R}_+$ and it is negative then $M=\mathfrak{M}_E$ is k-homogeneous for some fixed $k\geqq 2$ if and only if there exist a multiplicative function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ (i.e. $m(xy)=m(x)m(y)$ for $x,y\in\mathbf{R}_+$) and a positive constant $a$ such that either*

$$M_n(x_1,\ldots,x_n)=\exp\Big(\sum_{i=1}^n m(x_i)\ln x_i\Big/\sum_{i=1}^n m(x_i)\Big) \tag{2}$$

*for all* $n \in \mathbf{N}$ *and* $x_1, \ldots, x_n \in \mathbf{R}_+$, *or*

$$M_n(x_1, \ldots, x_n) = \Big(\sum_{i=1}^{n} m(x_i) x_i^a \Big/ \sum_{i=1}^{n} m(x_i)\Big)^{1/a} \tag{3}$$

*for all* $n \in \mathbf{N}$ *and* $x_1, \ldots, x_n \in \mathbf{R}_+$.

The aim of the present note is to prove this theorem without assuming the differentiability of $E$.

**2. Preliminary results.** Let $E$ be a fixed deviation on $\mathbf{R}_+$ throughout this paper and denote

$$f(x) = E(x, 1) \tag{4}$$

$$\mu(x) = \mathfrak{M}_{k,E}(x, \underbrace{1, \ldots, 1}_{k-1}) \tag{5}$$

*for* $x \in \mathbf{R}_+$.

Using these notations the following result can be verified (see [1] for the details):

**Theorem 2.** *If* $M = \mathfrak{M}_E$ *is a k-homogeneous mean for some fixed* $k \geqq 2$ *then there exists a function* $h\colon \mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$E(x, y) = h(y) f(x/y) \tag{6}$$

*for* $x, y \in \mathbf{R}_+$, *further, the functions f and* $\mu$ *satisfy the functional equation*

$$\frac{f(xy)}{f(x)f(y)} + \frac{k-1}{f(x)} = \frac{f(\mu(x)y)}{f(\mu(x))f(y)} \tag{7}$$

*for all* $x, y \in \mathbf{R}_+ \setminus \{1\}$.

Based on this assertion the following theorem was proved in [1, Th. 2].

**Theorem 3.** *If E is a differentiable deviation on* $\mathbf{R}_+$ *and* $M = \mathfrak{M}_E$ *is a k-homogeneous mean* ($k \geqq 2$ *is fixed*), *then f satisfies the functional equation*

$$2\frac{f(xy)}{f(x)f(y)} = \frac{f(x^2)}{f^2(x)} + \frac{f(y^2)}{f^2(y)} \tag{8}$$

*for any* $x, y \in \mathbf{R}_+ \setminus \{1\}$.

The proof of this theorem was the only point in [1] where the differentiability of $E$ was used. Therefore, if we can deduce functional equation (8) from Theorem 2 without assuming $E$ to be differentiable then we verify Theorem 1 for all deviations.

**3. Main results.** It follows from the definition of deviations and deviation means that

$$\operatorname{sign} f(x) = \operatorname{sign}\big(\mu(x) - 1\big) = \operatorname{sign}\big(x - \mu(x)\big) = \operatorname{sign}(x - 1) \tag{9}$$

for all $x \in \mathbf{R}_+$.

If $E$ generates a multiplicative mean then, by Theorem 2,

$$f(xy)/f(x) = E(y, 1/x)/E(1, 1/x) \quad (x, y \in \mathbf{R}_+, \quad x \neq 1).$$

Thus, by property E1) of deviations we can see that

$$x \to f(xy)/f(x) \tag{10}$$

is a continuous function on $\mathbf{R}_+ \setminus \{1\}$ for each fixed $y \in \mathbf{R}_+$.

Now we can state our main result:

**Theorem 4.** *Let $f$: $\mathbf{R}_+ \to \mathbf{R}$ and $\mu$: $\mathbf{R}_+ \to \mathbf{R}_+$ be arbitrary functions. Assume that*

(i) (9) *is satisfied if* $x \in \mathbf{R}_+$;

(ii) (10) *is a continuous function on* $\mathbf{R}_+ \setminus \{1\}$ *for all fixed* $y \in \mathbf{R}_+$;

(iii) (7) *holds for all* $x, y \in \mathbf{R}_+ \setminus \{1\}$.

*Then $f$ satisfies functional equation* (8) *for all* $x, y \in \mathbf{R}_+ \setminus \{1\}$.

PROOF. Let $y, z \in \mathbf{R}_+ \setminus \{1\}$ be fixed and define the function $G = G_{y,z}$ in the following way

$$G(x) = G_{y,z}(x) = \frac{f(xy)}{f(x)f(y)} - \frac{f(xz)}{f(x)f(z)}$$

for $x \in \mathbf{R}_+ \setminus \{1\}$.

We are going to show that $G$ is the constant function. First we deduce two identities for $G$. Put $z$ into (7) instead of $y$ and subtract the resulting equation from (7). Then we get

$$G(x) = G(\mu(x)) \tag{11}$$

for $x \in \mathbf{R}_+ \setminus \{1\}$.

Let $t \in \mathbf{R}_+$ with $1 \neq t \neq 1/\mu(x)$ and put $y = yt$ and $y = t$ into (7) to obtain

$$f(\mu(x))[f(xty) + (k-1)f(ty)] = f(x)f(\mu(x)ty)$$

and

$$f(x)f(\mu(x)t) = f(\mu(x))[f(xt) + (k-1)f(t)]. \tag{12}$$

Multiplying these inequalities and then dividing by $f(x)f(\mu(x)) \neq 0$, we have

$$f(\mu(x)t)[f(xty) + (k-1)f(ty)] = f(\mu(x)ty)[f(xt) + (k-1)f(t)]. \tag{13}$$

Since $f(x)f(\mu(x))f(\mu(x)t) \neq 0$, hence, by (12), $f(xt) + (k-1)f(t) \neq 0$. Therefore it follows from (13) that

$$\frac{f(xty)}{f(xt)f(y)} \frac{f(xt)}{f(xt)+(k-1)f(t)} + \frac{f(ty)}{f(t)f(y)} \frac{(k-1)f(t)}{f(xt)+(k-1)f(t)} = \frac{f(\mu(x)ty)}{f(\mu(x)t)f(y)}. \tag{14}$$

Putting $y = z$ into (14) and substracting the resulting equation from (14) we obtain

$$G(xt)\frac{f(xt)}{f(xt)+(k-1)f(t)} + G(t)\frac{(k-1)f(t)}{f(xt)+(k-1)f(t)} = G(\mu(x)t). \tag{15}$$

Now let $x_0 > 1$ be fixed and define

$$H_{x_0} = \{x > 1 | G(x) = G(x_0)\}.$$

By our assumption (ii), $G$ is a continuous function on $]1, \infty[$, therefore $H_{x_0}$ is a closed set in the interval $]1, \infty[$.

Based on (15), we shall prove that $H_{x_0}$ is convex. If $H_{x_0}$ were not convex then (since it is closed) there exists $t, s \in H_{x_0}$, $t<s$ such that $r \notin H_{x_0}$ if $t<r<s$. Let $x=s/t$ in (15). Then $G(xt)=G(s)=G(t)=G(x_0)$ since $s, t \in H_{x_0}$. Therefore $G(\mu(x)t)= =G(x_0)$, i.e.

$$r = \mu(x)\,t = \mu(s/t)\,t \in H_{x_0}.$$

However, by assumption (i),

$$t < \mu(s/t)\,t < s,$$

which is a contradiction.

The convexity of $H_{x_0}$ means that it is a subinterval of $]1, \infty[$. If

$$\inf H_{x_0} = x^* > 1$$

then (since $H_{x_0}$ is closed) $x^* \in H_{x_0}$. On the other hand, by (11), we have that $\mu(x^*)$ is also contained in $H_{x_0}$. But (i) implies that $1<\mu(x^*)<x^*$ which contradicts $x^*>1$. This contradiction proves that $x^*=1$.

To prove that $G(x)=G(x_0)$ for all $x \in ]1, \infty[$, write $x^*=\min(x_0, x)$. Now $x^* \in H_{x_0} \cap H_x$ since $]1, x_0] \subseteqq H_{x_0}$, $]1, x] \subseteqq H_x$. But then, by the definition of $H_{x_0}$ and $H_x$, $G(x_0)=G(x^*)=G(x)$.

Therefore we have seen that $G(x)=c'$ for $x>1$. A similar argument shows that $G(x)=c''$ for $0<x<1$ is also valid. We have only to verify that $c'=c''$. Let $x>1$ and choose $t \in \mathbf{R}_+$ so that $t<1$ and $tx, t\mu(x)>1$. Then, applying (15), we obtain $c'=c''$.

Now we can show that (8) holds. Let $x, y \in \mathbf{R}_+ \setminus \{1\}$. As we have proved, $G=G_{x,y}$ is a constant function. Therefore

$$G_{x,y}(x) = G_{x,y}(y)$$

i.e. (8) is satisfied.

Thus the proof of the theorem is complete.

Based on the above result, we can restate Theorem 3 in the following form

**Theorem 3*.** *If $E$ is a deviation on $\mathbf{R}_+$ and $M=\mathfrak{M}_E$ is a $k$-homogeneous mean ($k \geqq 2$ is fixed) then $f(x)=E(x, 1)$ satisfies the functional equation* (8) *for all $x, y \in \mathbf{R}_+ \setminus \{1\}$.*

Using the same method that was followed in [1], Theorem 3 implies

**Theorem 1*.** *If $M$ is a deviation mean then it is $k$-homogeneous for some fixed $k \geqq 2$ if and only if there exist a multiplicative function $m$: $\mathbf{R}_+ \to \mathbf{R}_+$ and a positive, constant $a$ such that either* (2) *or* (3) *is satisfied for all $n \in \mathbf{N}$ and $x_1, \dots, x_n \in \mathbf{R}_+$.*

## Reference

[1] Z. Daróczy—Zs. Páles, Generalized homogeneous deviation means, *Publ. Math. (Debrecen)* **33** (1986), 53—65.

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