Prolongations of G-structures to the frame bundle of second order

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Introduction

Let M be an n-dimensional differentiable manifold, TM its tangent bundle $T^r M$ its tangent bundle of order r, $r \geqq 2$, FM its frame bundle and $F^2 M$ its frame bundle of second order.

The theory of prolongations of G-structures on M to appropriate G-structures on TM and $T^r M$ has been introduced by A. MORIMOTO in [7] and [8], respectively. Recently, L. A. CORDERO and M. DE LEÓN [1] has developed a similar theory for FM.

In this paper, we extend this theory for $F^2 M$ and show, by studying the prolongations of some classical G-structures on M, how the different definitions of lifts given in [3], [4] and [13] fit nicely in this general framework.

In a forthcoming paper, we shall show that a similar theory can be developed for the diagonal lifts of tensor fields on M to $F^2 M$ introduced in [13].

§ 1. Preliminaires

1.1. *The tangent bundle of n^2-velocities.*

Let M be an n-dimensional manifold. We denote by $T^2_n M$ the set of all 2-jets at 0 of differentiable mappings $R^n \to M$. Let π^2_n: $T^2_n M \to M$ be the target projection, i.e., $\pi^2_n(j^2_0 f) = f(0)$. In $T^2_n M$ we consider the following manifold structure (see MORIMOTO [9]): Let N_2 denote the set of all n-tuplas $\nu = (\nu_1, \ldots, \nu_n)$ of non negative integers such that $|\nu| = \nu_1 + \ldots + \nu_n \leqq 2$. Every chart (U, x^i) on M induces a chart $\big((\pi^2_n)^{-1}(U) = T^2_n U, x^i_{(\nu)}: i = 1, \ldots, n;\ \nu \in N_2\big)$ on $T^2_n M$, called the induced chart, where

$$x^i_{(\nu)}(j^2_0 f) = \frac{1}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu} (x^i \circ f)_{t=0} = \frac{1}{\nu!}\,\frac{\partial^{\nu_1 + \ldots + \nu_n}}{\partial t_1^{\nu_1} \ldots \partial t_n^{\nu_n}} (x^i \circ f)_{t=0}$$

being $(t_1, \ldots, t_n)$ the natural coordinates of R^n and $\nu! = \nu_1! \ldots \nu_n!$.

The set N_2 has $1 + n + \dfrac{n(n+1)}{2}$ elements which can be ordered by the following bijection

$$N_2 \to \left\{0, 1, \ldots, n, \ldots, n+\frac{n(n+1)}{2}\right\}$$

$$\Psi(0, \ldots, 0) = 0,$$

$$\Psi(0, \ldots, 1, \ldots, 0) = i \quad \text{(we place 1 in } i\text{-th position)}$$

$$\Psi(0, \ldots, 1, \ldots, 1, \ldots, 0) = j + \frac{(2n-i)(i-1)}{2} + n,$$

$$\text{(we place 1 in } i\text{-th and } j\text{-th positions, } i < j)$$

$$\Psi(0, \ldots, 2, \ldots, 0) = i + \frac{(2n-i)(i-1)}{2} + n,$$

$$\text{(we place 2 in } i\text{-th position).}$$

If $\nu=(\nu_1, \ldots, \nu_n)$ and $\lambda=(\lambda_1, \ldots, \lambda_n)$ are two n-tuplas, then we writte $\nu\pm\lambda= =(\nu_1\pm\lambda_1, \ldots, \nu_n\pm\lambda_n)$ and $\nu=\lambda$ if $\nu_i=\lambda_i$ for $i=1, \ldots, n$.

It is clear that $x^i_{(0,\ldots,0)}(j^2_0 f)=x^i(f(0))$ and we shall denote these coordinates $x^i_{(0,\ldots,0)}$ by x^i. In the sequel, we shall denote by $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$ the induced coordinate system in $T^2_n U$, where $\lambda, \nu\in N_2$ with $|\lambda|=1$, $|\nu|=2$.

Let (U, x^i) and $(\overline{U}, \bar{x}^i)$ be two intersecting charts on M and the coordinate transformation in $U\cap\overline{U}$ given by $\bar{x}^i=\bar{x}^i(x)$. Then the transformation of induced coordinates in $T^2_n U\cap T^2_n\overline{U}=T^2_n(U\cap\overline{U})$ is given by

$$\bar{x}^i = \bar{x}^i \quad \bar{x}^i_{(\lambda)} = \frac{\partial \bar{x}^i}{\partial x^r} x^r_{(\lambda)}, \quad |\lambda| = 1,$$

(1.1)

$$\bar{x}^i_{(\nu)} = \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s} \sum_{\substack{|\lambda|=1\\ \lambda\in N_2}} x^r_{(\nu-\lambda)} x^s_{(\lambda)} + \frac{\partial \bar{x}^i}{\partial x^r} x^r_{(\nu)}, \quad |\nu| = 2.$$

The Jacobian of (1.1) is given by the matrix

(1,2)

$$\begin{pmatrix}
a & 0 & \cdots & \cdots & \cdots & 0\\
B_1 & a & & & & \vdots\\
\vdots & 0 & \ddots & & & \vdots\\
\vdots & \vdots & & \ddots & & \vdots\\
B_n & 0 & \cdots\; 0 & a & & \vdots\\
C_{n+1} & & & 0 & a & \vdots\\
\vdots & & (*) & & \ddots & 0\\
C_{n+\frac{n(n+1)}{2}} & & & 0 & \cdots\; 0 & a
\end{pmatrix}$$

being (*) the matrix

$$\left(\begin{array}{cccc}
B_1 & & & \\
\vdots & B_1 & & \\
\vdots & & \ddots \quad 0 & \\
\vdots & 0 & \ddots & \\
B_n & 0 & & B_1 \\
0 & B_2 & & \\
0 & \vdots & \ddots \quad 0 & \\
 & \vdots \quad 0 & \ddots & \\
 & B_n & & B_2 \\
\hline
 & B_k & & \\
 & 0 \quad \vdots & \ddots & 0 \\
 & \vdots & 0 \quad \ddots & \\
 & B_n & & B_k \\
\hline
 & 0 & B_{n-1} & 0 \\
 & & B_n & B_{n-1} \\
 & & 0 & B_n
\end{array}\right)$$

where

$$a = \left(\frac{\partial \bar{x}^i}{\partial x^t}\right)$$

$$B_k = \left(\frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^t} x^r_{\psi^{-1}(k)}\right), \quad k = 1, \dots, n,$$

$$C_{j+\frac{(2n-h)(h-1)}{2}+n} = \left(\frac{1}{2}\frac{\partial^3 \bar{x}^i}{\partial x^r \partial x^s \partial x^t} x^r_{(\psi^{-1}(j)} x^s_{\psi^{-1}(h))} + \right.$$

$$\left. + \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^t} x^r_{\psi^{-1}\left(j+\frac{(2n-h)(h-1)}{2}+n\right)}\right), \quad j \leq h,$$

and

$$x^r_{(\psi^{-1}(j)} x^s_{\psi^{-1}(h))} = x^r_{\psi^{-1}(j)} x^s_{\psi^{-1}(h)} + x^r_{\psi^{-1}(h)} x^s_{\psi^{-1}(j)}$$

Definition 1.1. The differentiable manifold $T^2_n M$ with projection π^2_n is called the tangent bundle of n^2-velocities of M.

Now, we shall recall some properties of the functor T^2_n:

(A) Let $f\colon M \to N$ be a map of a manifold M into another manifold N. Then the map f induces a map $T^2_n f\colon T^2_n M \to T^2_n N$ defined by $T^2_n f(j^2_0 g) = j^2_0(f \circ g)$. Moreover, if f has maximal rank, then $T^2_n f$ has maximal rank. Therefore, if f is an immersion (resp. diffeomorphism) then $T^2_n f$ is an immersion (resp. diffeomorphism), and if f is a diffeomorphism then $(T^2_n f)^{-1} = T^2_n(f^{-1})$.

(B) Let M and N be two arbitrary manifolds and $M \times N$ the manifold product. Then $T^2_n(M \times N)$ and $T^2_n M \times T^2_n N$ are canonically diffeomorphic.

(C) Let G be a Lie group with multiplication μ: $G\times G\to G$. Then $T_n^2 G$ is a Lie group with multiplication $T_n^2\mu$: $T_n^2 G\times T_n^2 G\to T_n^2 G$. Moreover, if h: $G\to G'$ is a homomorphism of Lie groups then $T_n^2 h$: $T_n^2 G\to T_n^2 G'$ is so also. Therefore, if G is a Lie subgroup of G', then $T_n^2 G$ is a Lie subgroup of $T_n^2 G'$.

(D) If a Lie group G operates on a manifold M differentiably and effectively then $T_n^2 G$ operates on $T_n^2 M$ differentiably and effectively.

(E) If $P(M, \pi, G)$ is a principal fibre bundle then $T_n^2 P(T_n^2 M, T_n^2 \pi, T_n^2 G)$ is a principal fibre bundle which be called the induced bundle. In fact, let U be a coordinate neighbourhood on M; if φ_U: $U\times G\to\pi^{-1}(U)$ is the local trivialization of P then $T_n^2(\varphi_U)$: $T_n^2 U\times T_n^2 G\to T_2^n(\pi^{-1}(U))$ is the local trivialization of $T_n^2 P$.

1.2. *The frame bundle of second order.*

Let M be an n-dimensional manifold. We denote by $F^2 M$ the set of all 2-jets at 0 of diffeomorphisms of open neighbourhoods of 0 in R^n onto open subsets of M. Let π^2: $F^2 M\to M$ be the target projection $\pi^2(j_0^2 f)=f(0)$. Then π^2: $F^2 M\to M$ is a principal fibre bundle with the structural group L_n^2 of all 2-jets with the source and the target at 0 of local diffeomorphisms of R^n. The group L_n^2 operates on $F^2 M$ on the right in the natural way $((j_0^2 f)(j_0^2 g)=j_0^2(f\circ g))$, where $j_0^2 f$ and $j_0^2 g$ belong to $F^2 M$ and L_n^2, respectively.

Let us remark that $F^2 M$ is an open and dense subset of $T_n^2 M$. If f: $M\to N$ is a local diffeomorphism then we define its prolongation f^2: $F^2 M\to F^2 N$ by $f^2(j_0^2 g)=$ $=j_0^2(f\circ g)$, and it is clear that $T_n^2 f|_{F^2 M}=f^2$.

Every chart (U, x^i) on M induces a chart $((\pi^2)^{-1}(U)=F^2 U, x^i_{(\nu)}$: $i=1, \dots, n$; $\nu\in N_2)$ on $F^2 M$, where

$$x^i_{(\nu)}(j_0^2 f) = \frac{1}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu}(x^i\circ f)_{t=0}.$$

We shall denote the induced coordinate system by $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$, where λ, $\nu\in N_2$, with $|\lambda|=1$, $|\nu|=2$.

Now, we can consider the restriction to $F^2 M$ of the γ-lifts $(\gamma\in N_2)$ of tensor fields on M to $T_n^2 M$ (see [3], [4]):

(A) *Lifts of functions.*

If f is a differentiable function on M, we define the 0-lift, (λ)-lift (ν)-lift $(\lambda, \nu\in N_2$: $|\lambda|=1$, $|\nu|=2)$ as the functions $f^0, f^{(\lambda)}, f^{(\nu)}$ on $F^2 M$ given by

$$f^0 = f^V = (\pi^2)^* f,$$

$$f^{(\lambda)} = x^i_{(\lambda)}\partial_i f, \tag{1.3}$$

$$f^{(\nu)} = \sum_{\substack{\lambda\in N_2\\ |\lambda|=1}} x^i_{(\nu-\lambda)} x^j_{(\lambda)}\partial_i\partial_j f + x^i_{(\nu)}\partial_i f,$$

where $\partial_i f = \dfrac{\partial^i f}{\partial x}$.

It is convenient to define $f^{(\gamma)}=0$ if $\gamma\notin N_2$.

(B) *Lifts of vector fields.*

If X is a vector field on M, we define the complete lift (or 0-lift), (λ)-lift and (ν)-lift to F^2M as the vector fields on F^2M given by

$$X^C=(X^i)^V\frac{\partial}{\partial x^i}+\sum_{\substack{\lambda\in N_2\\|\lambda|=1}}(X^i)^{(\lambda)}\frac{\partial}{\partial x^i_{(\lambda)}}+\sum_{\substack{\nu\in N_2\\|\nu|=2}}(X^i)^{(\nu)}\frac{\partial}{\partial x^i_{(\nu)}},$$

$$X^{(\lambda)}=(X^i)^V\frac{\partial}{\partial x^i_{(\lambda)}}+\sum_{\substack{\mu\in N_2\\|\mu|=1}}(X^i)^{(\mu)}\frac{\partial}{\partial x^i_{(\lambda+\mu)}},\tag{1.4}$$

$$X^{(\nu)}=(X^i)^V\frac{\partial}{\partial x^i_{(\nu)}},$$

where X^i are the local components of X.

It is convenient to define $X^{(\gamma)}=0$ if $\gamma\notin N_2$.

(C) *Lifts if l-forms.*

Similarly, if Θ is an i-form on M, we can define the corresponding lifts Θ^0, $\Theta^{(\lambda)}$, $\Theta^{(\nu)}$ to F^2M as the i-forms on F^2M given by

$$\Theta^0=\Theta^V=(\pi^2)^*\Theta,$$

$$\Theta^{(\lambda)}=(\Theta_i)\,dx^i_{(\lambda)}+(\Theta_i)^V\,dx^i,\tag{1.5}$$

$$\Theta^{(\nu)}=(\Theta_i)^V\,dx^i_{(\nu)}+\sum_{\substack{\mu\in N_2\\|\mu|=1}}(\Theta_i)^{(\nu-\mu)}\,dx^i_{(\mu)}+(\Theta_i)^V\,dx^i,$$

where Θ^i are the local components of Θ.

It is convenient to define $\Theta^{(\lambda)}=0$, if $\gamma\notin N_2$.

From (1.3), (1.4) and (1.5), we deduce

Proposition 1.2. *Let X and Θ be a vector field and l-form on M, respectively. Then we have*

(1) $\Theta^V(X^C)=(\Theta(X))^V,\quad \Theta^V(X^{(\lambda)})=0,\quad \Theta^V(X^{(\nu)})=0,$

(2) $\Theta^{(\alpha)}(X^C)=(\Theta(X))^{(\alpha)},\quad \Theta^{(\alpha)}(X^{(\lambda)})=\delta^{\alpha\lambda}(\Theta(X))^V,\quad \Theta^{(\alpha)}(X^{(\nu)})=0,$

(3) $\Theta^{(\beta)}(X^C)=(\Theta(X))^{(\beta)},\quad \Theta^{(\beta)}(X^{(\lambda)})=(\Theta(X))^{(\beta-\lambda)},\quad \Theta^{(\beta)}(X^{(\nu)})=\delta^{\beta\nu}(\Theta(X))^V,$

where $\alpha,\beta,\lambda,\nu\in N_2$, *with* $|\alpha|=|\lambda|=1$, $|\beta|=|\nu|=2$. #

(D) *Lifts of tensor fields of type* (1, 1).

If F is a tensor field of type (1, 1) on M, we define the (γ)-lift $(\gamma\in N_2)$ of F to F^2M as the tensor field $F^{(\gamma)}$ of type (1, 1) on F^2M given by

$$F^{(\gamma)}(X^C)=(FX)^{(\gamma)},\quad F^{(\gamma)}(X^{(\xi)})=(FX)^{(\gamma-\xi)},$$

for any vector field X on M, where $\xi\in N_2$, $1\leqq|\xi|\leqq2$.

We shall denote $F^{(0,\dots,0)}$ by F^C. F^C is called the complete lift of F to F^2M. In [3], Gancarzewicz proved that F^C is completely characterized by the identity $F^C(X^C)=(FX)^C$, for any vector field X on M.

(E) *Lifts of tensor fields of type* $(0, 2)$.

Finally, let g be a tensor field of type $(0, 2)$ on M. Then we define the (γ)-lift of g to F^2M, $\gamma\in N_2$, as the tensor field of type $(0, 2)$ on F^2M given by

$$g^{(\gamma)}(X^{(\alpha)}, Y^{(\beta)}) = \big(g(X, Y)\big)^{(\gamma-\alpha-\beta)},$$

for any vector field X on M, where $\alpha, \beta\in N_2$.

§ 2. Imbedding of $\mathbf{T}^2_n$ GL (n) into Gl (N), $\mathrm{N}=\mathbf{n}\left(1+n+\dfrac{n(n+1)}{2}\right)$.

Let R^n be the n-dimensional euclidean space, and consider the tangent bundle of n^2-velocities $T^2_nR^n$ of R^n. It is clear that $T^2_nR^n$ is a vector space of dimension N. In fact, for any two 2-jets j^2_0f and j^2_0g, we define their sum by: $j^2_0f+j^2_0g=j^2_0(f+g)$, where $(f+g)(t)=f(t)+g(t)$, for $t\in R^n$. For any $c\in R^n$ we define the scalar multiplication of j^2_0f by c as follows: $c\cdot(j^2_0f)=j^2_0(c\cdot f)$, where $(c\cdot f)(t)=c\cdot f(t)$, for $t\in R^n$.

Let (t^i) be the natural coordinate system on R^n and let $(t^i_{(\alpha)}: i=1, \dots, n; \alpha\in N_2)$ be the induced coordinate system on $T^2_nR^n$. Then the sum and scalar multiplication in $T^2_nR^n$ are as follows:

$$(t^i_{(\alpha)})+('t^i_{(\alpha)}) = (t^i_{(\alpha)}+'t^i_{(\alpha)}),\ c\cdot(t^i_{(\alpha)}) = (c\cdot t^i_{(\alpha)}).$$

Let $Gl(n)$ be the general linear group. Then $T^2_0Gl(n)$ is a Lie group. Let $\tilde{A}\in T^2_nGl(n)$ be the 2-jet defined by the map $A\colon R^n\to Gl(n)$, and $\big(A^h_i(t)\big)$ the matrix that represents to $A(t)$ for each $t\in R^n$. The element $\tilde{A}$ can be identified to the $\left(1+n+\dfrac{n(n+1)}{2}\right)$-tupla

$$(a;\ \dots, B^{(\lambda)}, \dots;\ \dots, C^{(\nu)}, \dots),\quad |\lambda| = 1, |\nu| = 2,$$

where

$$a = \big(A^h_i(0)\big), a\in Gl(n),$$

$$B^{(\lambda)} = \frac{1}{\lambda!}\left[\left(\frac{\partial}{\partial t}\right)^{\lambda}\big(A^h_i(t)\big)\right]_{t=0},\quad B^{(\lambda)}\in gl(n),$$

$$C^{(\nu)} = \frac{1}{\nu!}\left[\left(\frac{\partial}{\partial t}\right)^{\nu}\big(A^h_i(t)\big)\right]_{t=0},\quad C^{(\nu)}\in gl(n),$$

being $gl(n)$ the Lie algebra of $Gl(n)$.

Now, let $Gl(n)\times R^n\to R^n$ be the usual operation of the general linear group $Gl(n)$ on R^n and consider the induced operation

$$T^2_n\,Gl(n)\times T^2_nR^n \to T^2_nR^n$$
$$(\tilde{A}, \tilde{P}) \to \tilde{A}\cdot\tilde{P}$$

If $\tilde{A}=j^2_0A$, $\tilde{P}=j^2_0P$, where $A\colon R^n\to Gl(n)$, $P\colon R^n\to R^n$ are defined by $A(t)=\big(A^h_i(t)\big)$, $P(t)=\big(\xi^h(t)\big)$, then $\tilde{A}\cdot\tilde{P}=j^2_0(A\cdot P)$, being $A\cdot P\colon R^n\to R^n$ the map defined by

$(A\cdot P)(t)=A(t)\cdot P(t)$ for $t\in R^n$. Therefore, if $(A.P)(t)=('\xi^h(t)$, then $'\xi^h(t)=A^h_i(t)\,\xi^i(t)$ for $t\in R^n$. Let $(t^i_{(\alpha)})$ and $('t^i_{(\alpha)})$ the induced coordinates of $\tilde{P}$ and $\tilde{A}\cdot\tilde{P}$, respectively. Then we obtain

$$
\begin{aligned}
't^h &= a^h_i t^i \\
't^h_{(\lambda)} &= (B^{(\lambda)})^h_i t^i + a^h_i t^i_{(\lambda)}, \quad |\lambda| = 1, \\
't^h_{(\nu)} &= (C^{(\nu)})^h_i t^i + \sum_{\substack{\lambda\in N_2\\ |\lambda|=1}} (B^{(\nu-\lambda)})^h_i t^i_{(\lambda)} + a^h_i t^i_{(\nu)}, \quad |\nu| = 2,
\end{aligned}
\tag{2.1}
$$

from where deduce that $T^2_nGl(n)$ operates on $T^2_n R^n$ as a group of linear transformations. So, each element $\tilde{A}=j^2_0 A\in T^2_nGl(n)$ can be identified to a linear automorphism of $T^2_n R^n\cong R^N$. Therefore, we deduce the following proposition.

Proposition 2.1. *There is a canonical homomorphism ϱ^2_n of $T^2_nGl(n)$ into $Gl(N)$ which is given by the following equality:*

$$
\varrho^2_n(\tilde{A}) = \left(\begin{array}{l|ccc|ccc}
a & & & & & & \\
B_1 & 0 & \cdots & 0 & \cdots & & 0 \\
\vdots & a & & \vdots & & & \vdots \\
\vdots & 0 & \ddots & \vdots & & 0 & \vdots \\
B_n & \vdots & & \ddots & & & \vdots \\
C_{n+1} & 0 & \cdots & 0 & a & \cdots & 0 \\
\vdots & & & & \vdots & \ddots & \vdots \\
C_{n+\frac{n(n+1)}{2}} & & (*) & & 0 & \cdots & 0\ a
\end{array}\right)
\tag{2.2}
$$

where (*) *is the following matrix*

$$
\left(\begin{array}{cccc}
B_1 & & & \\
\vdots & B_1 & & 0 \\
\vdots & & \ddots & \\
\vdots & 0 & & \ddots \\
B_n & 0 & & B_1 \\
0 & B_2 & & \\
 & \vdots & \ddots & 0 \\
0 & \vdots & 0 & \ddots \\
 & B_n & & B_2 \\
\hline
 & B_k & & \\
0 & \vdots & \ddots & 0 \\
 & \vdots & 0 & \ddots \\
 & B_n & & B_k \\
\hline
 & & B_{n-1} & 0 \\
0 & & B_n & B_{n-1} \\
 & & 0 & B_n
\end{array}\right)
$$

being

$$B_i = B^{(0,\cdots,1,\cdots,0)}$$

(we place 1 *in th-position)*

$$C_{j+\frac{(2n-i)(i-1)}{2}+n} = C^{(0,\cdots,1,\cdots,1,\cdots,0)}$$

(we place 1 *in i-th and j-th position,* $j<i$*)*

$$C_{i+\frac{(2n-i)(i-1)}{2}+n} = C^{(0,\cdots,2,\cdots,n)}$$

(we place 2 *in i-the position).*

§3. Imbedding of T_n FM into FF^2 M.

Let $FM(M, \pi_M, Gl(n))$ be the frame bundle of M, $T_n^2 FM(T_n^2 M, T_n^2 \pi_M, T_n^2 Gl(n))$ the induced bundle and $FT_n^2 M(T_n^2 M, \pi_{T_n^2 M}, Gl(N))$ the frame bundle of $T_n^2 M$.

Theorem 3.1. *There exists a canonical injective homomorphism of principal bundles*

$$j_M^2\colon T_n^2 FM \to FT_n^2 M$$

over the identity of $T_n^2 M$, *with associate Lie group homomorphism* $\varrho_n^2\colon T_n^2 Gl(n) \to Gl(N)$.

Proof. Let U be a coordinate neighbourhood in M, and

$$\varphi_U\colon T_n^2 U \times T_n^2 Gl(n) \to T_n^2 FU, \quad \psi_U\colon T_n^2 U \times Gl(N) \to FT_n^2 U$$

the local trivializations of $T_n^2 FM$ and $FT_n^2 M$, respectively. Then we define $j_{M|U}^2\colon T_n^2 FU \to FT_n^2 U$ as the composition $j_{M|U}^2 = \psi_U \circ (1_{T_n^2 U} \times \varrho_n^2) \circ \varphi_U^{-1}$. In order to prove Theorem 3.1 it is sufficient to check the following identity

$$\tilde{J}_{U\bar{U}} = \varrho_n^2 \circ T_n^2 J_{U\bar{U}} \quad \text{on} \quad T_n^2 U \cap T_n^2 \bar{U}, \tag{3.1}$$

where $J_{U\bar{U}}\colon U \cap \bar{U} \to Gl(n)$ and $\tilde{J}_{U\bar{U}}\colon T_n^2 U \cap T_n^2 \bar{U} \to Gl(N)$ denote the Jacobian matrices of change of coordinates in M and $T_n^2 M$, respectively.

Let (x^i) and (y^i) be the coordinate functions in U and $\bar{U}$, respectively. We assume that $y^i = f^i(x^j)$ in $U \cap \bar{U}$. Let $j_0^2 g \in T_n^2 U \cap T_n^2 \bar{U} = T_n^2(U \cap \bar{U})$ and $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$ be the induced coordinates. Then $T_n^2 J(j_0^2 g) = j_0^2(J \circ g)$ is the 2-jet corresponding to the composition $(J \circ g)(t) = \left(\frac{\partial f^i}{\partial x^j}\right)_{|g(t)}$. Therefore, $j_0^2(J \circ g)$ is identified to the $\left(1+n+\frac{n(n+1)}{2}\right)$-tupla

$$(a;\ \ldots, B^{(\lambda)}, \ldots;\ \ldots, C^{(\nu)}, \ldots),$$

where

$$a = \left(\frac{\partial f^i}{\partial x^j}\Big|_x\right), \quad x = g(0),$$

$$B^{(\lambda)} = \left(\frac{\partial^2 f^i}{\partial x^j \partial x^r}\Big|_x x^r_{(\lambda)}\right), \quad |\lambda| = 1, \tag{3.2}$$

$$C^{(\nu)} = \left(\frac{1}{2}\frac{\partial^2 f^i}{\partial x^j \partial x^r \partial x^s}\Big|_x \sum_{\substack{\lambda\in N_2 \\ |\lambda|=1}} x^r_{(\nu-\lambda)} x^s_{(\lambda)} + \frac{\partial^2 f^i}{\partial x^j \partial x^r} x^r_{(\nu)}\right), \quad |\nu| = 2.$$

Now, from (3.2) one gets that the matrix $\varrho_n^2\big(j_0^2(J\circ g)\big)$ coincides with the Jacobian matrix (1.2). This proves the equality (3.1). #

Now, let $T_n^2 F^2 M_{|F^2M}$ be the restriction of $T_n^2 F^2 M$ to the open submanifold F^2M of $T_n^2 M$. Remark that the restriction $FT_n^2 M_{|F^2M}$ is canonically isomorphic to the frame bundle FF^2M of F^2M. Then, from Theorem 3.1, we deduce

Theorem 3.2. *j_M^2 induces a bundle homomorphism of $T_n^2 FM_{|F^2M}$ into FF^2M over the identity of F^2M and with associate Lie group homomorphism ϱ_n^2.* #

§ 4. Prolongations of G-structures to F^2M.

Let G be a subgroup of $Gl(n)$, and denote $G^{(2)} = \varrho_n^2(T_n^2 G)$. Then $G^{(2)}$ is a Lie subgroup of $Gl(N)$ isomorphic to $T_n^2 G$. Let $P(M, \pi, G)$ be a G-structure on M. Then we have

Theorem 4.1. *If M has a G-structure P, then F^2M has a canonical $G^{(2)}$-structure $P^{(2)}$.*

PROOF. Taking into account Theorem 3.2, it suffices to set $P^{(2)} = j_M^2(T_n^2 P_{|F^2M})$. #

Definition 4.2. $P^{(2)}$ will be called the prolongation of the G-structure P on M to the frame bundle of second order F^2M.

Let M and M' be n-dimensional manifolds, $f\colon M\to M'$ a diffeomorphism and $f^1\colon FM\to FM'$ the induced isomorphism of principal bundles (see [1]). Then we have

Theorem 4.3. *The following diagram is commutative*
Theorem 4.3. *The following diagram is commutative*

$$\begin{array}{ccc} T_n^2 FM & \xrightarrow{\quad j_M^2 \quad} & FT_n^2 M \\ {\scriptstyle T_n^2(f^1)}\downarrow & & \downarrow{\scriptstyle (T_n^2 f)^1} \\ T_n^2 FM' & \xrightarrow{\quad j_{M'}^2 \quad} & FT_n^2 M' \end{array}$$

# #

Theorem 4.4. *Let P and P' be G-structures on M and M', respectively, and $f\colon M\to M'$ a diffeomorphism. Then f is an isomorphism of P to P' if and only if f^2 is an isomorphism of $P^{(2)}$ to $P'^{(2)}$.*

PROOF. Suppose that f is an isomorphism of P to P'. Then by virtue of Theorem 4.3 and since $T_n^2(f^1)(T_n^2 P_{|F^2M}) = T_n^2 P'_{|F^2M}$, we have

$$(f^2)^1(P^{(2)}) = (f^2)^1\big(j_M^2(T_n^2 P_{|F^2M})\big) = j_{M'}^2\big(T_n^2(f^1)(T_n^2 P_{|F^2M})\big) = j_{M'}^2(T_n^2 P'_{|F^2M}) = P'^{(2)}.$$

Conversely, if

$$(f^2)^1(P^{(2)}) = P'^{(2)}, \quad \text{then} \quad (f^2)^1\big(j_M^2(T_n^2 P_{|F_2M})\big) = j_{M'}^2(T_n^2 P'_{|F^2M}).$$

On the other hand,

$$(f^2)^1\big(j_M^2(T_n^2 P_{|F^2M})\big) = j_{M'}^2\big(T_n^2(f^1)(T_n^2 P_{|F^2M})\big).$$

Then, since j_M^2 is injective, we deduce that $T_n^2(f^1)(T_n^2 P_{|F^2M}) = T_n^2 P'_{|F^2M}$. Hence $f^1(P) = P'$ that is, f is an isomorphism of P to P'. #

Corollary 4.5. *Let P be a G-structure on M and let f be a diffeomorphism of M into itself. Then f is an automorphism of P if and only if f^2 is an automorphism of $P^{(2)}$.* #

Corollary 4.6. *A vector field X on M is an infinitesimal automorphism of a G-structure P on M if and only if X^C is an infinitesimal automorphism of the prolongation $P^{(2)}$ of P to F^2M.*

PROOF. It is a consequence of the following result (see Gancarzewicz [3]): If φ_t is the local 1-parameter group generated by X then φ_t^2 is the local 1-parameter generated by X^C. #

§ 5. Integrability of the prolongation of G-structures

In this section, we shall prove that the prolongation of an integrable G-structure is also integrable, and viceversa.

Definition 5.1. Let $P(M, \pi, G)$ be a G-structure on M. P is said to be integrable if for each point $x \in M$ there is a coordinate system (U, x^i) with $x \in U$ such that the frame $\left(\frac{\partial}{\partial x^1}\Big|_y, \ldots, \frac{\partial}{\partial x^n}\Big|_y\right) \in P$ for every $y \in U$.

Theorem 5.2. *Let (U, x^i) be a coordinate system in M, and let $\Phi\colon U \to FM$ be a cross-section given by $\Phi(x) = \left(\Phi_j^i(x) \frac{\partial}{\partial x^i}\Big|_x\right)$, $x \in U$. Define $\Phi^2 = j_M^2 \circ T_n^2 \Phi\colon T_n^2 U \to FT_n^2 M$. Then Φ^2 is also a cross-section which is given at $p \in T_n^2 U$, with $\pi_n^2(p) = x$, by*

$$(5.1) \qquad \Phi^2(p) = (X_{i|p}^C, X_{i|p}^{(\lambda)}, X_{i|p}^{(\nu)}), \quad i = 1, \ldots, n;\ \lambda, \nu \in N_2, |\lambda| = 1, |\nu| = 2,$$

where X_i is the local vector field given on U by $X_{i|x} = \Phi_i^j(x) \frac{\partial}{\partial x^\nu}\Big|_x$ $x \in U$, and X_i^C, $X_i^{(\lambda)}$, $X_i^{(\nu)}$ are the complete lift, (λ)-lift and (ν)-lift of X_i to $T_n^2 M$.

PROOF. From Theorem 3.1, one easily prove that Φ^2 is a cross-section. Now, putting $f(x)=(\Phi^i_j(x))\in Gl(n)$, for $x\in U$, then we have $\varphi_U^{-1}\circ\Phi=(1_U,f)$, where φ_U is the local trivialization of FM. Hence

$$\Phi^2=\psi\circ(1_{T^2_nM}\times\varrho^2_n)\circ T^2_n(1_U,f)=\psi_U\circ(1_{T^2_nM}\times(\varrho^2_n\circ T^2_n f)),$$

where φ_U is the local trivialization of $T^2_n FM$.
If $j^2_0 h\in T^2_n U$ has coordinates $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$ then $T^2_n f(j^2_0 h)$ has coordinates

$$\left(\Phi^i_j(x), \frac{\partial\Phi^j_j}{\partial x^r}\Big|_x x^r_{(\lambda)}, \frac{1}{2}\frac{\partial^2\Phi^i_j}{\partial x^r\partial x^s}\Big|_x \sum_{\substack{\lambda\in N_2\\|\lambda|=1}} x^r_{(\nu-\lambda)}x^s_{(\lambda)}+\frac{\partial\Phi^i_j}{\partial x^r}\Big|_x x^r_{(\nu)}\right)$$

Then, from Proposition 2.1, one deduces (5.1). #

Actually, if $\Phi\colon U\to FM$ is a cross-section, then the restriction $\bar\Phi=\Phi^2{}_{|F^2U}\colon F^2U\to\to FF^2M$ is also a cross-section locally expressed by

$$\tilde\Phi(p)=(X^C_{i|p}, X^{(\lambda)}_{i|p}, X^{(\nu)}_{i|p}),\quad i=1,\dots n;\ \lambda,\nu\in N_2,\quad |\lambda|=1, |\nu|=2, \tag{5.2}$$

where $X_i=\Phi^j_i\dfrac{\partial}{\partial x^j}$ in U.

Proposition 5.3. *Let P be a G-structure on M. Then P is integrable if and only if the prolongation $P^{(2)}$ is integrable.*

PROOF. Suppose that P is integrable. From Definition 5.1 and Theorem 5.2 we deduce that $P^{(2)}$ is integrable.

Conversely, suppose that $P^{(2)}$ is integrable. Let x_0 be an arbitrary point in M, (U,x^i) a coordinate system with $x_0\in U$, and $\Phi\colon U\to P$ a local cross-section of P over U. Now, let $p_0\in F^2U$ be with coordinates $(x^i=x^i(x_0), x^i_{(\lambda)}=\delta^{i\Psi(\lambda)}, x^i_{(\nu)}=0)$. Since $P^{(2)}$ is integrable there exists a coordinate system $(\tilde U, y^i, y^i_{(\lambda)}, y^i_{(\nu)}\colon i=1,\dots,n;$ $|\lambda|=1$, $|\nu|=2)$ in F^2U with $p_0\in\tilde U$, $\tilde U\subset F^2U$, such that, if we define $\tilde\Phi_0$ by

$$\tilde\Phi_0(p)=\left(\frac{\partial}{\partial y^i}\Big|_p, \frac{\partial}{\partial y^i_{(\lambda)}}\Big|_p, \frac{\partial}{\partial y^i_{(\nu)}}\Big|_p\right),$$

then $\tilde\Phi_0$ is a cross-section of $P^{(2)}$ over $\tilde U$. Now, since $\bar\Phi$ and $\tilde\Phi_0$ are both cross-sections of $P^{(2)}$ over $\tilde U$, there exists a map $\tilde g\colon\tilde U\to G^{(2)}=\varrho^2_n(T^2_n G)$ such that $\tilde\Phi_0(p)=\bar\Phi(p)\cdot\tilde g(p)$ holds for $p\in\tilde U$. Then, using similar arguments as in the proof of Proposition 5.5 [7], Proposition 10. [8] and Proposition 4.5 [1], we deduce that P is integrable. #

§ 6. Prolongations of classical G-structures

Let P be a G-structure on M, (U,x^i) a local coordinate system in U, and $\Phi\colon U\to P$ a cross-section. Then Φ defines a local field of frames $\{X_1,\dots,X_n\}$ adapted to P and given by $X_i=\Phi^j_i\dfrac{\partial}{\partial x^j}$. Hence the local field of coframes $\{\theta^1,\dots,\theta^n\}$ dual to $\{X_1,\dots,X_n\}$ is given by

$$\theta^j=\psi^j_i\,dx^i, \tag{6.1}$$

where (ψ^j_i) denotes the inverse matrix of (Φ^i_j). Then Φ induces a cross-section $\bar{\Phi}\colon U\to P^{(2)}$ given by (5.2). There $\bar{\Phi}$ defines the local field of frames adapted to $P^{(2)}$ given by $\{X_i^C, X_i^{(\lambda)}, X_i^{(\nu)}\colon i=1,\dots,n;\ \lambda,\nu\in N_2, |\lambda|=1, |\nu|=2\}$. From (6.1), we deduce that the dual local field of coframes is $\{(\theta^i)^V, (\theta^i)^{(\lambda)}, (\theta^i)^{(\nu)}\colon i=1,\dots,n;\ \lambda,\nu\in N_2, |\lambda|=1, |\nu|=2\}$.

(I) *G*-structures defined by tensor fields of type (1, 1)

Let $\varrho\colon Gl(n)\to \mathrm{Aut}(R^n)$ be the canonical representation of $Gl(n)$ into R^n, $u\in \mathrm{End}(R^n)$ an arbitrary element and G_u the isotropy group of u with respecs to ϱ. Let $u^2=T^2_n u\in \mathrm{End}\ (R^N)$. R^N identified to $T^2_n R^n$. the induced ma! defined by $u^2(j^2_0 g)=j^2_0(u\circ g)$, $j^2_0 g\in T^2_n R^n$. Let $('x^i, 'x^i_{(\lambda)}, 'x^i_{(\nu)})$ and $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$ the induced coordinates of $j^2_0(u\circ g)$ and $j^2_0 g$, respectively. If $u=(u^i_j)$ is the matrix representation of u, then we obtain

$$'x^i = u^i_j x^j,\ 'x^i_{(\lambda)} = u^i_j x^j_{(\lambda)},\ 'x^i_{(\nu)} = u^i_j x^j_{(\nu)},\ \lambda,\nu\in N_2,\quad |\lambda| = 1,\quad |\nu| = 2,$$

and therefore the matrix representation of u^2 is

$$u^2 = \begin{pmatrix} (u^i_j) & & 0 \\ & \ddots & \\ 0 & & (u^i_j)\end{pmatrix} \tag{6.2}$$

From (6.2) we deduce

Lemma 6.1. *Let $u^2=T^2_n u\in \mathrm{End}(R^N)$ be the linear map induced by $u\in\mathrm{End}(R^n)$. If rank $u=r$, then rank $u^2=r\left(1+n+\dfrac{n(n+1)}{2}\right)$. Moreover, if u satisfies a polynomial equation $Q(u)=0$, then u^2 satisfies the same equation, that is, $Q(u^2)=0$.* #

Proposition 6.2. *Let G_{u^2} be the isotropy group of u^2 with respect to the canonical representation of $Gl(N)$ into R^N, and denote $(G_u)^{(2)}=\varrho^2_n(T^2_n G_u)$. Then $(G_u)^{(2)}\subset$ $\subset G_{u^2}$.* #

Theorem 6.3. *If M admits a G_u-structure, then F^2M admits a G_{u^2}-structure. Moreover, if the G-structure in M is integrable, then the induced G_{u^2}-structure on F^2M is so also.*

Proof. From Theorem 4.1 we deduce that F^2M admits a $(G_u)^{(2)}$-structure, which, by virtue of Proposition 6.2, can be extended to a G_{u^2}-structure. The assertion on the integrability follows from Proposition 5.3. #

Let P be a G_u-structure on M, and let F be the tensor field of type (1, 1) on M associated to P. If (U, x^i) is a coordinate system in M, and if $\{X_i\}$, $\{\theta^i\}$ are the local field of frames and coframes induced by a cross-section $\Phi\colon U\to P$, then F is locally given by

$$F = F^i_j\frac{\partial}{\partial x^i}\otimes dx^j = u^i_j X_i\otimes\theta^j. \tag{6.3}$$

Similarly, let $F^{(2)}$ be the tensor field of type (1, 1) on F^2M associated to $\tilde{P}^{(2)}$, extension of the prolongation $P^{(2)}$ of P (Theorem 6.3). Then, from (5.2) and (6.2), we have

$$F^{(2)} = u^i_j X^C_i \otimes (\theta^i)^V + \sum_{|\lambda|=|\mu|=1} \delta^{\lambda\mu} u^i_j X^{(\lambda)}_i \otimes (\theta^j)^{(\mu)} + \sum_{|\nu|=|\eta|=2} \delta^{\nu\eta} u^i_j X^{(\nu)}_i \otimes (\theta^j)^{(\eta)}. \tag{6.4}$$

From (6.4), one easily deduces that $F^{(2)}=F^C$.
Summing up, we can state

Theorem 6.4. *Let be $u\in \mathrm{End}\,(R^n)$, P a G_u-structure on M, and F the tensor field of type* (1, 1) *induced by P on M. Then the complete lift F^C of F to F^2M defines the G_{u^2}-structure on F^2M given in Theorem* 6.3. #

From Lemma 6.1 and Theorem 6.4, we deduce

Corolary 6.5. [3]. *If F defines on M a polynomial structure of rank r, then F^C defines on F^2M a polynomial structure of rank $r\left(1+n+\dfrac{n(n+1)}{2}\right)$ and same structural polynomial.* #

(II) G-structures defined by tensor fields of type (0, 2)

Let be $u\in \otimes_2(R^n)$,* $T^2_n u\colon T^2_n R^n \times T^2_n R^n \to T^2_n R$, the induced map, and let $p_1, p_2\colon T^2_n R \cong R^{N/n} \to R$ be the maps defined by

$$p_1(j^2_0 g) = \sum_{\substack{\lambda\in N_2\\ |\lambda|=1}} C_{(\lambda)}, \quad p_2(j^2_0 g) = \sum_{\substack{\nu\in N_2\\ |\nu|=2}} D_{(\nu)}$$

respectively, where $(g(0), C_{(\lambda)}, D_{(\nu)})$ are the induced coordinates of $j^2_0 g\in T^2_n R$, $g\colon R^n\to R$. Then we define

$$\tilde u^2_i\colon R^N\times R^N \to R$$

as the composition $\tilde u^2_i = p_i\circ T^2_n u$, $i=1, 2$.

Lemma 6.6. *$\tilde u^2_i \in \otimes_2(R^N)^*$, that is, $\tilde u^2_i$ is bilinear, $i=1, 2$. Moreover, if u is symmetric (resp. skew-symmetric) then $\tilde u^2_i$ is also symmetric (resp. skew-symmetric) and if rank $u=r$, then rank $\tilde u^2_1=2r$ and rank $\tilde u^2_2=3r$.*

PROOF. Suppose that (u_{ij}) is the matrix representation of u.
Let be $j^2_0 f, j^2_0 g\in T^2_n R^n\cong R^N$ with the induced coordinates $(x^i, x^i_{(\lambda)}, x^i_{(\nu)})$ and $(y^i, y^i_{(\lambda)}, y^i_{(\nu)})$, respectively. Then, since $u^2_1(j^2_0 f, j^2_0 g) = p_i(j^2_0(u\circ(f, g)))$ and $u\circ(f, g)(t) = f^i(t) u_{ij} g^j(t)$, $t\in R^n$, we have

$$\tilde u^2_1(j^2_0 f, j^2_0 g) = \sum_{\substack{\lambda\in N_2\\ |\lambda|=1}} (x^i_{(\lambda)} u_{ij} y^j + x^i u_{ij} y^j_{(\lambda)})$$

$$\tilde u^2_2(j^2_0 f, j^2_0 g) = \sum_{\substack{\nu\in N_2\\ |\nu|=1}} \Big\{x^i_{(\nu)} u_{ij} y^j + \sum_{\substack{\lambda\in N_2\\ |\lambda|=1}} x^i_{(\nu-\lambda)} u_{ij} y^j_{(\lambda)} + x^i u_{ij} y^j_{(\nu)}\Big\}$$

Therefore, the matrix representations of $\tilde{u}_1^2$ and $\tilde{u}_2^2$ are

(6.5)

$$\tilde{u}_1^2 = \begin{pmatrix} 0 & (u_{ij}) \dots (u_{ij}) & 0 \dots 0 \\ (u_{ij}) & & \\ \vdots & 0 & 0 \\ (u_{ij}) & & \\ 0 & & \\ \vdots & 0 & 0 \\ 0 & & \end{pmatrix}, \quad \tilde{u}_2^2 = \begin{pmatrix} 0 & 0 \;\dots\; 0 & (u_{ij}) \dots (u_{ij}) \\ 0 & (u_{ij}) \dots (u_{ij}) & \\ \vdots & \vdots \qquad \vdots & 0 \\ 0 & (u_{ij}) \dots (u_{ij}) & \\ (u_{ij}) & & \\ \vdots & 0 & 0 \\ (u_{ij}) & & \end{pmatrix},$$

respectively. Lemma 6.6 is now obvious. #

Proposition 6.7. *Let G_u (resp. $G_{\tilde{u}_i^2}$) be the isotropy group of $u \in \otimes_2(R^n)^*$ (resp. $\tilde{u}_i^2 \in \otimes_2(R^N)^*$) with respect to the canonical representation of $Gl(n)$ into R^n (resp. of $\mathrm{Gl}(N)$ into R^N), and denote $(G_u)^{(2)} = \varrho_n^2(T_n^2 G_u)$. Then*

$$(G_u)^2 \subset G_{\tilde{u}_i^2}, \quad i = 1, 2.$$

PROOF. Direct from (6.5). #

Theorem 6.8. *If M admits a G_u-structure, then F^2M admits a $G_{\tilde{u}_i^2}$-structure, $i=1, 2$. Moreover, if the G_u-structure is integrable then the induced $G_{u_i^2}$-structure is so also, $i=1, 2$.* #

Corollary 6.9. *If M has an almost symplectic (resp. symplectic) structure then F^2M has two induced almost presymplectic (resp. presymplectic) structures.* #

Let be $u \in \otimes_2(R^n)^*$, P a G_u-structure on M, and $\tilde{P}_i^{(2)}$, $i=1, 2$, the induced $G_{\tilde{u}_i^2}$-structure on F^2M, and g (resp. $\tilde{g}_i^{(2)}$, $i=1, 2$) the tensor field of type $(0, 2)$ on M (resp. on F^2M) associated to P (resp. to $\tilde{P}_i^{(2)}$, $i=1, 2$). Then we have in U

$$g = g_{ij}\, dx^i \otimes dx^j = u_{ij} \theta^i \otimes \theta^j$$

and from (6.5) and (6.6), we have in F^2U

$$\tilde{g}_1^{(2)} = \sum_{\substack{\lambda \in N_2 \\ |\lambda| = 1}} \{u_{ij} (\theta^i) \otimes (\theta^j)^{(\lambda)} + u_{ij} (\theta^i) \otimes (\theta^j)^V\},$$

$$\tilde{g}_2^{(2)} = \sum_{\substack{\nu \in N_2 \\ |\nu| = 2}} \{u_{ij} (\theta^i)^V \otimes (\theta^j)^{(\nu)} + \sum_{\substack{\lambda \in N_2 \\ |\lambda| = 1}} u_{ij} (\theta^i)^{(\nu-\lambda)} \otimes (\theta^j)^{(\lambda)} + u_{ij} (\theta^i)^{(\nu)} \otimes (\theta^j)^V\}$$

Then one easily deduces

$$\tilde{g}_1^{(2)} = \sum_{\substack{\lambda \in N_2 \\ |\lambda| = 1}} g^{(\lambda)}, \quad \tilde{g}_2^{(2)} = \sum_{\substack{\nu \in N_2 \\ |\nu| = 2}} g^{(\nu)}. \tag{6.7}$$

Remark. Let π_1^2: $F^2M \to FM$ be the projection defined by $\pi_1^2(j_0^2 f)=j_0^1 f$. Then, from (6.7) one obtains that $\tilde{g}_1^{(2)}=(\pi_1^2)^* g^C$, where g^C is the complete lift of g to FM defined by MOK [5].

As in [1], [7] and [8] one can obtain prolongations of $Gl(V, W)$ and $Sl(n, R)$-structures on M to F^2M.

§ 7. Lifts of G-connections to F^2M

Let G be a Lie subgroup of $Gl(n)$, $P(M, \pi, G)$ a G-structure on M and ∇ a linear connection on M. Let U be an arbitrary coordinate neighbourhood on M and $\{X_i\}$ a local field of frames adapted to P. Then, for any vector field Y on M assume that

$$\nabla_Y X_i = Y^k \Lambda_{ki}^h X_h \tag{7.1}$$

holds, the matrix $(Y^k \Lambda_{ki}^h)$ belonging to the Lie algebra $\underline{G}$ of G, where $Y=Y^k X_k$. Under these assumptions, ∇ is said to be a G-connection relative to the G-structure P and the coefficients Λ_{ki}^h in (7.1) are called the components of ∇ with respect to the adapted frame $\{X_i\}$.

Now, we consider the so called complete lift ∇^C to $T_n^2 M$ of a linear connection ∇ on M, which is defined as the unique linear connection on $T_n^2 M$ verifying

$$\nabla^C_{X^{(\alpha)}} Y^{(\beta)} = (\nabla_X Y)^{(\alpha+\beta)},$$

for any vector fields X and Y on M, $\alpha, \beta \in N_2$ (see MORIMOTO [9]). Now, we consider the restriction of ∇^C to F^2M, also denoted by ∇^C. Then ∇^C is characterized by the following identities

$$\begin{gathered}\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \quad \nabla^C_{X^C} Y^{(\lambda)} = \nabla^C_{X^{(\lambda)}} Y^C = (\nabla_X Y)^{(\lambda)},\\ \nabla^C_{X^C} Y^{(\nu)} = \nabla^C_{X^{(\nu)}} Y^C = (\nabla_X Y)^{(\nu)}, \quad \nabla^C_{X^{(\lambda)}} Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu)},\\ \nabla^C_{X^{(\lambda)}} Y^{(\nu)} = \nabla^C_{X^{(\nu)}} Y^{(\lambda)} = \nabla^C_{X^{(\nu)}} Y^{(\eta)} = 0,\end{gathered} \tag{7.2}$$

for any vector fields X and Y on M, and every $\lambda, \mu, \nu, \eta \in N_2$ such that $|\lambda|=|\mu|=1$, $|\nu|=|\eta|=2$.

Definition 7.1. ∇^C is called the complete lift of ∇ to F^2M.

Now, if X_i is a local field of frames on U adapted to a G-structure P, then $\{X_i^C, X_i^{(\lambda)}, X_i^{(\nu)}\}$ is a local field of frames on F^2U adapted to the prolongation $P^{(2)}$ of P to F^2M. Hence, from (7.1) and (7.2), we obtain

$$\nabla^C_{X_i^C} X_j^C = (\Lambda_{ij}^h)^V X_h^C + \sum_{\substack{\lambda \in N_2\\ |\lambda|=1}} (\Lambda_{ij}^h)^{(\lambda)} X_h^{(\lambda)} + \sum_{\substack{\nu \in N_2\\ |\nu|=2}} (\Lambda_{ij}^h)^{(\nu)} X_h^{(\nu)},$$

$$\nabla^C_{X_i^C} X_j^{(\lambda)} = \nabla^C_{X_i^{(\lambda)}} X_j^C = (\Lambda_{ij}^h)^V X_h^{(\lambda)} + \sum_{\substack{\mu \in N_2\\ |\mu|=2}} (\Lambda_{ij}^h)^{(\mu)} X_h^{(\mu+\lambda)},$$

$$\nabla^C_{X_i^C} X_j^{(\nu)} = \nabla^C_{X_i^{(\nu)}} X_j^C = (\Lambda_{ij}^h)^V X_h^{(\nu)},$$

$$\nabla^C_{X_i^{(\lambda)}} X_j^{(\nu)} = \nabla^C_{X_i^{(\nu)}} X_j^{(\lambda)} = \nabla^C_{X_i^{(\nu)}} X_j^{(\eta)} = 0.$$

Therefore, we deduce

Theorem 7.2. *Let ∇ be a G-connection relative to a G-structure P on M. Then the complete lift ∇^C of ∇ to F^2M is a $G^{(2)}$-connection relative to the prolongation $P^{(2)}$ of P to F^2M.* #

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